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Simultaneous stabilization of systems with zeros at infinity or zero[1]

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Abstract
It is shown that a class of linear, time-invariant, multi-input multi-output plants that all have zeros at infinity or at zero but do not have other unstable poles can be simultaneously stabilized. A procedure is proposed to design simultaneously stabilizing controllers. All simultaneously stabilizing controllers for these classes are also characterized in terms of a parameter matrix that has to satisfy a unimodularity condition.

1 Introduction

Controller design for simultaneous stabilization is a challenging problem. The well-known parametrization of all stabilizing controllers in the standard unity-feedback system leads to explicit necessary and sufficient conditions for existence of controllers that simultaneously stabilize two given plants [5]. These simple conditions require that a pseudo-plant associated with the two given plants has the parity-interlacing-property (PIP). However, there are no known necessary and sufficient conditions for existence of simultaneously stabilizing controllers for a general class of three or more arbitrary plants. Conditions restricted to checking the real-axis pole-zero locations are not sufficient to guarantee existence of simultaneously stabilizing controllers [1, 3]. In the absence of necessary and sufficient conditions applicable to a completely general class of (three or more) plants, it may be possible to conclusively answer the question of simultaneous stabilizability for some special classes [6, 2, 4]. As a special case, we consider the class $\mathcal{P} = \{P_o, P_1, \ldots, P_n\}$ of $n+1$ linear time-invariant (LTI), multi-input multi-output (MIMO) plants that have no other zeros in the region of instability except at infinity or at zero; the plants all have w blocking-zeros at $s = \infty$ and m blocking-zeros at $s = 0$, where w or m are non-negative integers. Although the class considered here includes finitely many MIMO plants as "centers" and proposes simultaneously stabilizing controller design that guarantees stabilization of these centers, "small" perturbations around these centers are also stabilized using the same controller as in standard robustness results. The main result (Proposition 2.2) is a simple design procedure based on calculating w positive real constants $k_1, \ldots, k_w$ and m positive real constants $f_1, \ldots, f_m$ that define a simultaneously stabilizing controller. All simultaneously stabilizing controllers can be obtained from this central controller in terms of a stable controller-parameter satisfying a restrictive unimodularity condition. Due to the algebraic framework, the results apply to continuous-time as well as discrete-time systems. A continuous-time setting was assumed throughout for simplicity; in the discrete-time case, all evaluations and discussions involving poles and zeros at $s = 0$ should be interpreted at $z = 1$.

Notation: Let $\mathcal{U}$ be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, rational functions (with real coefficients), proper and strictly-proper rational functions, proper rational functions that have no poles in the region of instability $\mathcal{U}$ are denoted by $\mathbb{R}$, $\mathcal{F}$, $\mathrm{R_p}$, $\mathrm{R_s}$, $\mathcal{R}$. The set of matrices whose entries are in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called stable iff $M \in \mathcal{M}(\mathcal{R})$; a square $M \in \mathcal{M}(\mathcal{R})$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. For $M \in \mathcal{M}(\mathcal{R})$, $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value, $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$.

2 Main Results

Consider the standard LTI, MIMO, unity-feedback system $\mathcal{S}(P_j, C)$, where $P_j : e_P \mapsto y$, $C : e \mapsto y_C$, $e = r - y$, $e_P = y_C + u$; $P_j \in \mathrm{R_p}^{n_y \times n_u}$ and $C \in \mathrm{R_p}^{n_u \times n_y}$ represent the transfer-functions of the plant and the controller. It is assumed that P_j and C have no hidden modes corresponding to eigenvalues in $\mathcal{U}$.

2.1. Assumptions: The plant $P_j \in \mathrm{R_p}^{n_y \times n_u}$ belongs to the class $\mathcal{P} := \{P_o, P_1, \ldots, P_n\}$; for $j \in \{0, 1, \ldots, n\}$, each $P_j \in \mathcal{P}$ satisfies the following assumptions: *i)* (normal) rank$P_j = n_y$; *ii)* P_j has exactly w blocking-zeros at ∞ and exactly m blocking-zeros at zero (i.e., $s^{w-1} P_j(\infty) = 0$, $s^w P_j(\infty) \neq 0$, $s^{-(m-1)} P_j(0) = 0$, $s^{-m} P_j(0) \neq 0$) but it has no other transmission-zeros in $\mathcal{U}$; w and m are non-negative integers; *iii)* when $w \neq 0$, $(s^w P_j)(\infty)\Delta_j = (s^w P_o)(\infty)$, for some symmetric positive-definite matrix $\Delta_j \in \mathbb{R}^{n_y \times n_y}$; *iv)* when $m \neq 0$, $(s^{-m} P_j)(0)\Theta_j = (s^{-m} P_o)(0)$, for some symmetric positive-definite matrix $\Theta_j \in \mathbb{R}^{n_y \times n_y}$. □

The system $\mathcal{S}(P_j, C)$ is said to be stable iff the transfer-function H from (r, u) to (y, y_C) is stable, i.e., $H \in \mathcal{M}(\mathcal{R})$. The controller C is said to be a stabilizing controller for the plant P_j (or C stabilizes P_j) iff $C \in \mathcal{M}(\mathrm{R_p})$ and the system $\mathcal{S}(P_j, C)$ is stable; C is

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said to simultaneously stabilize all $P_j \in \mathcal{P}$ iff the system $\mathcal{S}(P_j, C)$ is stable for all $P_j \in \mathcal{P}$.

A simultaneously stabilizing controller design procedure for the class $\mathcal{P}$ is given in Proposition 2.2. In addition to finding one such controller explicitly, all controllers can also be characterized based on one of the plants, P_o, called the nominal plant. The choice of the nominal plant in the class $\mathcal{P}$ is completely arbitrary.

2.2. Proposition: Let $P_j \in {\mathrm{R_p}}^{n_y \times n_y}$ belong to the class $\mathcal{P} := \{P_o, P_1, \ldots, P_n\}$. For $i = 1, \ldots, w$, let $-\alpha_i \in \mathbb{R} \backslash \mathcal{U}$; for $i = 1, \ldots, m$, let $-\beta_i \in \mathbb{R} \backslash \mathcal{U}$. For $j \in \{0, 1, \ldots, n\}$, define $P_j =: D_j^{-1} \frac{s^m}{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)}$, where $D_j \in \mathcal{M}(\mathcal{R})$, $\det D_j(\infty) \neq 0$. Let $k_1 \in \mathbb{R}$ be such that $k_1 > \max_{j \in \{0,\ldots,n\}} \| s(\Delta_j - D_j D_o^{-1}(\infty))\|$. For $v = 2, \ldots, w$, let $k_v \in \mathbb{R}$ be such that $k_v > \max_{j \in \{0,\ldots,n\}} \|s(I + D_j D_o^{-1}(\infty) \sum_{i=1}^{v-1} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell})^{-1} (I + D_j D_o^{-1}(\infty) \sum_{i=1}^{v-2} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell})\|$. Define $K := D_o^{-1}(\infty) \sum_{i=1}^{w} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell} \in \mathcal{M}(\mathcal{F})$, $W_j := \prod_{i=1}^{w} \frac{k_i}{(s+\alpha_i)} (I + D_j K) \in \mathcal{M}(\mathcal{R})$. Choose $X_1 \in \mathcal{M}(\mathcal{R})$ such that $X_1(\infty)$ is nonsingular. Let $f_1 \in \mathbb{R}$ be such that $0 < f_1 < \min_{j \in \{0,\ldots,n\}} \|s^{-1}(\Theta_j \prod_{i=1}^{w} \frac{\alpha_i}{k_i} - W_j^{-1} D_j D_o^{-1}(0)) - W_j^{-1} D_j D_o^{-1}(0) X_1\|^{-1}$. For $v = 2, \ldots, m$, let $f_v \in \mathbb{R}$ be such that $0 < f_v < \min_{j \in \{0,\ldots,n\}} \| s^{-1}(I + D_j(K + \prod_{i=1}^{w} \frac{(s+\alpha_i)}{k_i} D_o^{-1}(0)(I + sX_1) \sum_{i=1}^{v-1} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell))^{-1} (I + D_j(K + \prod_{i=1}^{w} \frac{(s+\alpha_i)}{k_i} D_o^{-1}(0)(I + sX_1) \sum_{i=1}^{v-2} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell))\|^{-1}$. All $P_j \in \mathcal{P}$ can be simultaneously stabilized by the controller

$$C_o = \frac{\prod_{i=1}^{m}(s+\beta_i)}{s^m} \Big(\frac{D_o^{-1}(\infty)}{\prod_{i=1}^{w}(s+\alpha_i)} \sum_{i=1}^{w} s^i \prod_{\ell=1}^{i} \frac{1}{k_\ell} + \frac{1}{\prod_{i=1}^{w} k_i} D_o^{-1}(0)(I + sX_1) \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell \Big)^{-1} \in {\mathrm{R_p}}^{n_y \times n_y}. \tag{1}$$

Furthermore, all controllers C that simultaneously stabilize all $P_j \in \mathcal{P}$, for $j \in \{1, \ldots, n\}$, are

$$C = \Big(C_o^{-1} - \frac{s^m\, Q}{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)} \Big)^{-1} (I + QD_o), \tag{2}$$

where $Q \in \mathcal{M}(\mathcal{R})$ is such that $G_j := I + Q(D_o - D_j)(I + C_o^{-1} D_j \frac{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)}{s^m})^{-1}$ is unimodular. $\square$

The stable controller-parameter Q in the simultaneously stabilizing controller characterization (2) must be so that G_j is unimodular. This condition is obviously satisfied for $Q = 0$, corresponding to the central controller C_o in (1). Some additional possible choices other than $Q = 0$ satisfying this condition can be obtained by choosing $Q \in \mathcal{M}(\mathcal{R})$ 'sufficiently small' as $\|Q\| < \min_{j \in \{1,\ldots,n\}} \|(D_j - D_o)(I + C_o^{-1} D_j \frac{\prod_{i=1}^{w}(s+\alpha_i) \prod_{i=1}^{m}(s+\beta_i)}{s^m})^{-1}\|^{-1}$. The simultaneously stabilizing controller C in (2) is strictly-proper if and only if $Q \in \mathcal{M}(\mathcal{R})$ is such that $Q(\infty) = -D_o^{-1}(\infty)$.

In Proposition 2.2, the plants $P_j \in \mathcal{P}$ all have w blocking-zeros at infinity and m blocking-zeros at zero. If $m = 0$, then by (1), all $P_j \in \mathcal{P}$ can be simultaneously stabilized by the controller $\widetilde{C}_o = K^{-1} \prod_{i=1}^{w}(s+\alpha_i)$. Furthermore, all controllers $\widetilde{C}$ that simultaneously stabilize all $P_j \in \mathcal{P}$ are $\widetilde{C} = (\widetilde{C}_o^{-1} - \frac{1}{\prod_{i=1}^{w}(s+\alpha_i)} Q)^{-1}(I + QD_o)$, where $Q \in \mathcal{M}(\mathcal{R})$ is such that $\widetilde{G}_j := I + Q(D_o - D_j)(I + \widetilde{C}_o^{-1} D_j \prod_{i=1}^{w}(s+\alpha_i))^{-1}$ is unimodular, for $j \in \{1, \ldots, n\}$. If $w = 0$, then by (1), all $P_j \in \mathcal{P}$ can be simultaneously stabilized by the controller $\hat{C}_o = \frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)}((I + sX_1) \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} f_\ell)^{-1} D_o(0)$. Furthermore, all controllers $\hat{C}$ that simultaneously stabilize all $P_j \in \mathcal{P}$ are $\hat{C} = (\hat{C}_o^{-1} - \frac{s^m}{\prod_{i=1}^{m}(s+\beta_i)} Q)^{-1}(I + QD_o)$, where $Q \in \mathcal{M}(\mathcal{R})$ is such that $\det(f_1 X_1 - Q)(\infty) \neq 0$ and $\hat{G}_j := I + Q(D_o - D_j)(I + \hat{C}_o^{-1} D_j \frac{\prod_{i=1}^{m}(s+\beta_i)}{s^m})^{-1}$ is unimodular, for $j \in \{1, \ldots, n\}$. If $m = 0$ and $w = 0$, then the plants in the class $\mathcal{P}$ have no transmission-zeros in $\mathcal{U}$. Using the notation of Proposition 2.2, each P_j can be written as $P_j = D_j^{-1}$, where $D_j \in \mathcal{M}(\mathcal{R})$. In this case, all controllers that simultaneously stabilize all P_j are $C = -Q^{-1}(I + QD_o)$, where $Q \in \mathcal{M}(\mathcal{R})$ is such that $I + Q(D_o - D_j)$ is unimodular and $\det Q(\infty) \neq 0$. The simultaneously stabilizing controller design procedure proposed here applies to a set of MIMO plants that have the same number of blocking-zeros at infinity or at zero (or both). It may be possible to extend this procedure to classes of plants that all have the same number of blocking-zeros at one fixed real-axis location in $\mathcal{U}$. With some minor additional assumptions on the plants, existence of simultaneously stabilizing controllers in such cases can be proved although explicit controller design remains an open problem.

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