Parameterization of Stabilizing Controllers with Integral Action

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***Abstract*— In the standard linear time-invariant multi-input/multi-output unity-feedback system, a parameterization of stabilizing controllers with integral action is obtained. These controllers guarantee asymptotic tracking of step reference inputs at each output channel with zero steady-state error.**

***Index Terms*—Asymptotic tracking, controller design, integral action.**

I. Introduction

Stabilizing controllers that achieve robust asymptotic tracking of general reference signals can be designed by using the well-known parameterization of all controllers that stabilize a given plant [4]. For asymptotic tracking of step reference signals, controllers are designed to have integral action (see, for example, [1] and [3]).

In this paper we parameterize controllers with integral action in the standard linear time-invariant (LTI), multi-input/multi-output (MIMO) unity-feedback system. We show that any stabilizing controller with integral action (as defined here) is expressed explicitly as the sum of two controllers: any arbitrary stabilizing controller and a controller with an integral term.

The paper is organized as follows: in Section II, following the problem description and the stability and integral action definitions, Lemma 1 claims that any stable (MIMO) system can be stabilized using an integral controller. Based on such an integral controller for the stable numerator factor of any coprime factorization of the plant, all stabilizing controllers with integral action are parameterized in the main result, Theorem 1. The special case of this parameterization for stable plants is given in Corollary 1. The parameterization of all stabilizing controllers with integral action can also be derived starting with state-space representations as explained in Comment 1. The proofs are given in the Appendix.

Due to the algebraic framework described in the following notation, the results apply to continuous-time as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at $s = 0$ would be interpreted at $z = 1$.

Notation: Let $\mathcal{U}$ be the extended closed right half-plane (for continuous-time systems) or the complement of the open unit disk (for discrete-time systems). The set of real numbers, the set of proper rational functions that have no poles in the region of instability $\mathcal{U}$, and the sets of proper and strictly proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathcal{R}$, $\mathrm{R_p}$, $\mathrm{R_s}$, respectively. The set of matrices whose entries are in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called stable iff $M \in \mathcal{M}(\mathcal{R})$ (a notation of the form $\mathcal{M} \in \mathcal{R}^{n\times m}$ is used where it is important to indicate the order of a matrix explicitly); a stable M is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. For $M \in \mathcal{M}(\mathcal{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s\in\partial\mathcal{U}} \overline{\sigma}(M(s))$, where $\overline{\sigma}$ and $\partial\mathcal{U}$ denote the maximum singular value and the boundary of $\mathcal{U}$, respectively. A right coprime-factorization (RCF) and a left coprime-factorization (LCF) of $P \in \mathrm{R_p}^{n_y\times n_u}$ are denoted by $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$, where $N, D, \tilde{N}, \tilde{D} \in \mathcal{M}(\mathcal{R})$, D, and $\tilde{D}$ are biproper. Let $\mathrm{rank}\ P = r$; $s_o \in \mathcal{U}$ is a (transmission) zero of P if and only if $\mathrm{rank}\ N(s_o) = \mathrm{rank}\ \tilde{N}(s_o) < r$; s_o is called a blocking-zero of P iff $P(s_o) = 0$; $s_o \in \mathcal{U}$ is a blocking-zero of P if and only if $N(s_o) = 0 = \tilde{N}(s_o)$.

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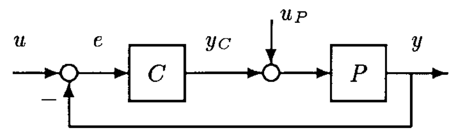

Fig. 1. The system $\mathcal{S}(P, C)$.

II. Main Results

Consider the standard LTI MIMO unity-feedback system $\mathcal{S}(P, C)$ shown in Fig. 1 where $\mathcal{S}(P, C)$ is a well-posed system and $P \in \mathrm{R_p}^{n_y\times n_u}$ and $C \in \mathrm{R_p}^{n_u\times n_y}$ represent the transfer functions of the plant and the controller. It is assumed that P and C have no hidden modes corresponding to eigenvalues in the region of instability $\mathcal{U}$. Let H_{eu} denote the (input-error) transfer function from u to e, and let H_{yu} denote the (input–output) transfer function from u to y.

Definitions 1:

1) *Stability:* The system $\mathcal{S}(P, C)$ is said to be stable iff the transfer function H from (u, u_P) to (y, y_C) is stable, i.e., $H \in \mathcal{M}(\mathcal{R})$.
2) *Integral Action:* The stable system $\mathcal{S}(P, C)$ is said to have integral action in each output channel iff the (input-error) transfer function $H_{eu}(s) = I - H_{yu}(s)$ has blocking-zeros at $s = 0$.
3) *Stabilizing Controller:* The controller C is said to be a stabilizing controller for the plant P (or C is said to stabilize P) iff $C \in \mathcal{M}(\mathrm{R_p})$ and the system $\mathcal{S}(P, C)$ is stable.
4) *Stabilizing Controller with Integral Action:* The controller C is said to be a stabilizing controller with integral action iff C stabilizes P and $D_C(s)$ has blocking-zeros at $s = 0$, where $D_C \in \mathcal{R}^{n_y\times n_y}$ is the denominator-matrix of any RCF $N_C D_C^{-1}$ of C. □

Let $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ be any RCF and LCF of $P \in \mathrm{R_p}^{n_y\times n_u}$. Let $C = N_C D_C^{-1}$ be any RCF of $C \in \mathrm{R}_p^{n_u\times n_y}$. The controller C stabilizes P if and only if $(\tilde{D}D_C + \tilde{N}N_C)$ is unimodular for any RCF $N_C D_C^{-1}$ of C [2], [4]. All stabilizing controllers for P are given by

$$C = (\tilde{U} + DQ)(\tilde{V} - NQ)^{-1} = (V - Q\tilde{N})^{-1}(U + Q\tilde{D}) \tag{1}$$

where $Q \in \mathcal{R}^{n_u\times n_y}$ is such that $(\tilde{V} - NQ)$ is biproper, which holds for all $Q \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$; in (1), $U, V, \tilde{U}, \tilde{V} \in \mathcal{M}(\mathcal{R})$ are stable matrices such that

$$\begin{bmatrix} V & U \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N & \tilde{V} \end{bmatrix} = \begin{bmatrix} I_{n_u} & 0 \\ 0 & I_{n_y} \end{bmatrix}. \tag{2}$$

Using the parameterization (1) of all stabilizing controllers, for any stabilizing controller C, the corresponding (input-error) transfer function $H_{eu} = (I_{n_y} + PC)^{-1}$ is given by

$$H_{eu} = I_{n_y} - N(U + Q\tilde{D}) = (\tilde{V} - NQ)\tilde{D}. \tag{3}$$

If $\mathcal{S}(P, C)$ is stable, the (input-error) transfer function $H_{eu}(0) = (I_{n_y} + PC)^{-1}(0) = I_{n_y} - PC(I_{n_y} + PC)^{-1}(0) = 0$ only if $\mathrm{rank}\ P = n_y \le n_u$. Also by (3), if $H_{eu}(0) = I_{n_y} - N(0)(U + Q\tilde{D})(0) = 0$, then $\mathrm{rank} N(0) = n_y \le n_u$. Therefore, it is clear that a crucial necessary condition on the plant P for the stable system $\mathcal{S}(P, C)$ to have integral action is that $\mathrm{rank}\ P = n_y \le n_u$ and P has no (transmission) zeros at $s = 0$.

In any arbitrary RCF $N_C D_C^{-1}$ of a stabilizing controller C, the factors (N_C, D_C) are given by $(N_C, D_C) = ((\tilde{U} + DQ)R, (\tilde{V} - NQ)R)$ for some unimodular $R \in \mathcal{M}(\mathcal{R})$; therefore, $D_C(0) = 0$ is equivalent to $(\tilde{V} - NQ)(0) = 0$. By Definition 1, C is a stabilizing controller with integral action if and only if $(\tilde{V} - NQ)(0) = 0$; therefore, if C is a stabilizing controller with integral action, then $H_{eu}(0) = (\tilde{V} - NQ)(0)\tilde{D}(0) = 0$ and hence the stable system $\mathcal{S}(P, C)$ has integral action in each output channel. Although designing the stabilizing controllers so that $D_C(0) = 0$ is a sufficient condition for the stable system $\mathcal{S}(P, C)$ to have integral action, it is clearly not necessary. However, when P has no poles at $s = 0$, and in particular when P is stable, $H_{eu}(0) = 0$ if and only if $D_C(0) = 0$, i.e., the stable system $\mathcal{S}(P, C)$ has integral action if and only if the controller C is a stabilizing controller with integral action.

A simple parameterization of all stabilizing controllers with integral action is given in Theorem 1. This parameterization is based on an arbitrary integral controller designed for the stable numerator matrix N in any RCF ND^{-1} of P. Lemma 1 guarantees existence of an integral controller K_i/s for any (MIMO) stable transfer function N.

Lemma 1—Existence of Integral Controllers for Stable Systems: Let $N \in \mathcal{R}^{n_y \times n_u}$ where $\mathrm{rank}\ N = N_y \le N_u$. There exists an integral controller K_i/s that stabilizes N, where $K_i \in \mathbb{R}^{n_u \times n_y}$, if and only if $\mathrm{rank}\ N(0) = n_y \le n_u$; equivalently, there exists a constant controller $K_i \in \mathbb{R}^{n_u \times n_y}$ that stabilizes N/s if and only if $\mathrm{rank}\ N(0) = n_y \le n_u$.

Theorem 1—All Stabilizing Controllers with Integral Action: Let $P \in \mathrm{R}_{\mathrm{p}}^{n_y \times n_u}$, where $\mathrm{rank}\ P = n_y \le n_u$. Let P have no (transmission) zeros at $s = 0$. Let $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ be any RCF and LCF of P. Let K_i/s be any integral controller that stabilizes N, where $K_i \in \mathbb{R}^{n_u \times n_y}$. Let $U, V, \tilde{U}, \tilde{V} \in \mathcal{M}(\mathcal{R})$ satisfy (2). The controller C is a stabilizing controller with integral action if and only if

$$C = (V - Q_1\tilde{N})^{-1}\left(U + Q_1\tilde{D} + \frac{K_i}{s}\right) \tag{4}$$

or equivalently

$$C = (\tilde{U} + DQ_1)(\tilde{V} - NQ_1)^{-1}\left(I_{n_y} + N\frac{K_i}{s}\right) + D\frac{K_i}{s} \tag{5}$$

where $Q_1 \in \mathcal{R}^{n_u \times n_y}$ is such that $(V - Q_1\tilde{N})$ is biproper (equivalently, $(\tilde{V} - NQ_1)$ is biproper), which holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$. The corresponding (input–output) transfer function H_{yu} of the stable system $\mathcal{S}(P, C)$ is $H_{yu} = (I_{n_y} + N(K_i/s))^{-1}N(U + Q_1\tilde{D} + K_i/s)$. □

If the plant is stable, then $P = PI_{n_u}^{-1} = I_{n_y}^{-1}P$ is an RCF and an LCF of P and a solution for (2) is given by $V = I_{n_u}$, $\tilde{V} = I_{n_y}$, $U = \tilde{U} = 0$. By (3), $H_{eu} = I_{n_y} - PQ$ and hence the stable system $\mathcal{S}(P, C)$ has integral action if and only if the controller C is a stabilizing controller with integral action. The parameterization (4) of all controllers with integral action is simplified for the special case of stable plants in Corollary 1.

Corollary 1—All Stabilizing Controllers with Integral Action for Stable Plants: Let $P \in \mathcal{R}^{n_y \times n_u}$, where $\mathrm{rank}\ P = n_y \le n_u$. Let P have no (transmission) zeros at $s = 0$, i.e., $\mathrm{rank}\ P(0) = n_y$. Let K_i/s be any integral controller that stabilizes P, where $K_i \in \mathbb{R}^{n_u \times n_y}$. The controller C is a stabilizing controller with integral action if and only if

$$C = (I_{n_u} - Q_1 P)^{-1}\left(Q_1 + \frac{K_i}{s}\right) \tag{6}$$

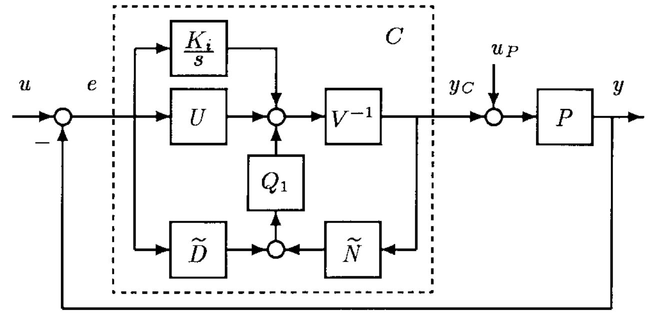

Fig. 2. The system $\mathcal{S}(P, C)$ where C is a stabilizing controller with integral action.

where $Q_i \in \mathcal{R}^{n_u \times n_y}$ is such that $(I_{n_u} - Q_1 P)$ is biproper, which holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$. The corresponding (input–output) transfer function H_{yu} of the stable system $S(P, C)$ is $H_{yu} = (I_{n_y} + P(K_i/s))^{-1}P(Q_1 + (K_i/s))$. □

Comments 1:

a) Simple interpretation of the parameterization of all stabilizing controllers with integral action: Theorem 1 states that any stabilizing controller with integral action is expressed as the sum of an arbitrary stabilizing controller $(V - Q_1\tilde{N})^{-1}(U + Q_1\tilde{D})$ and a controller with an integral term $(V - Q_1\tilde{N})^{-1}K_i/s$. The block-diagram of the system $\mathcal{S}(P, C)$ with the stabilizing controller C as in (4) is shown in Fig. 2.

b) The integral controller stabilizing N: In Theorem 1, $K_i \in \mathbb{R}^{n_u \times n_y}$ is such that K_i/s stabilizes N, equivalently, $K_i \in \mathbb{R}^{n_u \times n_y}$ is any constant controller that stabilizes N/s. Since $\mathrm{rank}\ N(0) = n_y$ by assumption, existence of such controllers is guaranteed by Lemma 1; in fact, as in the proof of Lemma 1, K_i can be chosen as $K_i = \beta N(0)^I$ for any positive $\beta \in \mathbb{R}$ satisfying (12), i.e.,

$$0 < \beta < \left\| \frac{N(s)N(0)^I - I_{n_y}}{s} \right\|^{-1} \tag{7}$$

where $N(0)^I$ denotes any right-inverse of $N(0)$. Note that $\mathrm{rank}\ K_i = n_y$ for any K_i that stabilizes N/s.

c) Full-Order observer-based realization of all stabilizing controllers with integral action: The parameterization in (4) of all stabilizing controllers with integral action can also be obtained by using the coprime factorizations of P obtained from a state-space representation as follows: Let $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ be a state-space representation of $P = \overline{C}(sI_n - \overline{A})^{-1}\overline{B} + \overline{D}$, where $(\overline{A}, \overline{B})$ is stabilizable and $(\overline{C}, \overline{A})$ is detectable. Let $K \in \mathbb{R}^{n_u \times n}$ and $L \in \mathbb{R}^{n \times n_y}$ be such that $A_K := (sI_n - \overline{A} + \overline{B}K)^{-1} \in \mathcal{M}(\mathcal{R})$, $A_L := (sI_n - \overline{A} + L\overline{C})^{-1} \in \mathcal{M}(\mathcal{R})$. Let $\mathrm{rank}\ N(0) = \mathrm{rank}\,(\overline{D} - (\overline{C} - \overline{D}K)(\overline{A} - \overline{B}K)^{-1}\overline{B}) = n_y \le n_u$. Let K_i/s be any integral controller that stabilizes $N = (\overline{C} - \overline{D}K)A_K\overline{B} + \overline{D}$, where $K_i \in \mathbb{R}^{n_u \times n_y}$. By Theorem 1, the controller C is a stabilizing controller with integral action if and only if

$$\begin{aligned} C = {} & (I_{n_u} + KA_L(\overline{B} - L\overline{D}) \\ & - Q_1(\overline{C}A_L(\overline{B} - L\overline{D}) + \overline{D}))^{-1} \\ & \times \left(KA_L L + Q_1(I_{n_y} - \overline{C}A_L L) + \frac{K_i}{s}\right) \end{aligned} \tag{8}$$

where $Q_1 \in \mathcal{R}^{n_u \times n_y}$ is such that $\det(I_{n_u} - Q_1(\infty)\overline{D}) \neq 0$, which holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$. The block-diagram of the system $\mathcal{S}(P, C)$ with the stabilizing controller C as in (8) is shown in Fig. 3.

Note again that, as in the proof of Lemma 1 and Comment b) above, K_i can be chosen as $K_i = \beta N(0)^I$ for any positive $\beta \in \mathbb{R}$

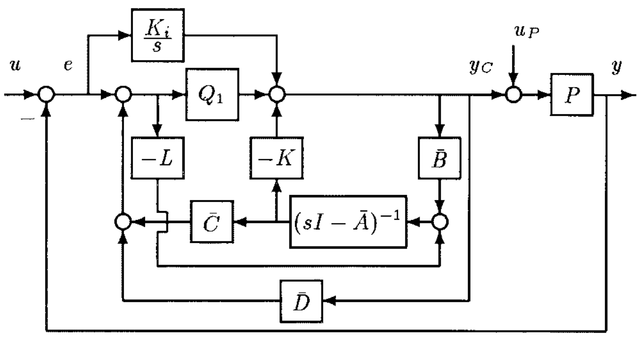

Fig. 3. The system $\mathcal{S}(P, C)$ with an observer-based stabilizing controller with integral action.

satisfying (7). In this case, a right-inverse $N(0)^I$ of $N(0)$ is given by

$$N(0)^I = (\overline{D} - (\overline{C} - \overline{D}K)(\overline{A} - \overline{B}K)^{-1}\overline{B})^I \tag{9}$$

where $(\overline{A} - \overline{B}K)^{-1}$ exists since $A_K \in \mathcal{M}(\mathcal{R})$ implies A_K has no poles at $s = 0$. Therefore, (7) becomes

$$\begin{aligned} 0 < \beta &< \left\| \frac{N(s)N(0)^I - I_{n_y}}{s} \right\|^{-1} \\ &= \|(\overline{C} - \overline{D}K)A_K(\overline{A} - \overline{B}K)^{-1}\overline{B}N(0)^I\|^{-1}. \end{aligned} \tag{10}$$

d) Simple algorithm for parameterizing all stabilizing controllers with integral action: Theorem 1, with the explanations in Comments b) and c) above, lead obviously to the following very simple algorithm for finding all stabilizing controllers with integral action.

Let the given plant $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ satisfy the assumptions in Theorem 1. Let $C \in \mathcal{M}(\mathrm{R_p})$ be any stabilizing controller for P. Find any LCF $C = \tilde{D}_C^{-1}\tilde{N}_C$ of C. Find any right-inverse $N(0)^I$ of $N(0)$. Let $K_i = \beta N(0)^I$, where β is any positive real constant satisfying (7). All stabilizing controllers with integral action are parameterized by $C = (\tilde{D}_C - Q_1\tilde{N})^{-1}(\tilde{N}_C + Q_1\tilde{D} + K_i/s)$, where $Q_1 \in \mathcal{M}(\mathcal{R})$ is any stable matrix of appropriate order such that $\det(\tilde{D}_C - Q_1\tilde{N})(\infty) \neq 0$.

We can restate the algorithm more explicitly in terms of a state-space representation of the given plant: given $P = \overline{C}(sI_n - \overline{A})^{-1}\overline{B} + \overline{D}$, where $(\overline{A}, \overline{B})$ is stabilizable and $(\overline{C}, \overline{A})$ is detectable.

Step 0: If

$$\operatorname{rank}\begin{bmatrix} -\overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = n_y + n$$

then go to Step 1 (P has no zeros at $s = 0$); else, stop (P has zeros at $s = 0$ and hence, stabilizing controllers with integral action do not exist).

Step 1: Choose any $K \in \mathbb{R}^{n_u \times n}$ and $L \in \mathbb{R}^{n \times n_y}$ such that

$$\begin{aligned} A_K &:= (sI_n - \overline{A} + \overline{B}K)^{-1} \in \mathcal{M}(\mathcal{R}) \\ A_L &:= (sI_n - \overline{A} + L\overline{C})^{-1} \in \mathcal{M}(\mathcal{R}). \end{aligned}$$

Step 2: Choose $K_i = \beta N(0)^I$, where $N(0)^I$ is given by (9) and $\beta \in \mathbb{R}$ satisfies (10).

Step 3: All stabilizing controllers with integral action are given by (8) (see Fig. 3).

APPENDIX
PROOFS

Proof of Lemma 1: Suppose K_i/s stabilizes N (equivalently, K_i stabilizes N/s). Then for any $\alpha \in \mathbb{R}$, $\alpha > 0$

$$\frac{s}{s+\alpha} I_{n_y} + N \frac{K_i}{s+\alpha} =: M \tag{11}$$

is unimodular. Therefore $\operatorname{rank} M(0) = \operatorname{rank}(\alpha^{-1}N(0)K_i) = n_y \le \min\{\operatorname{rank} N(0), \operatorname{rank} K_i\} \le \min\{n_y, n_u\}$ implies $\operatorname{rank} N(0) = n_y \le n_u$. To show the converse, let $\operatorname{rank} N(0) = n_y \le n_u$; then there exists a right-inverse $N(0)^I \in \mathbb{R}^{n_u \times n_y}$ of $N(0)$ such that $N(0)N(0)^I = I_{n_y}$. Let $K_i := \beta N(0)^I$, where $\beta \in \mathbb{R}$, $\beta > 0$ is such that

$$\beta < \left\| \frac{N(s)N(0)^I - I_{n_y}}{s} \right\|^{-1}. \tag{12}$$

Note that $K_i/s = ((\beta/s+\beta)N(0)^I)((s/s+\beta)I_{n_y})^{-1}$ is an RCF of K_i/s. With β as in (12), we have

$$\begin{aligned} &\frac{s}{s+\beta} I_{n_y} + N \frac{\beta}{s+\beta} N(0)^I \\ &\quad = \frac{s}{s+\beta} I_{n_y} + \frac{\beta}{s+\beta} N(0)N(0)^I \\ &\qquad + \frac{\beta}{s+\beta}(N - N(0))N(0)^I \\ &\quad = I_{n_y} + \frac{\beta s}{(s+\beta)} \frac{(N - N(0))N(0)^I}{s} \end{aligned} \tag{13}$$

is unimodular since $\|(\beta s/s+\beta)\| = \beta$. Therefore K_i/s stabilizes N. Equivalently, $N/s = ((s/s+\beta)I_{n_y})^{-1}(N/s+\beta)$ is an LCF of N/s and it follows similarly from (13) that K_i stabilizes N/s. $\square$

Proof of Theorem 1: Let $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ be any RCF and LCF of P. By Lemma 1, there exists $K_i \in \mathbb{R}^{n_u \times n_y}$ such that K_i/s stabilizes N. Now K_i/s stabilizes N if and only if, for any $\alpha \in \mathbb{R}$, $\alpha > 0$, the matrix $M \in \mathcal{M}(\mathcal{R})$ in (11) is unimodular, equivalently

$$\begin{aligned} &\begin{bmatrix} I_{n_u} & \dfrac{K_i}{s+\alpha} \\ -N & \dfrac{s}{s+\alpha} I_{n_y} \end{bmatrix} \begin{bmatrix} I_{n_u} - \dfrac{K_i}{s+\alpha} M^{-1}N & -\dfrac{K_i}{s+\alpha} M^{-1} \\ M^{-1}N & M^{-1} \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{n_u} & 0 \\ 0 & I_{n_y} \end{bmatrix}. \end{aligned} \tag{14}$$

By (1), (N_C, D_C) in any arbitrary RCF $N_C D_C^{-1}$ of a stabilizing controller C are given by $(N_C, D_C) = ((\tilde{U} + DQ)R, (\tilde{V} - NQ)R)$ for some unimodular $R \in \mathcal{M}(\mathcal{R})$; therefore, $D_C(0) = 0$ if and only if $(\tilde{V} - NQ)(0) = 0$. By Definition 1, C is a stabilizing controller with integral action if and only if $(\tilde{V} - NQ)(0) = 0$; hence, finding all stabilizing controllers with integral action is equivalent to finding all solutions for $Q \in \mathcal{R}^{n_u \times n_y}$ and $\hat{D}_c \in \mathcal{R}^{n_y \times n_y}$ of the equality

$$\tilde{V} - NQ = \frac{s}{s+\alpha} \hat{D}_c \tag{15}$$

for any $\alpha \in \mathbb{R}$, $\alpha > 0$. But (15) is equivalent to

$$\begin{bmatrix} I_{n_u} & \dfrac{K_i}{s+\alpha} \\ -N & \dfrac{s}{s+\alpha} I_{n_y} \end{bmatrix} \begin{bmatrix} -Q \\ \hat{D}_c \end{bmatrix} = \begin{bmatrix} -Q_1 \\ \tilde{V} \end{bmatrix}$$

for some $Q_1 \in \mathcal{R}^{n_u \times n_y}$. By (14), all solutions of (15) are given

$$\begin{bmatrix} -Q \\ \hat{D}_c \end{bmatrix} = \begin{bmatrix} -Q_1 - \dfrac{K_i}{s+\alpha} M^{-1}(\tilde{V} - NQ_1) \\ M^{-1}(\tilde{V} - NQ_1) \end{bmatrix}. \tag{16}$$

Plugging $Q = Q_1 + (K_i/s + \alpha)M^{-1}(\tilde{V} - NQ_1) = Q_1 + (K_i/s)(I_{n_y} + N(K_i/s))^{-1}(\tilde{V} - NQ_1) \in \mathcal{M}(\mathcal{R})$ given by (16) into (1), the controller becomes

$$\begin{aligned} C &= (\tilde{U}+DQ)(\tilde{V}-NQ)^{-1} \\ &= \left(\tilde{U}+DQ_1+D\,\frac{K_i}{s+\alpha}\,M^{-1}(\tilde{V}-NQ_1)\right) \\ &\quad \cdot (\tilde{V}-NQ_1)^{-1}M\,\frac{s+\alpha}{s} \\ &= (\tilde{U}+DQ_1)(\tilde{V}-NQ_1)^{-1}\left(I_{n_y}+N\,\frac{K_i}{s}\right)+D\,\frac{K_i}{s} \\ &= (\tilde{U}+DQ_1)(\tilde{V}-NQ_1)^{-1}+(V-Q_1\tilde{N})^{-1}\,\frac{K_i}{s} \\ &= (V-Q_1\tilde{N})^{-1}(U+Q_1\tilde{D})+(V-Q_1\tilde{N})^{-1}\,\frac{K_i}{s} \end{aligned}$$

as claimed in (4) and (5). The controller C is proper if and only if $(\tilde{V}-NQ)=(s/s+\alpha)\hat{D}_c=(s/s+\alpha)M^{-1}(\tilde{V}-NQ_1)$ is biproper, equivalently, $(\tilde{V}-NQ_1)$ is biproper since M is unimodular. Using the same $Q\in\mathcal{M}(\mathcal{R})$ in $H_{yu}=N(U+Q\tilde{D})= I_{n_y}-(\tilde{V}-NQ_1)\tilde{D}$, the corresponding (input–output) transfer function of the stable system $\mathcal{S}(P,C)$ becomes $H_{yu}=I_{n_y}-(I_{n_y}+N(Ki/s))^{-1}(\tilde{V}-NQ_1)\tilde{D}=(I_{n_y}+N(K_i/s))^{-1}N(U+Q\tilde{D}+(K_i/s))$ as claimed. □

REFERENCES

[1] P. J. Campo and M. Morari, "Achievable closed-loop properties of systems under decentralized control: Conditions involving the steady-state gain," *IEEE Trans. Automat. Contr.,* vol. 39, pp. 932–943, 1994.

[2] A. N. Gündeş and C. A. Desoer, *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators,* Lecture Notes in Control and Information Sciences, vol. 142. Berlin, Germany: Springer-Verlag,1990.

[3] M. Morari and E. Zafiriou, *Robust Process Control.* Englewood Cliffs, NJ: Prentice Hall, 1989.

[4] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: M.I.T. Press, 1985.

Robustness of Nonlinear Control Systems with Respect to Unmodeled Dynamics

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***Abstract*—In theory, it can be established that nonlinear control laws for linear or nonlinear plants can be used to meet strict performance requirements. The success of these control designs in practical situations will very much depend on whether they can still meet the expected performance characteristics in the presence of inevitable modeling errors. In this paper, we develop necessary and sufficient conditions for a general class of nonlinear control laws in the presence of high-frequency unmodeled dynamics, under which global signal boundedness or asymptotic stability is guaranteed. We show that a wide class of nonlinear control laws does not satisfy these conditions and therefore does not guarantee global stability in the presence of high-frequency unmodeled dynamics.**

***Index Terms*—Global stability, nonlinear control system, robustness, unmodeled dynamics.**

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I. INTRODUCTION

During the recent years, the design and analysis of nonlinear control systems has been pursued by several investigators [5], [9]–[11]. These designs and results are based on the assumption that the nonlinearities are known and the plant is free of disturbances and unmodeled dynamics. More recent efforts are focused on adaptive techniques to deal with parametric uncertainties and techniques to deal with unknown disturbances and classes of unknown nonlinearities [3], [2], [8], [6], [7], [1], [15], [12], [16]. The issue of unmodeled dynamics has been addressed in [13] and [14], where global results are obtained under the assumption that the "input unmodeled dynamics" are linear time invariant and small in all frequencies. In practice, however, unmodeled dynamics are often small in the low-frequency range, which is usually the range of interest, and are allowed to be large relative to the modeled part in the high-frequency range. Obviously if the unmodeled dynamics are large in the frequency range of interest, then they should be part of the model. It is therefore of interest to examine whether nonlinear control systems that are developed to guarantee global stability for a nonlinear system in the absence of modeling errors can maintain such property in the presence of a general class of unmodeled dynamics that are likely to appear in applications.

In this paper, we develop necessary and sufficient conditions for a general class of nonlinear control laws in the presence of high-frequency unmodeled dynamics, under which global signal boundedness or asymptotic stability is guaranteed. We show that a wide class of nonlinear control laws that guarantees global stability in the absence of unmodeled dynamics does not satisfy these conditions and therefore does not guarantee global stability in the presence of high-frequency unmodeled dynamics. Moreover, These nonlinear controllers can lead to unbounded solutions in the presence of high-frequency unmodeled dynamics that are arbitrarily small in the low frequency range. These controllers, however, guarantee local stability provided the unmodeled dynamics are small in the low-frequency range.

II. ROBUSTNESS OF A FIRST-ORDER SYSTEM

In this section, we consider the robustness of a first-order system.

A. A Simple Linear System

Let us consider the linear time-invariant (LTI) system

$$x = G(s)(1+\Delta_m(s))u \tag{1}$$

where $G(s)=(1/s)$ is the plant nominal transfer function, $\Delta_m(\mu,s)=(-2\mu s/(1+\mu s))$ is a multiplicative uncertainty, and $\mu>0$ is a small constant.

The above system can be written in the following state space form:

$$\begin{aligned} \dot{x} &= u+\eta \\ \mu\dot{\eta} &= -\eta-2\mu\dot{u}. \end{aligned} \tag{2}$$

We note that the multiplicative uncertainty $\Delta_m(\mu,s)$ is small for small μ in the low frequency range but is large in the high-frequency range and has a 180° phase shift. Moreover, $\Delta_m(\mu,s)$ changes the high-frequency gain and its sign of the modeled plant $G(s)$, rendering the overall plant being nonminimum phase.

Let us consider the reduced-order system

$$\dot{x} = u \tag{3}$$