Abstract— In the standard linear time-invariant multi-input/multi-output unity-feedback system, a parameterization of stabilizing controllers with integral action is obtained. These controllers guarantee asymptotic tracking of step reference inputs at each output channel with zero steady-state error.

Index Terms—Asymptotic tracking, controller design, integral action.

I. INTRODUCTION

Stabilizing controllers that achieve robust asymptotic tracking of general reference signals can be designed by using the well-known parameterization of all controllers that stabilize a given plant [4]. For asymptotic tracking of step reference signals, controllers are designed to have integral action (see, for example, [1] and [3]).

In this paper we parameterize controllers with integral action in the standard linear time-invariant (LTI), multi-input/multi-output (MIMO) unity-feedback system. We show that any stabilizing controller with integral action (as defined here) is expressed explicitly as the sum of two controllers: any arbitrary stabilizing controller and a controller with an integral term.

The paper is organized as follows: in Section II, following the problem description and the stability and integral action definitions, Lemma 1 claims that any stable (MIMO) system can be stabilized using an integral controller. Based on such an integral controller for the stable numerator factor of any coprime factorization of the plant, all stabilizing controllers with integral action are parameterized in the main result, Theorem 1. The special case of this parameterization for stable plants is given in Corollary 1. The parameterization of all stabilizing controllers with integral action can also be derived starting with state-space representations as explained in Comment 1. The proofs are given in the Appendix.

Due to the algebraic framework described in the following notation, the results apply to continuous-time as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at \( s = 0 \) would be interpreted at \( z = 1 \).

Notation: Let \( \mathcal{U} \) be the extended closed right half-plane (for continuous-time systems) or the complement of the open unit disk (for discrete-time systems). The set of real numbers, the set of proper rational functions that have no poles in the region of instability \( \mathcal{U} \), and the sets of proper and strictly proper rational functions with real coefficients are denoted by \( \mathbb{R} \), \( \mathbb{R}_p \), \( \mathbb{R}_s \), respectively. The set of matrices whose entries are in \( \mathbb{R} \) is denoted by \( \mathcal{M}(\mathbb{R}) \); \( M \) is called stable iff \( M \in \mathcal{M}(\mathbb{R}) \) (a notation of the form \( M \in \mathbb{R}^{m \times n} \) is used where it is important to indicate the order of a matrix explicitly); a stable \( M \) is called unimodular iff \( M \in \mathcal{M}(\mathbb{R}) \) (a notion of \( M \) is used). For \( M \in \mathcal{M}(\mathbb{R}) \), the norm \( \| \cdot \| \) is defined as \( \| M \| = \sup_{\| u \|=1} \| M u \| \), where \( \| \cdot \| \) and \( \partial \mathcal{U} \) denote the maximum singular value and the boundary of \( \mathcal{U} \), respectively. A right coprime-factorization (RCF) and a left coprime-factorization (LCF) of \( P \in \mathbb{R}_p^{n \times m} \) are denoted by \( P = N D^{-1} = D^{-1} N \), where \( N, D, N, D \in \mathcal{M}(\mathbb{R}) \), \( D \), and \( D \) are biproper. Let rank \( P = r \); \( s \in \mathcal{U} \) is a (transmission) zero of \( P \) if and only if rank \( N(s) = \text{rank } N(s) < r \); \( s \) is called a blocking-zero of \( P \) iff \( P(s) = 0 \); \( s \in \mathcal{U} \) is a blocking-zero of \( P \) if and only if \( N(s) = 0 = \bar{N}(s) \).

II. MAIN RESULTS

Consider the standard LTI MIMO unity-feedback system \( S(P,C) \) shown in Fig. 1, where \( S(P,C) \) is a well-posed system and \( P \in \mathbb{R}_p^{n \times n} \) and \( C \in \mathbb{R}_s^{n \times n} \) represent the transfer functions of the plant and the controller. It is assumed that \( P \) and \( C \) have no hidden modes corresponding to eigenvalues in the region of instability \( \mathcal{U} \). Let \( H_{u,v} \) denote the (input-error) transfer function from \( u \) to \( e \), and let \( H_{u,y} \) denote the (input-output) transfer function from \( u \) to \( y \).

Definitions I:

1) Stability: The system \( S(P,C) \) is said to be stable iff the transfer function \( H \) from \( u \) to \( y \) is stable, i.e., \( H \in \mathcal{M}(\mathbb{R}) \).

2) Integral Action: The stable system \( S(P,C) \) is said to have integral action in each output channel iff the (input-error) transfer function \( H_{u,v}(s) = I + H_{y,v}(s) \) has blocking-zeros at \( s = 0 \).

3) Stabilizing Controller: The controller \( C \) is said to be a stabilizing controller for the plant \( P \) (or \( C \) is said to stabilize \( P \)) iff \( C \in \mathcal{M}(\mathbb{R}) \) and the system \( S(P,C) \) is stable.

4) Stabilizing Controller with Integral Action: The controller \( C \) is said to be a stabilizing controller with integral action iff \( C \) stabilizes \( P \) and \( D_C(s) \) has blocking-zeros at \( s = 0 \), where \( D_C \in \mathbb{R}_s^{n \times n} \) is the denominator-matrix of any RCF \( N \in \mathcal{M}(\mathbb{R}) \).

Let \( P = N D^{-1} \), \( D^{-1} \bar{N} \) be any RCF and LCF of \( P \) in \( \mathbb{R}_p^{n \times n} \). Let \( C = N D^{-1} \) be any RCF of \( C \in \mathbb{R}_s^{n \times n} \). The controller \( C \) stabilizes \( P \) if and only if \( (D_C + N C) \) is unimodular for any RCF \( N \in \mathcal{M}(\mathbb{R}) \) of \( C \) [2], [4]. All stabilizing controllers for \( P \) are given by

\[
C = (\bar{U} + D Q) (\bar{V} - N Q)^{-1} = (V - Q \bar{N})^{-1} (U + Q \bar{D})
\]

(1)

where \( Q \in \mathbb{R}_s^{n \times n} \) is such that \( \bar{V} - Q \bar{N} \) is biproper, which holds for all \( Q \in \mathcal{M}(\mathbb{R}) \) when \( P \in \mathcal{M}(\mathbb{R}) \); in (1), \( U, V, \bar{U}, \bar{V} \in \mathcal{M}(\mathbb{R}) \).

Let \( \bar{U} = \begin{bmatrix} V & U \\ -\bar{N} & \bar{D} \end{bmatrix} \) and \( \bar{V} = \begin{bmatrix} I_{n_u} & 0 \\ 0 & I_{n_y} \end{bmatrix} \).

\[
\begin{bmatrix} V & U \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} D & \bar{U} \\ \bar{N} & \bar{D} \end{bmatrix} = \begin{bmatrix} I_{n_u} & 0 \\ 0 & I_{n_y} \end{bmatrix}.
\]

(2)

Using the parameterization (1) of all stabilizing controllers, for any stabilizing controller \( C \), the corresponding (input-error) transfer function \( H_{u,v} = (I_{n_y} + PC)^{-1} \) is given by

\[
H_{u,v} = I_{n_y} - N(U + Q \bar{D}) = (V - N Q) \bar{D}.
\]

(3)

\[ S(P,C) \text{ is stable, the (input-error) transfer function } H_{u,v}(0) = (I_{n_y} + PC)^{-1}(0) = I_{n_y} - PC(I_{n_y} + PC)^{-1}(0) = 0 \text{ only if rank } P = n_u \leq n_a. \]

Also by (3), if \( H_{u,v}(0) = I_{n_y} - N(0)(U + Q \bar{D})(0) = 0 \), then rank \( N(0) = n_y \leq n_a. \) Therefore, it is clear that
where $Q_i \in \mathbb{R}^{n_u \times n_u}$ such that $(I_{n_u} - Q_1 P)$ is biproper, which holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$. The corresponding (input-output) transfer function $H_{y_u}$ of the stable system $S(P, C)$ is

$$H_{y_u} = (I_{n_y} + P(K_i/s))^{-1} P(Q_1 + (K_i/s)).$$

**Comments:**

a) Simple interpretation of the parameterization of all stabilizing controllers with integral action: Theorem 1 states that any stabilizing controller with integral action is expressed as the sum of an arbitrary stabilizing controller $(V - Q_1 N)^{-1}(U + Q_1 D)$ and a controller with an integral term $(V - Q_1 N)K_i/s$. The block-diagram of the system $S(P, C)$ with the stabilizing controller $C$ as in (4) is shown in Fig. 2.

b) The integral controller stabilizing $N$: In Theorem 1, $K_i \in \mathbb{R}^{n_u \times n_u}$ such that $K_i/s$ stabilizes $N$, equivalently, $K_i \in \mathbb{R}^{n_u \times n_u}$ is any constant controller that stabilizes $N/s$. Since rank $N(0) = n_u$ by assumption, existence of such controllers is guaranteed by Lemma 1; in fact, as in the proof of Lemma 1, $K_i$ can be chosen as $K_i = \beta N(0)^T$ for any positive $\beta \in \mathbb{R}$ satisfying (12), i.e.,

$$0 < \beta < \left\| N(s)N(0)^T - I_{n_y} \right\|_s^{-1}$$

(7)

where $N(0)^T$ denotes any right-inverse of $N(0)$. Note that rank $K_i = n_u$ for any $K_i$ that stabilizes $N/s$.

c) Full-Order observer-based realization of all stabilizing controllers with integral action: The parameterization in (4) of all stabilizing controllers with integral action can also be obtained by using the coprime factorizations of $P$ obtained from a state-space representation as follows: Let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be a state-space representation of $P \equiv \tilde{C}(sI_n - \tilde{A})^{-1}\tilde{B} + \tilde{D}$, where $(\tilde{A}, \tilde{B})$ is stabilizable and $(\tilde{C}, \tilde{D})$ is detectable. Let $K \in \mathbb{R}^{n_u \times n_u}$ and $L \in \mathbb{R}^{n_x \times n_x}$ be such that $A_K := (sI_n - \tilde{A} + KL) \in \mathcal{M}(\mathcal{R})$, $A_{L} := (sI_n - \tilde{A} + L\tilde{C}) \in \mathcal{M}(\mathcal{R})$. Let rank $N(0) = \text{rank} (\tilde{D} - (\tilde{C} - \tilde{D}K)(\tilde{A} - \tilde{D}K))^{-1} = n_u$. Let $K_{i/s}$ be any integral controller that stabilizes $N \equiv (\tilde{C} - \tilde{D}K)A_{L}\tilde{B} + \tilde{D}$, where $K_{i/s} \in \mathbb{R}^{n_u \times n_u}$. By Theorem 1, the controller $C$ is a stabilizing controller with integral action if and only if

$$C = (I_{n_u} + K_A L + K_i/s) - Q_1 (\tilde{C}(L + LD) + Q_1))^{1}$$

where $Q_1 \in \mathbb{R}^{n_u \times n_u}$ is such that det $(I_{n_u} - Q_1(\infty)) \neq 0$, which holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathcal{R})$. The block-diagram of the system $S(P, C)$ with the stabilizing controller $C$ as in (8) is shown in Fig. 3.

Note again that, as in the proof of Lemma 1 and Comment b) above, $K_i$ can be chosen as $K_i = \beta N(0)^T$ for any positive $\beta \in \mathbb{R}$.
By (14), all solutions of (15) are given by

\[
\frac{s}{s+\beta}I_{ny} + \frac{\beta}{s+\beta}N(0)\gamma = \frac{\beta}{s+\beta}I_{ny} + \frac{\beta}{s+\beta}(N - N(0))N(0)\gamma.
\]

is unimodular since \(\|(s/s+\beta)\| = \beta\). Therefore \(K_i/s\) stabilizes \(N\). Equivalently, \(N/s = (s/s+\beta)I_{ny}\gamma\) is an RCF of \(N/s\) and it follows similarly from (13) that \(K_i/s\) stabilizes \(N/s\).

Proof of Theorem 1: Let \(P = ND^{-1} = \tilde{D}^{-1}N\) be any RCF and LCF of \(P\). By Lemma 1, there exists \(K_i \in \mathbb{R}^{nu \times ny}\) such that \(K_i/s\) stabilizes \(N\). Now \(K_i/s\) stabilizes \(N\) if and only if, for any \(\alpha \in \mathbb{R}, \alpha > 0\), the matrix \(M \in \mathcal{M}(\mathbb{R})\) in (11) is unimodular, equivalently

\[
\begin{bmatrix}
I_{nu} & K_i \\
\frac{s}{s+\alpha} & \frac{1}{s+\alpha}
\end{bmatrix} \begin{bmatrix}
I_{ny} & K_i \\
\frac{s}{s+\alpha} & \frac{1}{s+\alpha}
\end{bmatrix} M^{-1} N = \frac{K_i}{s+\alpha} M^{-1}
\]

for some \(Q_1 \in \mathbb{R}^{nu \times ny}\). By (14), all solutions of (15) are given by

\[
\begin{bmatrix}
-I_{ny} & K_i \\
\frac{s}{s+\alpha} & \frac{1}{s+\alpha}
\end{bmatrix} \begin{bmatrix}
-Q_1 \\
\frac{s}{s+\alpha}
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{s}{s+\alpha} I_{ny}
\end{bmatrix}
\]

for some \(Q_1 \in \mathbb{R}^{nu \times ny}\). By (16), all solutions of (15) are given by

\[
\begin{bmatrix}
Q_1 \\
\frac{s}{s+\alpha} I_{ny} + \frac{K_i}{s+\alpha} M^{-1}(\bar{V} - NQ_1)
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{s}{s+\alpha} I_{ny} + \frac{K_i}{s+\alpha} M^{-1}(\bar{V} - NQ_1)
\end{bmatrix}
\]

for some \(Q_1 \in \mathbb{R}^{nu \times ny}\). By (15), all solutions of (15) are given by (16).
into (1), the controller becomes
\[
C = (\hat{V} + D)(\hat{V} - Q)^{-1} \\
= \left(\hat{V} + DQ_1 + \frac{K_1}{s + \alpha} M^{-1}(\hat{V} - Q_1)\right) \\
\cdot (\hat{V} - Q_1)^{-1} M \frac{s + \alpha}{s} \\
= (\hat{V} + DQ_1)(\hat{V} - Q_1)^{-1}(I_{ay} + N \frac{K_1}{s}) + D \frac{K_1}{s} \\
= (\hat{V} + D)(\hat{V} - Q_1)^{-1} + (V - Q_1, \hat{N})^{-1} \frac{K_1}{s} \\
= (V - Q_1, \hat{N})^{-1}(U + Q_1, \hat{D}) + (V - Q_1, \hat{N})^{-1} \frac{K_1}{s}
\]
as claimed in (4) and (5). The controller \( C \) is proper if and only if
\( (\hat{V} - Q) = (s + \alpha)D \) is biproper, equivalently, \( (\hat{V} - Q_1) \) is biproper since \( M \) is unimodular.

Using the same \( Q \in M(\mathbb{R}) \) in \( H_{u,v} = N(U + Q, \hat{D}) = I_{uv} - (\hat{V} - Q_1, \hat{D}) \), the corresponding (input–output) transfer function of the stable system \( S(P, C) \) becomes \( H_{u,v} = I_{uv} + (N(K_1/s)^{-1}(V - Q_1, \hat{D}) = (I_{uv} + N(K_1/s)^{-1}N(U + Q, \hat{D}) + (K_1/s)) \) as claimed. \( \square \)

REFERENCES


II. ROBUSTNESS OF A FIRST-ORDER SYSTEM

In this section, we consider the robustness of a first-order system.

A. A Simple Linear System

Let us consider the linear time-invariant (LTI) system
\[
x = G(s)(1 + \Delta_m(s))u
\]
where \( G(s) = \frac{1}{s} \) is the plant nominal transfer function, \( \Delta_m(s) = \frac{-2\mu s/(1 + s)}{s} \) is a multiplicative uncertainty, and \( \mu > 0 \) is a small constant.

The above system can be written in the following state space form:
\[
\dot{x} = u + \eta \\
\mu \dot{\eta} = -\eta - 2\mu \dot{u}.
\]

We note that the multiplicative uncertainty \( \Delta_m(s) \) is small for small \( \mu \) in the low frequency range but is large in the high-frequency range and has a 180° phase shift. Moreover, \( \Delta_m(s) \) changes the high-frequency gain and its sign of the modeled plant \( G(s) \), rendering the overall plant being nonminimum phase.

Let us consider the reduced-order system
\[
\dot{x} = u
\]