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# MIMO controller design with type-m integral action [1]

A. N. Gündeş and M. G. Kabuli
Electrical and Computer Engineering, University of California, Davis, CA 95616
gundes@ece.ucdavis.edu kabuli@ece.ucdavis.edu

**Abstract**

A parametrization of all stabilizing controllers with at least $m$ integrators achieving type-$m$ integral action is obtained for linear time-invariant, multi-input multi-output systems. These controllers are expressed as integral terms added to any stabilizing controller.

## 1 Introduction

We consider the problem of designing stabilizing controllers with integral action for linear time-invariant (LTI), multi-input multi-output (MIMO) systems. The objective is to achieve closed-loop stability with (at least) type-$m$ integral action in each output channel so that polynomial references up to order $m-1$ applied at each input would be asymptotically tracked (with zero steady-state error). The design method developed here is motivated by the well-known parametrization of all stabilizing controllers [2]. Robust asymptotic tracking is achieved by choosing the controller's poles appropriately (in this case at zero) [2], [1]. A parametrization of all stabilizing controllers with at least $m$ integrators achieving type-$m$ integral action is obtained in Theorem 2.3; these controllers are expressed as an arbitrary stabilizing controller $\widetilde{D}_c^{-1}\widetilde{N}_c$ added to integral terms involving $m$ constant matrices. An alternate parametrization using only one constant matrix is given in Corollary 2.4.

Due to the algebraic framework, the results apply to continuous-time and discrete-time systems. Although a continuous-time setting was assumed throughout for simplicity, all evaluations and discussions involving poles and zeros at $s=0$ should be interpreted at $z=1$ in the case of discrete-time systems.

**Notation and algebraic framework:** Let $\mathcal{U}$ be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The set of real numbers, the set of proper rational functions that have no poles in the region of instability $\mathcal{U}$, the sets of proper and strictly-proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathcal{R}$, $\mathrm{R_p}$, $\mathrm{R_s}$, respectively. The set of matrices whose entries are in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; $M$ is called stable iff $M \in \mathcal{M}(\mathcal{R})$; a stable $M$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. For $M \in \mathcal{M}(\mathcal{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s\in\partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$. A right-coprime-factorization (RCF) and a left-coprime-factorization (LCF) of $P \in \mathrm{R_p}^{n_y \times n_u}$ are denoted by $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$, where $N, D, \widetilde{N}, \widetilde{D} \in \mathcal{M}(\mathcal{R})$, $D$ and $\widetilde{D}$ are biproper. Let $\mathrm{rank}P = r$; $s_o \in \mathcal{U}$ is a (transmission) zero of $P$ if and only if $\mathrm{rank}N(s_o) = \mathrm{rank}\widetilde{N}(s_o) < r$; $s_o$ is called a blocking-zero of $P$ iff $P(s_o)=0$; $s_o \in \mathcal{U}$ is a blocking-zero of $P$ if and only if $N(s_o) = 0 = \widetilde{N}(s_o)$.

## 2 Main Results

Consider the LTI, MIMO, unity-feedback system, where $P: e_P \mapsto y$, $C: e \mapsto y_C$, $e = u-y$, $e_P = y_C + u_P$; $P \in \mathrm{R_p}^{n_y\times n_u}$ and $C \in \mathrm{R_p}^{n_u \times n_y}$ represent the transfer-functions of the plant and the controller, respectively. It is assumed that $P$ and $C$ have no hidden modes corresponding to eigenvalues in $\mathcal{U}$. Let $H_{eu}$ denote the (input-error) transfer-function from $u$ to $e$, $H_{yu}$ denote the (input-output) transfer-function from $u$ to $y$, $H$ denote the transfer-function from $(u, u_P)$ to $(y, y_C)$.

**2.1. Definitions: a)** The system $\mathcal{S}(P,C)$ is said to be *stable* iff the transfer-function $H$ from $(u,u_P)$ to $(y,y_C)$ is stable, i.e., $H \in \mathcal{M}(\mathcal{R})$. **b)** The stable system $\mathcal{S}(P,C)$ is said to have *integral action* in each output channel iff $H_{eu}$ has blocking-zeros at zero, i.e., $H_{eu}(0) = 0$. The system $\mathcal{S}(P,C)$ is said to have *type-m integral action* (where $m \geq 1$ is an integer) iff $H_{eu}$ has (at least) $m$ blocking-zeros at zero, i.e., if $(s^{-(m-1)}H_{eu})(0) = 0$. **c)** The controller $C$ is said to be a *stabilizing controller* for the plant $P$ (or $C$ is said to stabilize $P$) iff $C \in \mathcal{M}(\mathrm{R_p})$ and the system $\mathcal{S}(P,C)$ is stable. **d)** The controller $C$ is said to be a *stabilizing controller with integral action* iff $C$ stabilizes $P$, and $D_c$ has blocking-zeros at zero, where $D_c \in \mathcal{R}^{n_y\times n_y}$ is the denominator-matrix of any RCF $N_c D_c^{-1}$ of $C$. The controller $C$ is said to be a *stabilizing controller with type-m integral action* iff $C$ stabilizes $P$ and $D_c$ has (at least) $m$ blocking-zeros at zero. □

Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ be any RCF and LCF of $P \in \mathrm{R_p}^{n_y\times n_u}$. Let $C = N_c D_c^{-1}$ be any RCF of $C \in \mathrm{R_p}^{n_u\times n_y}$. The controller $C$ stabilizes $P$ if and only if $(\widetilde{D}D_c + \widetilde{N}N_c)$ is unimodular [2]. All stabilizing controllers for $P$ are $C = (\widetilde{U} + DR)(\widetilde{V} - NR)^{-1} = (V - R\widetilde{N})^{-1}(U + R\widetilde{D})$, where $R \in \mathcal{R}^{n_u\times n_y}$ is such that $(\widetilde{V} - NR)$ is biproper (holds for all $R \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$); $U, V, \widetilde{U}, \widetilde{V} \in \mathcal{M}(\mathcal{R})$ are such that $VD + UN = I$, $\widetilde{D}\widetilde{V} + \widetilde{N}\widetilde{U} = I$, $V\widetilde{U} = U\widetilde{V}$. For any stabilizing

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controller $C$, the corresponding (input-error) transfer-function $H_{eu} = (I_{n_y}+PC)^{-1} = I_{n_y}-PC(I_{n_y}+PC)^{-1} = I_{n_y} - N(U+R\widetilde{D}) = (\widetilde{V} - NR)\widetilde{D}$. If $\mathcal{S}(P,C)$ is stable, $H_{eu}(0) = 0$ only if rank$P = n_y \leq n_u$ and rank$N(0) = n_y \leq n_u$. Therefore, in order for the stable system $\mathcal{S}(P,C)$ to have integral action, the necessary conditions on the plant $P$ are that rank$P = n_y \leq n_u$ and $P$ has no zeros at zero.

In any arbitrary RCF $C = N_c D_c^{-1}$ of a stabilizing controller, $D_c(0) = 0$ if and only if $(\widetilde{V} - NR)(0) = 0$. By Definition 2.1, if $C$ is a stabilizing controller with integral action, then $H_{eu}(0) = (\widetilde{V} - NR)(0)\widetilde{D}(0) = 0$ and hence, the stable system $\mathcal{S}(P,C)$ has integral action in each output channel. Although designing the stabilizing controllers so that $D_c(0) = 0$ is sufficient for the stable system $\mathcal{S}(P,C)$ to have integral action, it is clearly not necessary since $H_{eu}(0) = 0$ also when $\widetilde{D}(0) = 0$. However, when $P$ has no poles at zero, and in particular when $P$ is stable, $H_{eu}(0) = 0$ if and only if $D_c(0) = 0$; in these cases, the stable system $\mathcal{S}(P,C)$ has integral action if and only if the controller $C$ is a stabilizing controller with integral action. Similarly, if $(s^{-(m-1)}D_c)(0) = 0$, then $H_{eu}$ has $m$ blocking-zeros at zero; this is again a sufficient condition for type-$m$ integral action in $\mathcal{S}(P,C)$.

**2.2. Lemma:** Let $G \in \mathcal{R}^{n_y \times n_u}$, where rank$G = n_y \leq n_u$. There exists a constant controller $K_i \in \mathbb{R}^{n_u \times n_y}$ that stabilizes $\frac{G}{s}$ if and only if rank$G(0) = n_y \leq n_u$.

**2.3. Theorem:** Let $P \in \mathrm{R_p}^{n_y \times n_u}$, rank$P = n_y \leq n_u$. Let $P$ have no zeros at zero. Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ be any RCF and LCF. Let $U, V, \widetilde{U}, \widetilde{V} \in \mathcal{M}(\mathcal{R})$ satisfy $VD + UN = I$, $\widetilde{D}\widetilde{V} + \widetilde{N}\widetilde{U} = I$, $V\widetilde{U} = U\widetilde{V}$. Let $K_1 \in \mathbb{R}^{n_u \times n_y}$ be any constant controller that stabilizes $\frac{N}{s}$. Let $m$ be an integer; for $n = 2, \ldots, m$, let $K_n \in \mathbb{R}^{n_y \times n_y}$ be any constant controller that stabilizes $\frac{N_n}{s}$, where

$$N_n := \Big( I + N \sum_{j=1}^{n-1} \frac{1}{s^j} \prod_{\ell=1}^{j} K_\ell \Big)^{-1} \frac{N}{s^{n-1}} \prod_{j=1}^{n-1} K_j \in \mathcal{R}^{n_y \times n_y}.$$

Under these assumptions, $C$ is a stabilizing controller with type-$m$ integral action if and only if

$$C = (V - Q\widetilde{N})^{-1} \Big( U + Q\widetilde{D} + \sum_{j=1}^{m} \frac{1}{s^j} \prod_{\ell=1}^{j} K_\ell \Big),$$

where $Q \in \mathcal{R}^{n_u \times n_y}$ is such that $(V - Q\widetilde{N})$ is biproper (holds for all $Q \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$). □

**2.4 Corollary:** Under the assumptions of Theorem 2.3, $C$ is a stabilizing controller with type-$m$ integral action if and only if

$$C = (V - \tilde{Q}\widetilde{N})^{-1} \Big( U + \tilde{Q}\widetilde{D} + \frac{K_1}{s} \sum_{j=1}^{m} \big( I + N \frac{K_1}{s} \big)^{j-1} \Big),$$

where $\tilde{Q} \in \mathcal{R}^{n_u \times n_y}$ is such that $(V - \tilde{Q}\widetilde{N})$ is biproper (holds for all $\tilde{Q} \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$). □

**Comments: a)** *(All controllers with type-m integral action for stable plants)*: The parametrization in Theorem 2.3 (similarly in Corollary 2.4) is simplified for stable plants by substituting $N = \widetilde{N} = P$, $D = I_{n_u}$, $\widetilde{D} = I$, $V = I$, $\widetilde{V} = I$, $U = 0 = \widetilde{U}$. **b)** *(Construction of the constant matrices $K_1, \ldots, K_n$):* By Lemma 2.2, for any stable system $G \in \mathcal{R}^{n_y \times n_u}$ such that rank$G = n_y \leq n_u$, there exists a constant controller $K_i \in \mathbb{R}^{n_u \times n_y}$ that stabilizes $s^{-1}G$ if and only if rank$G(0) = n_y$. Note that rank$K_n = n_y$ for any $K_n$ that stabilizes $s^{-1}N_n$. A choice for $K_1 \in \mathbb{R}^{n_u \times n_y}$ that stabilizes $s^{-1}N$ is $K_1 = \beta_1 N(0)^I$, where $N(0)^I$ denotes any right-inverse of $N(0)$ and $\beta_1 \in \mathbb{R}$ is any positive real constant satisfying $0 < \beta_1 < \left\| s^{-1}(N(s)N(0)^I - I) \right\|^{-1}$. For $n = 2, \ldots, m$, each $N_n \in \mathcal{R}^{n_y \times n_y}$ in Theorem 2.3 is a stable system such that $N_n(0) = I$. Since $N_n(0)^I = N_n(0)^{-1} = I$, a choice for $K_n \in \mathbb{R}^{n_y \times n_y}$ that stabilizes $s^{-1}N_n$ is $K_n = \beta_n I_{n_y}$, where $\beta_n \in \mathbb{R}$ is any positive real constant satisfying $0 < \beta_n < \left\| s^{-1}(N_n(s) - I) \right\|^{-1}$. The constant $K_1 \in \mathbb{R}^{n_u \times n_y}$ that stabilizes $s^{-1}N$ in Corollary 2.4 can also be chosen as above. **c)** *(Observer-based realization of all stabilizing controllers with type-m integral action):* The parametrization in Theorem 2.3 and equivalently Corollary 2.4 can also be derived by using the coprime factorizations of $P$ obtained from a state-space representation [2]. We state the parametrization in Corollary 2.4 as a simple algorithm to design stabilizing controllers with type-$m$ integral action:

**Given:** A state-space representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of $P(s) = \bar{C}(sI_n - \bar{A})^{-1}\bar{B} + \bar{D} \in \mathrm{R_p}^{n_y \times n_u}$, $n_y \leq n_u$, where $(\bar{A}, \bar{B})$ is stabilizable and $(\bar{C}, \bar{A})$ is detectable. **Step 0:** If rank $\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = n + n_y$, then go to step 1 ($P(s)$ has no zeros at $s = 0$); else, stop ($P(s)$ has zeros at $s = 0$ and hence, stabilizing controllers with integral action do not exist). **Step 1:** Choose any $K \in \mathbb{R}^{n_u \times n}$ and $L \in \mathbb{R}^{n \times n_y}$ such that $A_K := (sI_n - \bar{A} + \bar{B}K)^{-1} \in \mathcal{M}(\mathcal{R})$, $A_L := (sI_n - \bar{A} + L\bar{C})^{-1} \in \mathcal{M}(\mathcal{R})$. **Step 2:** Choose any $K_1 \in \mathbb{R}^{n_u \times n_y}$ that stabilizes $\frac{N}{s}$, where $N = (\bar{C} - \bar{D}K)A_K\bar{B} + \bar{D}$. **Step 3:** The controller $C$ is a stabilizing controller with integral action if and only if $C = \Big( I + (K - \tilde{Q}\bar{C})A_L(\bar{B} - L\bar{D}) - \tilde{Q}\bar{D} \Big)^{-1} \cdot \Big( KA_L L + \tilde{Q}(I - \bar{C}A_L L) + \frac{K_1}{s}\sum_{j=1}^{m} \Big( I + N\frac{K_1}{s} \Big)^{j-1} \Big)$ where $\tilde{Q} \in \mathcal{R}^{n_u \times n_y}$ is such that $\det(I - \tilde{Q}(\infty)\bar{D}) \neq 0$. At step 2, $K_1$ can be chosen as $K_1 = \beta_1 N(0)^I$, where $N(0)^I = \big(\bar{D} - (\bar{C} - \bar{D}K)(\bar{A} - \bar{B}K)^{-1}\bar{B}\big)^I$; $(\bar{A} - \bar{B}K)^{-1}$ exists since $A_K$ has no poles at zero; $\beta_1$ satisfies $0 < \beta_1 < \|(\bar{C} - \bar{D}K)A_K(\bar{A} - \bar{B}K)^{-1}\bar{B}N(0)^I\|^{-1}$.

## References


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[2] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, M.I.T. Press, 1985.