Proof: First of all, notice that $(\nu^{1/2}A, B)$ is stabilizable if and only if $(\nu^{1/2}A, \nu^{1/2}B)$ is stabilizable. From Lemma A1, $(\hat{C}, \nu^{1/2}\hat{A})$ is detectable. Notice now that (5) can be written as

$$\begin{aligned} X &= \hat{J}'\hat{J} + (\nu^{1/2}A')X(\nu^{1/2}A) \\ &\quad - ((\nu^{1/2}A')X(\nu^{1/2}B) + \hat{J}'\hat{D}) \\ &\quad \cdot (\hat{D}'\hat{D} + (\nu^{1/2}B')X(\nu^{1/2}B))^{-1} \\ &\quad \cdot ((\nu^{1/2}B')X(\nu^{1/2}A) + \hat{D}'\hat{J}) \end{aligned}$$

and the lemma follows from standard results on ARE's (see [6, p. 263 and Appendix B]). □

Proof of Proposition 2.3: Consider in (5) $Y = \hat{Y} \geq \tilde{Y}$. From Lemma A.2, if $(\nu^{1/2}A, B)$ is stabilizable and $(\tilde{C}, \nu^{1/2}\tilde{A})$ is detectable, then there exists a unique positive semidefinite solution $\hat{X}$ to (5), and it is such that $\nu^{1/2}(A + B\hat{K})$ is stable, where $\hat{K}$ is as in (A1) with $X = \hat{X}$ and $Y = \hat{Y}$. If we now set $Y = \tilde{Y}$ we can conclude similarly the existence of a unique semipositive solution $\tilde{X}$ of (5) and a $\tilde{K}$ given by (A1) with $X = \tilde{X}$ and $Y = \tilde{Y}$. After some algebraic manipulation, we get that

$$\begin{aligned} &(\hat{X} - \tilde{X}) - (A + B\hat{K})((\hat{Y} + \nu\hat{X}) - (Y + \nu\tilde{X}))(A + B\hat{K}) \\ &\quad = (\tilde{K} - \hat{K})'(D'D + B'(\tilde{Y} + \nu\tilde{X})B)(\tilde{K} - \hat{K}) \end{aligned}$$

and thus

$$\begin{aligned} &(\hat{X} - \tilde{X}) - \nu(A + B\hat{K})'(\hat{X} - \tilde{X})(A + B\hat{K}) \\ &\quad = (A + B\hat{K})'(\hat{Y} - \tilde{Y})(A + B\hat{K}) \\ &\qquad + (\tilde{K} - \hat{K})(D'D + B'(\tilde{Y} + \nu\tilde{X})B)^{-1}(\tilde{K} - \hat{K}) \end{aligned}$$

and it follows from stability of $\nu^{1/2}(A + B\hat{K})$ that $\hat{X} - \tilde{X} \geq 0$. □

For a general account on the positive semidefinite partial ordering of maximal solutions of discrete-time ARE's, see [16].

References

[1] H. Abou-Kandil, G. Freiling, and G. Jank, "On the solution of discrete-time Markovian jump linear quadratic control problems," *Automatica*, vol. 31, no. 5, pp. 765–768, 1995.

[2] M. Ait-Rami and L. E. Ghaoui, "LMI optimization for nonstandard Riccati equations arising in stochastic control," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1666-1671, Nov. 1996.

[3] O. L. V. Costa, J. B. R. do Val, and J. C. Geromel, "A convex programming approach to $\mathcal{H}_2$-control of discrete-time Markovian jump linear systems," *Int. J. Contr.*, vol. 66, pp. 557–579, 1997.

[4] O. L. V. Costa and M. D. Fragoso, "Stability results for discrete-time linear systems with Markovian jumping parameters," *J. Math. Analysis and Appl.*, vol. 179, pp. 154–178, 1993.

[5] ——, "Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 2076–2088, 1995.

[6] M. H. A. Davis and R. B. Vinter, *Stochastic Modeling and Control*. London, U.K.: Chapman and Hall, 1985.

[7] L. El Ghaoui, R. Nikoukhah, and F. Delebecque, *LMITOOL: A Front-End for LMI Optimization—User's Guide*. Available via anonymous ftp from ftp.ensa.fr /pub/elghaoui/lmitool, 1995.

[8] M. D. Fragoso, J. B. Ribeiro do Val, and D. L. Pinto Jr., "Jump linear $\mathcal{H}_\infty$-control: The discrete-time case," *Contr. Th. and Adv. Tech.*, vol. 10, pp. 1459–1474, 1995.

[9] Z. Gajic and I. Borno, "Lyapunov iterations for optimal control of jump linear systems at steady state," *IEEE Trans. Automat. Contr.*, vol. 40, no. 11, pp. 1971–1075, 1995.

[10] J. C. Geromel, P. L. D. Peres, and S. R. Souza, "$\mathcal{H}_2$-guaranteed cost control for uncertain discrete-time linear systems," *Int. J. Contr.*, vol. 57, pp. 853–864, 1993.

[11] Y. Ji and H. J. Chizeck, "Controllability, observability and discrete-time Markovian jump linear quadratic control," *Int. J. Contr.*, vol. 48, pp. 481–498, 1988.

[12] Y. Ji, H. J. Chizeck, X. Feng, and K. A. Loparo, "Stability and control of discrete-time jump linear systems," *Control Th. and Adv. Tech.*, vol. 7, pp. 247–270, 1991.

[13] A. Laub, "Algebraic aspects of generalized eigenvalue problems for solving Riccati equations," in *Computational and Combinatorial Methods in Systems Theory*, C. Byrnes and A. Lindquist, Eds. Amsterdam, The Netherlands: Elsevier, 1986, pp. 213–227.

[14] M. Mariton, *Jump Linear Systems in Automatic Control*. New York: Marcel Dekker, 1990.

[15] L. Vandenberghe and S. Boyd, *Software for Semidefinite Programming—User's Guide*. Available via anonymous ftp from isl.stanford.edu /pub/boyd/semidef_prog, 1994.

[16] H. K. Wimmer, "Monotonicity and maximality of solutions of discrete-time algebraic Riccati equations," *J. Math. Syst., Estimation and Contr.*, vol. 2, no. 2, pp. 219–235, 1992.

Reliable Decentralized Stabilization of Linear Systems

A. N. Gündeş

***Abstract*— Reliable stabilization of linear time-invariant multi-input/multi-output plants is considered using a two-channel decentralized controller configuration. Necessary and sufficient conditions are obtained for existence of reliable controllers that maintain stability under the possible failure of either one of the two controllers. All decentralized controllers that achieve reliable stabilization are characterized.**

***Index Terms*—Controller design, decentralized control, reliable stabilization.**

I. Introduction

We consider reliable stabilization of linear time-invariant (LTI), multi-input/multi-output (MIMO) plants under possible sensor or actuator failures using a two-channel decentralized feedback control configuration. The goal is to maintain closed-loop stability when both controllers act together and when either one of the two controllers acts alone. It is assumed that the failure of a controller is recognized and it is taken out of service (i.e., the states in the controller implementation are all set to zero, the initial conditions and outputs of the channel that failed are set to zero for all inputs). Since the introduction of multi-controller systems in [5] and [6], reliable stabilization has been studied for various failure models using full-feedback [2], [8] and decentralized configurations [4], [7]. Conditions for existence of reliable decentralized controllers were given for a class of reliable stabilization problems using genericity arguments in [4]. The reliable stabilization problem considered in this paper is based on the two-channel decentralized configuration and failure model in [7]. The necessary and sufficient conditions here for existence of reliable decentralized controllers include generalizations of the sufficient conditions in [7].

The main results in this paper are the explicit existence conditions for reliable decentralized controllers. Theorem 2 gives an important interpretation of these conditions in terms of the strong stabilizability

Manuscript received January 10, 1996; revised February 7, 1997 and August 1, 1997. This work was supported by the NSF under Grant ECS-9257932.

The author is with Electrical and Computer Engineering Department, University of California, Davis, CA 95616 USA (e-mail gundes@ece.ucdavis.edu).

Publisher Item Identifier S 0018-9286(98)09444-6.

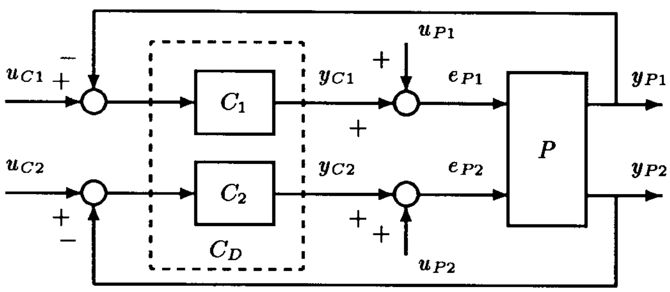

Fig. 1. The decentralized system $\mathcal{S}(P, C_D)$.

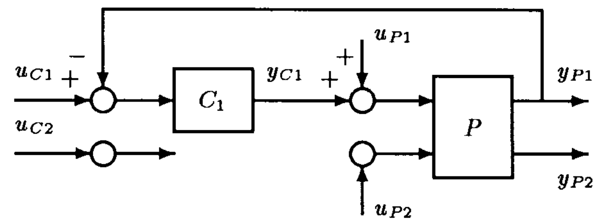

Fig. 2. The system $\mathcal{S}(P, C_1)$.

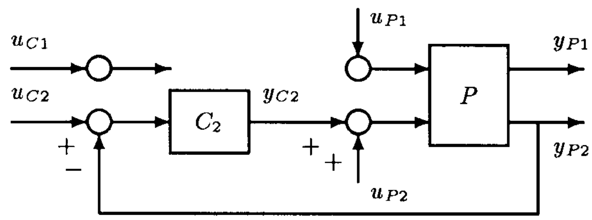

Fig. 3. The system $\mathcal{S}(P, C_2)$.

of an associated system and states that strong stabilizability of two of the subblocks of the plant is necessary. Proposition 1 gives a parameterization of all reliable decentralized controllers for stable plants, and Theorem 3 establishes explicit existence conditions when one channel is single-input/single-output (SISO). Theorem 4 gives important sufficient conditions when all channels are MIMO. The following notation is used throughout; due to the input–output approach, the setting can be continuous-time or discrete-time.

Notation: Let $\mathcal{U}$ denote the region of instability; $\mathcal{U}$ contains the extended closed right half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). Let R_p (R_s) denote proper (strictly proper) rational functions with real coefficients; $\mathcal{R}$ denotes proper rational functions with no poles in $\mathcal{U}$; $\mathcal{M}(\mathcal{R})$ denotes the set of matrices with entries in $\mathcal{R}$; M is called $\mathcal{R}$-stable iff $M \in \mathcal{M}(\mathcal{R})$; and $M \in \mathcal{M}(\mathcal{R})$ is called $\mathcal{R}$-stable unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. A right-coprime factorization (RCF), a left-coprime factorization (LCF), and a bicoprime factorization (BCF) of $P \in \mathcal{M}(R_p)$ are denoted by $P = ND^{-1} = \tilde{D}^{-1}\tilde{N} = N_{br}D_b^{-1}N_{bl} + G_b$; $N, D, \tilde{N}, \tilde{D}, N_{br}, D_b, N_{bl}, G_b \in \mathcal{M}(\mathcal{R})$, D, $\tilde{D}$, and D_b are biproper. Let $\operatorname{rank} P = r$; $s_o \in \mathcal{U}$ is called a $\mathcal{U}$-zero (blocking $\mathcal{U}$-zero) of P iff $\operatorname{rank} P(s_o) < r$ ($P(s_o) = 0$); the poles of P in $\mathcal{U}$ are called its $\mathcal{U}$-poles. For $M \in \mathcal{M}(\mathcal{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s \in \partial\mathcal{U}} \overline{\sigma}(M(s))$; $\overline{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$.

II. Main Results

Consider the LTI, MIMO, and two-channel decentralized control system $\mathcal{S}(P, C_D)$ shown in Fig. 1. The plant and the decentralized controller are represented by their transfer functions P and C_D, respectively

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R_p^{n_o \times n_i}, \qquad P_{jj} \in R_p^{n_{oj} \times n_{ij}}$$
$$C_D = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \in R_p^{n_i \times n_o}, \qquad C_j \in R_p^{n_{ij} \times n_{oj}} \tag{1}$$

$n_o = n_{o1} + n_{o2}$, $n_i = n_{i1} + n_{i2}$. It is assumed that P and C_D have no hidden modes corresponding to eigenvalues in $\mathcal{U}$ and that $\mathcal{S}(P, C_D)$ is well-posed. The failure of the jth controller channel is represented by setting $C_j = 0$; the corresponding jth channel output y_{Cj} is also set to zero. When the second (first) channel fails, the system is called $\mathcal{S}(P, C_1)$ shown in Fig. 2 ($\mathcal{S}(P, C_2)$ shown in Fig. 3).

Using any RCF $P = ND^{-1}$, any LCF $C_j = \tilde{D}_{c_j}^{-1}\tilde{N}_{c_j}$, $j = 1, 2$, $\tilde{D}_C = \operatorname{diag}[\tilde{D}_{C1}, \tilde{D}_{C2}]$, $\tilde{N}_C = \operatorname{diag}[\tilde{N}_{C1}, \tilde{N}_{C2}]$, $D\xi_P = e_P$, $N\xi_P = y_P$, $u_P = [u_{P1}^T u_{P2}^T]^T$, $u_C = [u_{C1}^T u_{C2}^T]^T$, $y_P = [y_{P1}^T y_{P2}^T]^T$, $y_C = [y_{C1}^T y_{C2}^T]^T$, $e_P = [e_{P1}^T e_{P2}^T]^T$, the system $\mathcal{S}(P, C_D)$ is described in (2) as $D_D\xi_P = N_L u$, $N_R\xi_P + Gu = y$; $\mathcal{S}(P, C_D)$ is well-posed, i.e., the transfer-function $H = N_R D_D^{-1} N_L + G$ from (u_P, u_C) to (y_P, y_C) is proper, if and only if D_D is biproper. The description of $\mathcal{S}(P, C_1)$ as $D_{D1}\xi_1 = N_{L1}u_1$, $N_{R1}\xi_1 = y_1$ is given by (3); a similar description can be obtained for $\mathcal{S}(P, C_2)$. The system $\mathcal{S}(P, C_2)$ is described as $\tilde{D}_{D2}\tilde{\xi}_2 = \tilde{N}_{L2}u_2$, $\tilde{N}_{R2}\tilde{\xi}_2 = y_2$ in (4); the description for $\mathcal{S}(P, C_1)$ is similar. For $j = 1, 2$, the transfer function of $\mathcal{S}(P, C_j)$ from (u_P, u_{Cj}) to (y_P, y_{Cj}) is $H_j = N_{Rj}D_{Dj}^{-1}N_{Lj} = \tilde{N}_{Rj}\tilde{D}_{Dj}^{-1}\tilde{N}_{Lj}$

$$(\tilde{D}_C D + \tilde{N}_C N)\xi_P = [\tilde{D}_C \quad \tilde{N}_C] \begin{bmatrix} u_P \\ u_C \end{bmatrix}$$
$$\begin{bmatrix} N \\ D \end{bmatrix} \xi_P + \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} u_P \\ u_C \end{bmatrix} = \begin{bmatrix} y_P \\ y_C \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} D & \begin{bmatrix} -I_{n_{i1}} \\ 0 \end{bmatrix} \\ [\tilde{N}_{C1} \quad 0]N & \tilde{D}_{C1} \end{bmatrix} \begin{bmatrix} \xi_P \\ y_{C1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \tilde{N}_{C1} \end{bmatrix} \begin{bmatrix} u_P \\ u_{C1} \end{bmatrix}$$
$$\begin{bmatrix} N & 0 \\ 0 & I_{n_{i1}} \end{bmatrix} \begin{bmatrix} \xi_P \\ y_{C1} \end{bmatrix} = \begin{bmatrix} y_P \\ y_{C1} \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} \tilde{D} & -\tilde{N} & \begin{bmatrix} 0 \\ N_{C2} \end{bmatrix} \\ [0 \quad I_{no2}] & & \tilde{D}_{C2} \end{bmatrix} \begin{bmatrix} y_P \\ \xi_{C2} \end{bmatrix} = \begin{bmatrix} \tilde{N} & 0 \\ 0 & I_{n_{o2}} \end{bmatrix} \begin{bmatrix} u_P \\ u_{C2} \end{bmatrix}$$
$$\begin{bmatrix} I & 0 \\ 0 & N_{C2} \end{bmatrix} \begin{bmatrix} y_P \\ \xi_{C2} \end{bmatrix} = \begin{bmatrix} y_P \\ y_{C2} \end{bmatrix}. \tag{4}$$

Reliable Stability: The system $\mathcal{S}(P, C_D)$ is said to be $\mathcal{R}$-stable iff $H \in \mathcal{M}(\mathcal{R})$; similarly, $\mathcal{S}(P, C_j)$ is $\mathcal{R}$-stable iff $H_j \in \mathcal{M}(\mathcal{R})$. The decentralized controller C_D is said to be an $\mathcal{R}$-stabilizing controller for P iff C_D is proper and $\mathcal{S}(P, C_D)$ is $\mathcal{R}$-stable. The pair (C_1, C_2) is called a *reliable decentralized controller pair* iff C_1, C_2 are proper and the systems $\mathcal{S}(P, C_D)$, $\mathcal{S}(P, C_1)$, $\mathcal{S}(P, C_2)$ are all $\mathcal{R}$-stable.

Lemma 1 gives necessary and sufficient conditions for $\mathcal{R}$-stability of $\mathcal{S}(P, C_D)$ under normal operation and under the complete failure of one of the controllers. We assume that the coprime factorizations are in canonical forms; the denominator-matrix of any RCF, LCF can be put into upper (lower) triangular Hermite forms by elementary column (row) operations [9], [1]. Without loss of generality, it is assumed that the RCF and LCF $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ are given by

$$P = ND^{-1} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}^{-1}$$
$$= \tilde{D}^{-1}\tilde{N} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}. \quad (5)$$

Lemma 1—Decentralized Stability: Let $ND^{-1} = \tilde{D}^{-1}\tilde{N}$ be any RCF, LCF of $P \in R_p^{n_o \times n_i}$; let $\tilde{D}_C^{-1}\tilde{N}_C = N_C D_C^{-1}$ be any LCF, RCF of C_D, $\tilde{D}_C = \operatorname{diag}[\tilde{D}_{C1}, \tilde{D}_{C2}]$, $\tilde{N}_C = \operatorname{diag}[\tilde{N}_{C1}, \tilde{N}_{C2}]$, $N_C = \operatorname{diag}[N_{C1}, N_{C2}]$, $D_C = \operatorname{diag}[D_{C1}, D_{C2}]$. The system $\mathcal{S}(P, C_D)$ is $\mathcal{R}$-stable if and only if $D_D := (\tilde{D}_C D + \tilde{N}_C N)$ is $\mathcal{R}$-unimodular. Let the RCF and LCF $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ be as in (5); $S(P, C_D)$ is $\mathcal{R}$-stable if and only if (6) holds. Let $C_2 = \tilde{D}_{C2}^{-1}\tilde{N}_{C2} = N_{C2}D_{C2}^{-1}$ be any LCF, RCF; the system $\mathcal{S}(P, C_2)$ is $\mathcal{R}$-stable if and only if (7) holds. Let $C_1 = \tilde{D}_{C1}^{-1}\tilde{N}_{C1} = N_{C1}D_{C1}^{-1}$ be any LCF and RCF; $\mathcal{S}(P, C_1)$ is $\mathcal{R}$-stable if and only if (8) holds

$$\begin{bmatrix} \tilde{D}_{C1}D_{11} + \tilde{N}_{C1}N_{11} & \tilde{N}_{C1}N_{12} \\ \tilde{D}_{C2}D_{21} + \tilde{N}_{C2}N_{21} & \tilde{D}_{C2}D_{22} + \tilde{N}_{C2}N_{22} \end{bmatrix} \text{is } \mathcal{R}\text{-unimodular} \quad (6)$$

$$D_{11} \quad \text{is } \mathcal{R}\text{-unimodular,}$$
$$\text{and} \quad (\tilde{D}_{C2}\, D_{22} + \tilde{N}_{C2}\, N_{22}) \quad \text{is } \mathcal{R}\text{-unimodular} \quad (7)$$
$$\begin{bmatrix} \tilde{D}_{C1}D_{11} + \tilde{N}_{C1}N_{11} & \tilde{N}_{C1}N_{12} \\ D_{21} & D_{22} \end{bmatrix} \text{ is } \mathcal{R}\text{-unimodular.} \quad (8)$$

□

To obtain a parameterization of all reliable decentralized controller pairs, we use the following characterization of all *admissible plants for stability using one controller* [1], [3]: Let $P \in R_p^{n_o \times n_i}$ be partitioned as in (1). There exists a decentralized controller $C_D = \operatorname{diag}[C_1, C_2]$ such that $\mathcal{S}(P, C_D)$ and $\mathcal{S}(P, C_2)$ are $\mathcal{R}$-stable if and only if P has an RCF and LCF $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ of the form

$$P = ND^{-1} = \begin{bmatrix} N_{11} & N_{12} \\ \tilde{V}_2\tilde{N}_{21} & N_{22} \end{bmatrix} \begin{bmatrix} I_{n_{i1}} & 0 \\ -\tilde{U}_2\tilde{N}_{21} & D_{22} \end{bmatrix}^{-1}$$
$$= \tilde{D}^{-1}\tilde{N} = \begin{bmatrix} I_{n_{o1}} & -N_{12}U_2 \\ 0 & \tilde{D}_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12}V_2 \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} \quad (9)$$

where $N_{11} \in \mathcal{R}^{n_{o1} \times n_{i1}}$, $N_{12} \in \mathcal{R}^{n_{o1} \times n_{i2}}$, $\tilde{N}_{21} \in \mathcal{R}^{n_{o2} \times n_{i1}}$, (N_{22}, D_{22}) is right-coprime, $(\tilde{D}_{22}, \tilde{N}_{22})$ is left-coprime, and $\tilde{U}_2, \tilde{V}_2, U_2, V_2 \in \mathcal{M}(\mathcal{R})$ satisfy (10); equivalently, $P_{11} - P_{12}D_{22}U_2P_{21} \in \mathcal{M}(\mathcal{R})$, $P_{12}D_{22} \in \mathcal{M}(\mathcal{R})$, $\tilde{D}_{22}P_{21} \in \mathcal{M}(\mathcal{R})$, where $N_{22}D_{22}^{-1}$ is an RCF and $\tilde{D}_{22}^{-1}\tilde{N}_{22}$ is an LCF of P_{22} and U_2 satisfies (10)

$$\begin{bmatrix} V_2 & U_2 \\ -\tilde{N}_{22} & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} D_{22} & -\tilde{U}_2 \\ N_{22} & \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} I_{n_{i2}} & 0 \\ 0 & I_{n_{o2}} \end{bmatrix}. \quad (10)$$

Theorem 1—Stabilizing Controllers: Let $P \in \mathcal{R}^{n_o \times n_i}$ have an RCF and LCF $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ satisfying (9). The system $S(P, C_2)$ is $\mathcal{R}$-stable if and only if C_2 is given by (11), $Q_2 \in \mathcal{R}^{n_{i2} \times n_{o2}}$ is such that $\tilde{D}_{C2}$ is biproper [holds for all $Q_2 \in \mathcal{M}(\mathcal{R})$ when $P_{22} \in \mathcal{M}(R_s)$]; $\mathcal{S}(P, C_D)$ and $\mathcal{S}(P, C_2)$ are both $\mathcal{R}$-stable if and only if C_2 and C_1 are given by (11) and (12), and $Q_1 \in \mathcal{R}^{n_{i1} \times n_{o1}}$ is such that $\tilde{D}_{C1}$ is biproper [holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $(N_{11} - N_{12}Q_2\tilde{N}_{21}) \in \mathcal{M}(R_s)$]; $\mathcal{S}(P, C_D)$, $\mathcal{S}(P, C_2)$, and $\mathcal{S}(P, C_1)$ are all $\mathcal{R}$-stable (i.e., (C_1, C_2) is a reliable decentralized controller pair) if and only if C_2 and C_1 are given by (11) and (12), where $Q_1 \in \mathcal{R}^{n_{i1} \times n_{o1}}$, $Q_2 \in \mathcal{R}^{n_{i2} \times n_{o2}}$ satisfy condition (13), or equivalently (14), and $\tilde{D}_{C1}$, $\tilde{D}_{C2}$ are biproper

$$\begin{aligned} C_2 &= \tilde{D}_{C2}^{-1}\tilde{N}_{C2} \\ &= (V_2 - Q_2\tilde{N}_{22})^{-1}(U_2 + Q_2\tilde{D}_{22}) \\ &= N_{C2}D_{C2}^{-1} \\ &= (\tilde{U}_2 + D_{22}Q_2)(\tilde{V}_2 - N_{22}Q_2)^{-1} \end{aligned} \quad (11)$$

$$C_1 = \tilde{D}_{C1}^{-1}\tilde{N}_{C1} = (I - Q_1(N_{11} - N_{12}Q_2\tilde{N}_{21}))^{-1}Q_1 \quad (12)$$

$$D_{22} + (\tilde{U}_2 + D_{22}Q_2)\tilde{N}_{21}Q_1N_{12} \quad \text{is } \mathcal{R}\text{-unimodular} \quad (13)$$

$$\tilde{D}_{22} + \tilde{N}_{21}\, Q_1\, N_{12}\, (U_2 + Q_2\tilde{D}_{22}) \quad \text{is } \mathcal{R}\text{-unimodular.} \quad (14)$$

□

Conditions (13) and (14) lead to the conditions that P must satisfy for the existence of reliable decentralized controller pairs as stated in Theorem 2. Strong $\mathcal{R}$-stabilizability of pseudo-systems related to P are important for existence of reliable decentralized controllers. The following are well known [9]: An LTI system $\hat{P}$ is said to be strongly $\mathcal{R}$-stabilizable iff an $\mathcal{R}$-stable $\mathcal{R}$-stabilizing controller $\hat{C}$ exists for $\hat{P}$. In the standard full-feedback system $\mathcal{S}(\hat{P}, \hat{C})$, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if it has an even number of $\mathcal{U}$-poles between consecutive pairs of real blocking $\mathcal{U}$-zeros. Let $\hat{P} = N_pD_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p = N_{br}D_b^{-1}N_{bl} + G_b$ be any RCF, LCF, and BCF of $\hat{P}$; let $\hat{C} = \tilde{D}_c^{-1}\tilde{N}_c$ be any LCF; $\hat{C} \in \mathcal{M}(\mathcal{R})$ if and only if $\tilde{D}_c$ is unimodular. Therefore, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if $\tilde{X} \in \mathcal{M}(\mathcal{R})$ exists such that $D_p + \tilde{X}N_p$ is $\mathcal{R}$-unimodular; equivalently, $X \in \mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_p + \tilde{N}_pX$ is $\mathcal{R}$-unimodular; equivalently, $X_b \in \mathcal{M}(\mathcal{R})$ exists such that

$$\begin{bmatrix} I + X_bG_b & X_bN_{br} \\ -N_{bl} & D_b \end{bmatrix} \quad \text{is } \mathcal{R}\text{-unimodular.}$$

Lemma 2—Coprime Factorizations and Strong Stabilizability: Let $P \in R_p^{n_o \times n_i}$, partitioned as in (1), have an RCF and LCF $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ of the form (9). Let $N_{12}D_{22}^{-1}$ be an RCF, $\tilde{Y}_{12}^{-1}\tilde{X}_{12}$ be an LCF of P_{12}; let $\tilde{D}_{22}^{-1}\tilde{N}_{21}$ be an LCF and $X_{21}Y_{21}^{-1}$ be an RCF of P_{21}. Define $\hat{P} := P_{12}(\tilde{U}_2 + D_{22}Q_2)\tilde{D}_{22}P_{21} = P_{12}D_{22}(U_2 + Q_2\tilde{D}_{22})P_{21}$.

1) $N_{12}D_{22}^{-1}(\tilde{U}_2 + D_{22}Q_2)\tilde{N}_{21}$ is a BCF, $N_{12}(U_2 + Q_2\tilde{D}_{22})X_{21}Y_{21}^{-1}$ is an RCF, and $\tilde{Y}_{12}^{-1}\tilde{X}_{12}(\tilde{U}_2 + D_{22}Q_2)\tilde{N}_{21}$ is an LCF of $\hat{P}$.
2) $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if $\det\, \tilde{D}_{22}$ has the same sign at all real blocking $\mathcal{U}$-zeros of $N_{12}(U_2 + Q_2\tilde{D}_{22})X_{21}$, equivalently, $\det\, D_{22}$ has the same sign at all real blocking $\mathcal{U}$-zeros of $\tilde{X}_{12}(\tilde{U}_2 + D_{22}Q_2)\tilde{N}_{21}$. □

Theorem 2—Conditions for Reliable Decentralized Stabilizability: Let $P \in R_p^{n_o \times n_i}$ be partitioned as in (1).

1) If there exists a reliable decentralized controller pair (C_1, C_2), then the following four necessary conditions hold: 1) P has an RCF and LCF $ND^{-1} = \tilde{D}^{-1}\tilde{N}$ satisfying (9); 2) in (9), $N_{12}D_{22}^{-1}$ is an RCF of P_{12}, $\tilde{D}_{22}^{-1}\tilde{N}_{21}$ is an LCF of P_{21}; 3) P_{12}, P_{21} are strongly $\mathcal{R}$-stabilizable; and 4) the sign of $\det\, D_{22}$ is the same at all real blocking $\mathcal{U}$-zeros of P_{12} and at all real blocking $\mathcal{U}$-zeros of P_{21}.
2) Let P have an RCF and LCF $ND^{-1} = \tilde{D}^{-1}\tilde{N}$ satisfying (9); let $N_{12}D_{22}^{-1}$ be an RCF of P_{12}, $\tilde{D}_{22}^{-1}\tilde{N}_{21}$ be an LCF of P_{21}.
    a) There exist $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ satisfying (13), or equivalently (14), if and only if $\hat{P} = P_{12}(\tilde{U}_2 + D_{22}Q_2)\tilde{D}_{22}P_{21} = P_{12}D_{22}(U_2 + Q_2\tilde{D}_{22})P_{21}$ is strongly $\mathcal{R}$-stabilizable for some $Q_2 \in \mathcal{M}(\mathcal{R})$.

b) Let $P_{22} \in \mathcal{M}(R_s)$ and let P_{12} or P_{21} be strictly proper. There exists a reliable decentralized controller pair (C_1, C_2) if and only if $\hat{P}$ is strongly $\mathcal{R}$-stabilizable for some $Q_2 \in \mathcal{M}(\mathcal{R})$. □

Condition 4) of Theorem 2-1) implies 3); the two conditions are equivalent when $P_{12}, P_{21} \in \mathcal{M}(R_s)$. By Theorem 2, if a reliable decentralized controller pair exists, then $Q_2 \in \mathcal{M}(\mathcal{R})$ exists such that $\hat{P} = P_{12}(\tilde{U}_2 + D_{22}Q_2)\tilde{D}_{22}P_{21}$ is strongly $\mathcal{R}$-stabilizable. When $P \in \mathcal{M}(R_s)$, strong $\mathcal{R}$ stabilizability of $\hat{P}$ becomes necessary and sufficient for existence of reliable decentralized controller pairs. We parameterize all reliable decentralized controller pairs for $\mathcal{R}$-stable plants in Proposition 1. Explicit necessary and sufficient conditions for existence of reliable decentralized controller pairs are stated in Theorem 3 for the important special case when at least one control channel has only one input and one output.

Proposition 1—Reliable Decentralized Stabilization for Stable Plants: Let $P \in \mathcal{R}^{n_o \times n_i}$ be $\mathcal{R}$-stable. Then there exists a reliable decentralized controller pair. Furthermore, all reliable decentralized controller pairs (C_1, C_2) are parameterized by (15), where C_j is proper if and only if $Q_j \in \mathcal{M}(\mathcal{R})$ is such that $(I - Q_jP_{jj})$ is biproper, which holds for all $Q_j \in \mathcal{M}(\mathcal{R})$ when $P_{jj} \in \mathcal{M}(R_s)$

$$\{(C_1, C_2) | C_j = (I - Q_jP_{jj})^{-1}Q_j,\ Q_j \in \mathcal{M}(\mathcal{R}),\ j = 1, 2,\ I - Q_1P_{12}Q_2P_{21} \text{ is } \mathcal{R}\text{-unimodular}\}. \tag{15}$$

□

By Theorem 2, if reliable decentralized controllers exist, then P has an RCF and LCF satisfying (9), $P_{22} = N_{22}D_{22}^{-1}$ is an RCF, $P_{12} = N_{12}D_{22}^{-1}$ is an RCF, and $P_{21} = \tilde{D}_{22}^{-1}\tilde{N}_{21}$ is an LCF. Suppose $P_{22} = 0$, or $P_{12} = 0$ or $P_{21} = 0$; i.e., $N_{22} = 0$ or $N_{12} = 0$ or $\tilde{N}_{21} = 0$. The pair $(0, D_{22})$ is right-coprime if and only if D_{22} is $\mathcal{R}$-unimodular, i.e., $P \in \mathcal{M}(\mathcal{R})$. So when P_{22}, P_{12}, or P_{21} is zero, reliable decentralized controllers exist if and only if P is $\mathcal{R}$-stable and the parameterization of all reliable decentralized controller pairs is given by (15), where $(I - Q_1P_{12}Q_2P_{21})$ is $\mathcal{R}$-unimodular for all $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ if P_{12} or P_{21} is zero.

Theorem 3—Necessary and Sufficient Conditions When P_{22} Is SISO: Let $P \in R_p^{n_o \times n_i}$, $P_{11} \in R_p^{n_{o1} \times n_{i1}}$, $P_{12} \in R_p^{n_{o1} \times 1}$, $P_{21} \in R_p^{1 \times n_{i1}}$, $P_{22} \in R_p$ (i.e., $n_{o2} = n_{i2} = 1$). Let $P_{22} = N_{22}D_{22}^{-1}$ be any coprime factorization. Let P_{12} or P_{21} be strictly proper. Let $\rho_1, \rho_2, \cdots, \rho_\gamma$ (arranged in ascending order) denote the distinct real $\mathcal{U}$-zero poles of P_{22} and let $\rho_{j_1}, \rho_{j_2}, \cdots, \rho_{j_\ell}$ (arranged in ascending order) denote those distinct real $\mathcal{U}$-poles of P_{22} for which the sign of $N_{22}(\rho_{j_k})$ is not equal to the sign of $N_{22}(\rho_{j_k+1})$, $1 \le k \le \ell$. There exists a reliable decentralized controller pair (C_1, C_2) if and only if the four necessary conditions of Theorem 2-1) hold and P_{22} has an even number of real $\mathcal{U}$-poles in each of the intervals $(\rho_{j_k}, \rho_{j_{k+1}+1})$, $1 \le k \le \ell - 1$, and (ρ_{j_ℓ}, ∞). □

Corollary 1—Sufficient Conditions When P_{22} is SISO: Let $P \in R_p^{n_o \times n_i}$, $P_{11} \in R_p^{n_{o1} \times n_{i1}}$, $P_{12} \in R_p^{n_{o1} \times 1}$, $P_{21} \in R_p^{1 \times n_{i1}}$; let $P_{22} = N_{22}D_{22}^{-1} \in R_p$ be any coprime factorization. Let the four conditions of Theorem 2-1) hold: 1) Let the sign of D_{22} at the real blocking $\mathcal{U}$-zeros of P_{12} and P_{21} be the same as the sign of $D_{22}(\infty)$. Reliable decentralized controllers exist if P_{22} has even number of $\mathcal{U}$-zeros between any pairs of its real $\mathcal{U}$-poles; 2) Reliable decentralized controllers exist if the sign of D_{22} is the same at all real $\mathcal{U}$-zeros of P_{22} as the sign of $D_{22}(\infty)$. □

In Corollary 1-1), if P_{12} or P_{21} is strictly proper, then the sign of D_{22} at the real blocking $\mathcal{U}$-zeros of P_{12} and of P_{21} being the same as the sign of $D_{22}(\infty)$ follows from the necessary conditions 3) and 4) of Theorem 2-1). When $P_{22} \in \mathcal{M}(R_s)$, the sufficient condition in Corollary 1-2) is equivalent to P_{22} being strongly $\mathcal{R}$-stabilizable; when $P_{22} \notin \mathcal{M}(R_s)$, this condition *implies* P_{22} is strongly $\mathcal{R}$-stabilizable.

Theorem 4—Conditions for MIMO Channels: Let $P \in R_p^{n_o \times n_i}$, $P_{11} \in R_p^{n_{o1} \times n_{i1}}$, $P_{12} \in R_p^{n_{o1} \times n_{i2}}$, $P_{21} \in R_p^{n_{o2} \times n_{i1}}$, $P_{22} \in R_p^{n_{o2} \times n_{i2}}$. Let the four necessary conditions of Theorem 2-1) hold.

1) Let $P_{12} \in \mathcal{M}(R_s)$ or $P_{21} \in \mathcal{M}(R_s)$. Let $n_{o2} = n_{i2} > 1$; let the sign of $\det D_{22}$ be the same at all common real $\mathcal{U}$-zeros of P_{12} and P_{21} as the sign of $\det D_{22}(\infty)$. Reliable decentralized controller pairs exist if $\operatorname{rank} P_{12} = n_{i2} \le n_{o1}$, $\operatorname{rank} P_{21} = n_{o2} \le n_{i1}$.
2) Let $P_{12} \in \mathcal{M}(R_s)$ or $P_{21} \in \mathcal{M}(R_s)$. Let $P_{22} \in R_s^{n_{o2} \times n_{o2}}$; let $\operatorname{rank} P_{22} = n_{o2} = n_{i2}$, $\operatorname{rank} P_{12} + \operatorname{rank} P_{21} > n_{o2} = n_{i2}$; let the sign of $\det D_{22}$ be the same at all real (transmission) $\mathcal{U}$-zeros of P_{12} and of P_{21} as the sign of $\det D_{22}(\infty)$. Reliable decentralized controller pairs exist if the number of real (transmission) $\mathcal{U}$-zeros of $P_{22} = \tilde{D_{22}}^{-1}\tilde{N}_{22}$ between any pair of real blocking $\mathcal{U}$-zeros of $\tilde{D}_{22}$ is even.
3) Let $P_{22} \in R_p^{n_{o2} \times n_{i2}}$, where n_{o2} and n_{i2} are not both equal to one. Reliable decentralized controller pairs exist if $P_{12} \in R_p^{n_{o1} \times n_{i2}}$ has an $\mathcal{R}$-stable left-inverse $P_{12}^I \in \mathcal{R}^{n_{i2} \times n_{o1}}$ and if $P_{21} \in R_p^{n_{o2} \times n_{i1}}$ has an $\mathcal{R}$-stable right-inverse $P_{21}^I \in \mathcal{R}^{n_{i1} \times n_{o2}}$.
4) Let $P_{21} \in R_p^{n_{o2} \times n_{i1}}$ have an $\mathcal{R}$-stable right-inverse $P_{21}^I \in \mathcal{R}^{n_{i1} \times n_{o2}}$. Let $P_{11} \in \mathcal{M}(R_s)$, $P_{12} \in \mathcal{M}(R_s)$. Let P_{22} be strongly $\mathcal{R}$-stabilizable. Reliable decentralized controller pairs exist if $\tilde{L}P_{12} = P_{22}$ for some $\tilde{L} \in \mathcal{R}^{n_{o2} \times n_{o1}}$.
5) Let $P_{12} \in R_p^{n_{o1} \times n_{i2}}$ have an $\mathcal{R}$-stable left-inverse $P_{12}^I \in \mathcal{R}^{n_{i2} \times n_{o1}}$. Let $P_{11} \in \mathcal{M}(R_s)$, $P_{12} \in \mathcal{M}(R_s)$. Let P_{22} be strongly $\mathcal{R}$-stabilizable. Reliable decentralized controller pairs exist if $P_{21}\tilde{R} = P_{22}$ for some $\tilde{R} \in \mathcal{R}^{n_{i1} \times n_{i2}}$.
6) Let $P_{22} \in R_s^{n_{o2} \times n_{i2}}$. Reliable decentralized controller pairs exist if $\tilde{L}P_{12} = P_{22}$ for some $\tilde{L} \in \mathcal{R}^{n_{o2} \times n_{o1}}$ and $P_{21}\tilde{R} = P_{22}$ for some $\tilde{R} \in \mathcal{R}^{n_{i1} \times n_{i2}}$. □

Other sufficient conditions for existence of reliable decentralized controllers can be derived from the six general cases in Theorem 4. For example, under the assumptions of case 2), reliable decentralized controllers exist if either P_{21} has an $\mathcal{R}$-stable right-inverse or P_{12} has an $\mathcal{R}$-stable left-inverse since $\operatorname{rank} P_{21} = n_{o2}$ or $\operatorname{rank} P_{12} = n_{i2} = n_{o2}$ implies $\operatorname{rank} P_{12} + \operatorname{rank} P_{21} > n_{o2}$.

III. Conclusions

We considered the design of reliable decentralized controllers that stabilize a given plant P when both controllers act together and when either one of the controllers acts alone. We showed that reliable decentralized controllers exist only if the subblocks P_{12} and P_{21} of P are strongly stabilizable. We established necessary and sufficient conditions for existence of reliable decentralized controllers when P_{22} is SISO and gave sufficient conditions when all subblocks of P are MIMO. We characterized all reliable decentralized controllers in the parameterizations (11), (12). Extensions to decentralized systems with more than two channels would require additional constraints on the plant.

Appendix

A. Proofs

The proof of Lemma 1 follows from (2)–(4) using standard arguments. The proof of Theorem 1 follows by Lemma 1 from the assumption that P has an RCF ND^{-1} of the form (9).

Proof of Lemma 2:

1) By (10), (D_{22}, N_{C2}) is left-coprime. By assumption, $(\tilde{D}_{22}, \tilde{N}_{21})$ is left-coprime, (X_{21}, Y_{21}) is right-coprime,

and $V_{21}, U_{21}, \tilde{V}_{21}, \tilde{U}_{21} \in \mathcal{M}(\mathcal{R})$ exist such that $V_{21}Y_{21} + U_{21}X_{21} = I$, $\tilde{D}_{22}\tilde{V}_{21} + \tilde{N}_{21}\tilde{U}_{21} = I$, $V_{21}\tilde{U}_{21} = U_{21}\tilde{V}_{21}$, $\tilde{N}_{21}Y_{21} = \tilde{D}_{22}X_{21}$. Hence

$$\begin{bmatrix} V_{21} + U_{21}D_{C2}\tilde{N}_{21} & U_{21}N_{22} \\ -N_{C2}\tilde{N}_{21} & D_{22} \end{bmatrix} \times \begin{bmatrix} Y_{21} & -\tilde{U}_{21}\tilde{N}_{22} \\ \tilde{N}_{C2}X_{21} & \tilde{D}_{C2} + \tilde{N}_{C2}\tilde{V}_{21}\tilde{N}_{22} \end{bmatrix} = I$$

implies $(D_{22}, N_{C2}\tilde{N}_{21})$ is left-coprime for all $Q_2 \in \mathcal{M}(\mathcal{R})$. Let $\hat{P} := P_{12}N_{C2}\tilde{D}_{22}P_{21}$; since (N_{12}, D_{22}) is right-coprime and $(D_{22}, N_{C2}\tilde{N}_{21})$ is left-coprime, $\hat{P} = N_{12}D_{22}^{-1}N_{C2}\tilde{N}_{21}$ is a BCF. For $P_{12} = N_{12}D_{22}^{-1} = \tilde{Y}_{12}^{-1}\tilde{X}_{12}$, there are $V_{12}, U_{12}, \tilde{V}_{12}, \tilde{U}_{12} \in \mathcal{M}(\mathcal{R})$ such that $V_{12}D_{22} + U_{12}N_{12} = I$, $\tilde{X}_{12}\tilde{U}_{12} + \tilde{Y}_{12}\tilde{V}_{12} = I$, $V_{12}\tilde{U}_{12} = U_{12}\tilde{V}_{12}$. Hence

$$\begin{bmatrix} V_{21} + U_{21}(D_{C2} + N_{22}V_{12}N_{C2})\tilde{N}_{21} & U_{21}N_{22}U_{12} \\ -\tilde{X}_{12}N_{C2}\tilde{N}_{21} & \tilde{Y}_{12} \end{bmatrix} \times \begin{bmatrix} Y_{21} & -\tilde{U}_{21}\tilde{N}_{22}\tilde{U}_{12} \\ N_{12}\tilde{N}_{C2}X_{21} & \tilde{V}_{12} + N_{12}(\tilde{D}_{C2} + \tilde{N}_{C2}\tilde{V}_{21}\tilde{N}_{22})\tilde{U}_{12} \end{bmatrix} = I$$

implies $(N_{12}\tilde{N}_{C2}X_{21}, Y_{21})$ is right-coprime, $(\tilde{Y}_{12}, \tilde{X}_{12}N_{C2}\tilde{N}_{21})$ is left-coprime.

2) Since $N_{12}\tilde{N}_{C2}X_{21}Y_{21}^{-1} = \tilde{Y}_{12}^{-1}\tilde{X}_{12}N_{C2}\tilde{N}_{21}$ is an RCF and LCF of $\hat{P}$, because $P_{22} = N_{22}D_{22}^{-1} = \tilde{D}_{22}^{-1}\tilde{N}_{22}$, $P_{21} = \tilde{D}_{22}^{-1}\tilde{N}_{21} = X_{21}Y_{21}^{-1}$, $P_{12} = N_{12}D_{22}^{-1} = \tilde{Y}_{12}^{-1}\tilde{X}_{12}$, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if $\det Y_{21}$, equivalently $\det \tilde{Y}_{12}$, $\det \tilde{D}_{22}$, or $\det D_{22}$, has the same sign at all real $\mathcal{U}$-zeros of $N_{12}\tilde{N}_{C2}X_{21}$, equivalently, of $\tilde{X}_{12}N_{C2}\tilde{N}_{21}$. □

Proof of Theorem 2:

1) If reliable decentralized controllers exist, Condition 1) holds by (9). By Theorem 1, (13) and (14) hold; (13) and (14) imply (N_{12}, D_{22}) is right-coprime, and $(\tilde{D}_{22}, \tilde{N}_{21})$ is left-coprime. By (9), Condition 2) holds since $P_{12} = N_{12}D_{22}^{-1}$, $P_{21} = \tilde{D}_{22}^{-1}\tilde{N}_{21}$. Conditions 3) and 4) are shown as follows: P_{12} is strongly $\mathcal{R}$-stabilizable if and only if for any RCF $P_{12} = N_{12}D_{22}^{-1}$, $\tilde{X} \in \mathcal{M}(\mathcal{R})$ exists such that $D_{22} + \tilde{X}N_{12}$ is $\mathcal{R}$-unimodular; with $\tilde{X} = N_{C2}\tilde{N}_{21}Q_1$, (13) implies P_{12} is strongly $\mathcal{R}$-stabilizable. Similarly, P_{21} is strongly $\mathcal{R}$-stabilizable if and only if for any LCF $P_{21} = \tilde{D}_{22}^{-1}\tilde{N}_{21}$, $X \in \mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_{22} + \tilde{N}_{21}X$ is $\mathcal{R}$-unimodular; with $X = Q_1N_{12}\tilde{N}_{C2}$, (14) implies P_{21} is strongly $\mathcal{R}$-stabilizable. Since (13) implies $\det(D_{22}+N_{C2}\tilde{N}_{21}Q_1N_{12})$ has no $\mathcal{U}$-zeros, $\det D_{22}(z_{12})$ has the same sign as $\det D_{22}(z_{21})$ for all real $z_{12}, z_{21} \in \mathcal{U}$, $N_{12}(z_{12}) = 0$ and $\tilde{N}_{21}(z_{21}) = 0$.

2)
 a) By Lemma 2, $\hat{P} = N_{12}D_{22}^{-1}N_{C2}\tilde{N}_{21}$ is a BCF; therefore, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if $\hat{X} \in \mathcal{M}(\mathcal{R})$ exists such that $(D_{22} + N_{C2}\tilde{N}_{21}\hat{X}N_{12})$ is $\mathcal{R}$-unimodular, equivalently (13) holds.

 b) By Theorem 1-3), reliable decentralized controllers exist if and only if $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ exist satisfying (13), such that $\tilde{D}_{C1}, \tilde{D}_{C2}$ are biproper. It was shown that $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ satisfying (13) exist if and only if $\hat{P}$ is strongly $\mathcal{R}$-stabilizable. Since $P_{22} \in \mathcal{M}(R_s)$, $(V_2 - Q_2\tilde{N}_{22})$ is biproper. If $P_{12} \in \mathcal{M}(R_s)$ or $P_{21} \in \mathcal{M}(R_s)$, then $\hat{P} = P_{12}D_{22}\tilde{N}_{C2}P_{21} \in \mathcal{M}(R_s)$; hence, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if for any $-a \in \mathbb{R}\backslash\mathcal{U}$, $(s+a)^{-1}\hat{P}$ is strongly $\mathcal{R}$-stabilizable, equivalently, $\tilde{Q}_1 \in \mathcal{M}(\mathcal{R})$ exists such that $D_{22} + N_{C2}\tilde{N}_{21}(s+a)^{-1}\tilde{Q}_1N_{12}$ is $\mathcal{R}$-unimodular. Let $Q_1 := (s+a)^{-1}\tilde{Q}_1 \in \mathcal{M}(R_s)$; then (13) holds and $\tilde{D}_{C1}$ is biproper. □

Proof of Proposition 1: By Lemma 1, (7) and (8) hold if and only if $C_j = \tilde{D}_{Cj}^{-1}\tilde{N}_{Cj}$ is an $\mathcal{R}$-stabilizing controller for P_{jj}, $j = 1, 2$. Therefore, all C_1, C_2 are given by (15). By (6), $\mathcal{S}(P, C_D)$ is also $\mathcal{R}$-stable if and only if $I - Q_1P_{12}Q_2P_{21}$ is $\mathcal{R}$-unimodular. The controllers are proper if and only if $\tilde{D}_{C_j}$ is biproper. We give a solution for $Q_j \in \mathcal{M}(\mathcal{R})$ such that $I - Q_1P_{12}Q_2P_{21}$ is $\mathcal{R}$-unimodular and $(I - Q_jP_{jj})$ is biproper. Choose $\overline{Q_1}$, $Q_2 \in \mathcal{M}(\mathcal{R})$ strictly proper; let $Q_1 = \alpha^{-1}\overline{Q_1}$, where $\alpha \in \mathbb{R}$, $|\alpha| > \|\overline{Q_1}P_{12}Q_2P_{21}\|$; choosing strictly proper Q_1, Q_2 is sufficient to make $(I - Q_jP_{jj})$ biproper, and choosing α that guarantees $\|\alpha^{-1}\overline{Q_1}P_{12}Q_2P_{21}\| < 1$ is sufficient to satisfy (6). This shows existence of reliable decentralized controllers for any $\mathcal{R}$-stable P. The expression for C_1 in (15) is equivalent to (12). By (12), $C_1 = (I - \hat{Q}_1(P_{11} - P_{12}Q_2P_{21}))^{-1}\hat{Q}_1$; by (13), $\Phi := (I + \hat{Q}_1P_{12}Q_2P_{21})$ is $\mathcal{R}$-unimodular. With $Q_1 = \Phi^{-1}\hat{Q}_1$, $C_1 = (I + \hat{Q}_1P_{12}Q_2P_{21} - \hat{Q}_1P_{11})^{-1}\hat{Q}_1 = (I - \Phi^{-1}\hat{Q}_1P_{11})^{-1}\Phi^{-1}\hat{Q}_1$ is equivalent to (15) and $\Phi^{-1} = (I - \Phi^{-1}\hat{Q}_1P_{12}Q_2P_{21})$ is $\mathcal{R}$-unimodular if and only if $I - Q_1P_{12}Q_2P_{21}$ is $\mathcal{R}$-unimodular. □

Proof of Theorem 3: Since $P_{22} = N_{22}D_{22}^{-1} = \tilde{D}_{22}^{-1}\tilde{N}_{22}$ is scalar, N_{22}, $\tilde{N}_{22}$, D_{22}, and $\tilde{D}_{22}$ are used interchangeably. By Theorem 2, $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ satisfying (13) and (14) exist if and only if $\hat{P} = P_{12}D_{22}\tilde{N}_{C2}P_{21}$ is strongly $\mathcal{R}$-stabilizable for some $Q_2 \in \mathcal{R}$, i.e., $\tilde{D}_{22}$ has the same sign at all real blocking $\mathcal{U}$-zeros of $N_{12}\tilde{N}_{C2}X_{21}$ by Lemma 2-2). Since the only $\mathcal{U}$-zeros of $P_{12} \in \mathcal{R}^{n_{o1}\times 1}$ and $P_{21} \in \mathcal{R}^{1\times n_{i1}}$ are their blocking $\mathcal{U}$-zeros, $s_o \in \mathcal{U}$ is a blocking $\mathcal{U}$-zero of $P_{12}\tilde{N}_{C2}X_{21}$ if and only if it is a blocking $\mathcal{U}$-zero of P_{12}, i.e., $N_{12}(s_o) = 0$, or of $\tilde{N}_{C2}$ or of P_{21}, i.e., $X_{21}(s_o) = 0$. The four conditions of Theorem 2-1) are necessary for the existence of reliable decentralized controllers. The $\mathcal{U}$-poles of P_{22} are the $\mathcal{U}$-zeros of D_{22}. By (10), the signs of U_2 and N_{22} are the same at all real $\mathcal{U}$-zeros of $\tilde{D}_{22}$. Suppose $\{j_1, \cdots, j_\ell\}$ is empty. Then $\tilde{Q}_2 \in \mathcal{R}$ exists such that $(U_2 + \tilde{Q}_2\tilde{D}_{22})$ has no $\mathcal{U}$-zeros [9]. Since the only blocking $\mathcal{U}$-zeros of $N_{12}(U_2 + \tilde{Q}_2\tilde{D}_{22})X_{21}$ are those of P_{12} and P_{21}, the conditions of Theorem 2-1) are sufficient for (13). If $P_{22} \in \mathcal{M}(R_s)$, then $(V_2 - \tilde{Q}_2\tilde{N}_{22})$ is biproper. If $\tilde{N}_{22}(\infty) \neq 0$, let $Q_2 := \tilde{Q}_2 + \hat{Q}_2$, $\hat{Q}_2 \in \mathcal{R}$, $\hat{Q}_2(\infty) \neq (V_2 - \tilde{Q}_2\tilde{N}_{22})\tilde{N}_{22}^{-1}(\infty)$, $\|\hat{Q}_2\| < \|\tilde{D}_{22}(U_2 + \tilde{Q}_2\tilde{D}_{22})^{-1}\|^{-1}$. Then $\tilde{D}_{C2}$ is biproper, i.e., $C_2 \in \mathcal{M}(R_p)$, and $\tilde{N}_{C2}$ is a unit in $\mathcal{R}$. Since $P_{12} \in \mathcal{M}(R_s)$ or $P_{21} \in \mathcal{M}(R_s)$, Q_1 satisfying (13) can be chosen strictly proper so that $\tilde{D}_{C1}$ is biproper. Therefore, the conditions of Theorem 2-1) are sufficient for the existence of reliable decentralized controllers. Suppose $\{j_1, \cdots, j_\ell\}$ is not empty; then N_{22} has an odd number of zeros in each interval $(\rho_{j_k}, \rho_{j_k+1})$, $1 \le k \le \ell$. By (10), $\tilde{N}_{C2}(\rho_{j_k})N_{22}(\rho_{j_k}) = 1$ at the $\mathcal{U}$-zeros ρ_{j_k} of $\tilde{D}_{22}$; hence, $\tilde{N}_{C2} = (U_2 + Q_2\tilde{D}_{22})$ has an odd number of zeros because $\tilde{N}_{C2}$ has even number of zeros in $(\rho_{j_k}, \rho_{j_k+1})$, $1 \le k \le \ell$. Note that ρ_{j_k} is the first zero of $\tilde{D}_{22}$ immediately to the left of the real $\mathcal{U}$-zero of $\tilde{N}_{C2}$ in $(\rho_{j_k}, \rho_{j_k+1})$ and $\rho_{j_{k+1}+1}$ is the first zero of $\tilde{D}_{22}$ immediately to the right of the real $\mathcal{U}$-zero of $\tilde{N}_{C2}$ in $(\rho_{j_{k+1}}, \rho_{j_{k+1}+1})$. If $Q_2 \in \mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_{22}$ has the same sign at all real $\mathcal{U}$-zeros of $\tilde{N}_{C2}$, then $\tilde{D}_{22}$ must have an even number of zeros between ρ_{j_k} and $\rho_{j_{k+1}+1}$ since $\tilde{N}_{C2}$ has at least one real $\mathcal{U}$-zero in each of these intervals. Since either $P_{12}(\infty) = 0$ or $P_{21}(\infty) = 0$, the sign of $\tilde{D}_{22}$ at the $\mathcal{U}$-zero of $\tilde{N}_{C2}$ in the last interval $(\rho_{j_\ell}, \rho_{j_\ell+1})$ must agree with the sign of $\tilde{D}_{22}(\infty)$; hence, $\tilde{D}_{22}$ must have even number of zeros in (ρ_{j_ℓ}, ∞). This proves necessity. For any $Q_2 \in\mathcal{R}$, the minimum number of $\mathcal{U}$-zeros of $\tilde{N}_{C2}$ is ℓ, which is the number of intervals where N_{22} has an odd number of zeros between real $\mathcal{U}$-zeros of $\tilde{D}_{22}$. There is $\tilde{Q}_2 \in \mathcal{R}$ such that $(U_2+\tilde{Q}_2\tilde{D}_{22})$ has exactly ℓ real $\mathcal{U}$-zeros, with exactly one in each of $(\rho_{j_k}, \rho_{j_k+1})$, $1 \le k \le \ell$, because $(U_2 + \tilde{Q}_2\tilde{D}_{22})$ has an odd number of zeros in each of these intervals. If $\tilde{D}_{22}$ has even number of real $\mathcal{U}$-zeros in each of $(\rho_{j_k}, \rho_{j_{k+1}+1})$, $1 \le k \le \ell-1$, and (ρ_{j_ℓ}, ∞), then $\hat{P} = P_{12}D_{22}(U_2 + \tilde{Q}_2\tilde{D}_{22})P_{21}$ is strongly $\mathcal{R}$-stabilizable. Let

$Q_1 \in \mathcal{M}(\mathcal{R})$ be such that $M_1 := \tilde{D}_{22} + \tilde{N}_{21} Q_1 N_{12}(U_2 + \tilde{Q}_2 \tilde{D}_{22})$ is $\mathcal{R}$-unimodular; Q_1 can be chosen strictly proper. If $\tilde{N}_{22}(\infty) \neq 0$, let $Q_2 := \tilde{Q}_2 + \hat{Q}_2$, $\hat{Q}_2 \in \mathcal{R}$, $\hat{Q}_2(\infty) \neq (\tilde{V}_2 - N_{22}\tilde{Q}_2)(\infty)\tilde{N}_{22}^{-1}(\infty)$, $\|\hat{Q}_2\| < \|\tilde{D}_{22} M_1^{-1} \tilde{N}_{21} Q_1 N_{12}\|^{-1}$. Then $\tilde{D}_{C2}$, $\tilde{D}_{C1}$ are biproper, i.e., C_1, C_2 are proper. Since $\tilde{D}_{22} + \tilde{N}_{21} Q_1 N_{12} \tilde{N}_{C2} = M_1 + \tilde{N}_{21} Q_1 N_{12} \hat{Q}_2 \tilde{D}_{22}$ is $\mathcal{R}$-unimodular, (C_1, C_2) is a reliable decentralized controller pair. □

Proof of Corollary 1:

1) If P_{22} has an even number of real $\mathcal{U}$-zeros between consecutive real $\mathcal{U}$-poles, the sign of N_{22} is the same at all real $\mathcal{U}$-zeros of $\tilde{D}_{22}$. Since $\{j_1, \cdots, j_\ell\}$ is empty, $Q_2 \in \mathcal{R}$ exists such that $\tilde{N}_{C2}$ has no $\mathcal{U}$-zeros (the only blocking $\mathcal{U}$-zeros of $N_{12}\tilde{N}_{C2}X_{21}$ are those of P_{12}, P_{21}) and $\tilde{D}_{C2}$ is biproper, i.e., C_2 is proper. For this Q_2, $\hat{P} = P_{12} D_{22} \tilde{N}_{C2} P_{21}$ is strongly $\mathcal{R}$-stabilizable. Since $(s+a)^{-1}\hat{P}$ is strongly $\mathcal{R}$-stabilizable for any $-a \in \mathbb{R}\backslash\mathcal{U}$, $\hat{Q}_1 \in \mathcal{M}(\mathcal{R})$ exists such that $D_{22} + (\tilde{U}_2 + D_{22}Q_2)\tilde{N}_{21}(s+a)^{-1}\hat{Q}_1 N_{12}$ is $\mathcal{R}$-unimodular; $Q_1 = (s+a)^{-1}\hat{Q}_1$ satisfies (13) and $\tilde{D}_{C1}$ is biproper.
2) By assumption, the sign of D_{22} is the same at all real $\mathcal{U}$-zeros of P_{12}, P_{21}, and P_{22} as the sign of $D_{22}(\infty)$; hence, P_{22} is strongly $\mathcal{R}$-stabilizable. Let $Q_2 \in \mathcal{R}$ be such that $\tilde{D}_{C2}$ is $\mathcal{R}$-unimodular, then $\tilde{D}_{22}$ has the same sign at all real $\mathcal{U}$-zeros of $\tilde{N}_{C2}N_{22}$. The sign of D_{22} at the real $\mathcal{U}$-zeros of $\tilde{N}_{C2}$ is the same as that of $D_{22}(\infty)$; hence, $\hat{P} = P_{12}D_{22}\tilde{N}_{C2}P_{21}$ is strongly $\mathcal{R}$-stabilizable. As in the proof of 1), $(s+a)^{-1}\hat{P}$ is also strongly $\mathcal{R}$-stabilizable, since Q_1 can be chosen strictly proper, the controllers are proper. □

Proof of Theorem 4: Let $P_{22} \in R_p^{n_{o2} \times n_{i2}}$, $\operatorname{rank} P_{22} =: r$, $\Lambda := \operatorname{diag}[\lambda_1 \cdots \lambda_r]$, and $\Psi := \operatorname{diag}[\psi_1 \cdots \psi_r]$; there exist $\mathcal{R}$-unimodular $L \in \mathcal{R}^{n_{o2} \times n_{o2}}$, $R \in \mathcal{R}^{n_{i2} \times n_{i2}}$ satisfying (16), where λ_j, $\psi_j \in \mathcal{R}$, ψ_j is biproper, (λ_j, ψ_j) is coprime, i.e., u_j, $v_j \in \mathcal{R}$ exist satisfying $v_j\psi_j + u_j\lambda_j = 1$, $j = 1, \cdots, r$; λ_j divides λ_{j+1}, ψ_{j+1} divides ψ_j, $j = 1, \cdots, r-1$ (see [9], Smith–McMillan form). By (16), any RCF and LCF $P_{22} = N_{22}D_{22}^{-1} = \tilde{D}_{22}^{-1}\tilde{N}_{22}$ are given in (17) and (18) for some $\mathcal{R}$-unimodular M, $\tilde{M} \in \mathcal{M}(\mathcal{R})$; let $U_D := \operatorname{diag}[u_1 \cdots u_r]$, $V_D := \operatorname{diag}[v_1 \cdots v_r]$; then $(V_D\Psi + U_D\Lambda) = I_r$; U_2, V_2 in (9) are given by (19), where $\hat{A} \in \mathcal{R}^{n_{i2} \times n_{o2}}$

$$\begin{aligned} P_{22} &= L \operatorname{diag}[\Lambda, 0_{(n_{o2}-r)\times(n_{i2}-r)}] \operatorname{diag}[\Psi^{-1}, I_{(n_{i2}-r)}]R \\ &= L \operatorname{diag}[\Psi^{-1}, I_{(n_{o2}-r)}] \operatorname{diag}[\Lambda, 0]R \end{aligned} \tag{16}$$

$$\begin{aligned} (N_{22}, D_{22}) = (&L \operatorname{diag}[\Lambda, 0_{(n_{o2}-r)\times(n_{i2}-r)}]M, \\ &R^{-1}\operatorname{diag}[\Psi, I_{(n_{i2}-r)}]M) \end{aligned} \tag{17}$$

$$\begin{aligned} (\tilde{D}_{22}, \tilde{N}_{22}) = (&\tilde{M}\operatorname{diag}[\Psi, I_{(n_{o2}-r)}]L^{-1}, \\ &\tilde{M}\operatorname{diag}[\Lambda, 0_{(n_{o2}-r)\times(n_{i2}-r)}]R) \end{aligned} \tag{18}$$

$$\begin{aligned} U_2 &= M^{-1}\operatorname{diag}[U_D, 0_{(n_{i2}-r)\times(n_{o2}-r)}]L^{-1} + \hat{A}\tilde{D}_{22} \\ V_2 &= M^{-1}\operatorname{diag}[V_D, I_{(n_{i2}-r)}]R - \hat{A}\tilde{N}_{22}. \end{aligned} \tag{19}$$

Let $Q_{11} \in \mathcal{R}^{r \times r}$ be any upper-triangular matrix whose nondiagonal entries $q_{ij} \neq 0$ are constants, for $i, j = 1, \cdots, r$, $j > i$. For $j = 1, \cdots, r$, choose $q_{jj} \in \mathcal{R}$ as follows: Let $\mathcal{Z}_{12}$, $\mathcal{Z}_{21}$ be the sets of all real $\mathcal{U}$-zeros of P_{12} and of P_{21}, respectively; let $\mathcal{Z} := \mathcal{Z}_{12} \bigcup \mathcal{Z}_{21} = \{z_1, \cdots, z_\ell\}$. Let $\mathcal{Z}_j = \{z_{j1}, \cdots, z_{j\ell_j}\} \subset \mathcal{Z}$ be such that $u_j(z) \neq 0$ for $z \in \mathcal{Z}_j$. Define $q_{jj} \in \mathcal{R}$ as $q_{jj} = q_{jj}(\infty)\prod_{k=1}^{\ell_j}(s - z_{jk})(s+a)^{-1}$; $-a \in \mathbb{R}\backslash\mathcal{U}$, $q_{jj}(\infty) \in \mathbb{R}\backslash\{0\}$ is such that $(v_j - q_{jj}\lambda_j)(\infty) \neq 0$; this holds for all $q_{jj}(\infty)$ when $\lambda_j \in R_s$; when $\lambda_j \notin R_s$, take $q_{jj}(\infty) \neq v_j\lambda_j^{-1}(\infty)$. If $\mathcal{Z}_j$ is empty, then $q_{jj} = q_{jj}(\infty) \in \mathbb{R}$. With this $q_{jj} \in \mathcal{R}$, $(u_j + q_{jj}\psi_j)$ does not have zeros at any of the real $\mathcal{U}$-zeros of P_{12} or P_{21}. If $u_j = 0$, then ψ_j is a unit in $\mathcal{R}$ and if $u_1 = 0$, then $P_{22} \in \mathcal{M}(\mathcal{R})$.

1) Choose
$$Q_2 = -\hat{A} + M^{-1}\begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix}\tilde{M}^{-1}$$
$Q_{12} \in \mathcal{R}^{r \times (n_{o2}-r)}$, $Q_{22} \in \mathcal{R}^{(n_{o2}-r)\times(n_{o2}-r)}$ has no real blocking $\mathcal{U}$-zeros and no zeros at any real $\mathcal{U}$-zeros of P_{12} and P_{21}; obvious choices for Q_{22} are any $\mathcal{R}$-unimodular matrix or the identity $I_{n_{o2}-r}$. By construction, $\tilde{N}_{C2} = (U_2 + Q_2\tilde{D}_{22})$ has no real blocking $\mathcal{U}$-zeros and no $\mathcal{U}$-zeros coinciding with any real $\mathcal{U}$-zeros of either P_{12} or P_{21}. Since $\operatorname{rank} P_{12} = n_{i2} = n_{o2}$, $z_o \in \mathcal{U}$ is a $\mathcal{U}$-zero of $P_{12} = N_{12}D_{22}^{-1}$ if and only if $\operatorname{rank} N_{12}(z_o) < n_{o2}$. Similarly, $z_o \in \mathcal{U}$ is a $\mathcal{U}$-zero of $P_{21} = X_{21}Y_{21}^{-1}$ if and only if $\operatorname{rank} X_{21}(z_o) < n_{o2}$. By Lemma 2, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if $\det\ \tilde{D}_{22}$ has the same sign at all real blocking $\mathcal{U}$-zeros of $N_{12}\tilde{N}_{C2}X_{21}$, which are the real blocking $\mathcal{U}$-zeros of P_{12}, of P_{21} and possibly some of the real $\mathcal{U}$-zeros common to P_{12} and P_{21}. We prove that no other real blocking $\mathcal{U}$-zeros exist by contradiction: Suppose $z_o \in \mathbb{R} \cap \mathcal{U}$ is such that $N_{12}\tilde{N}_{C2}X_{21}(z_o) = 0$ but $N_{12}(z_o) \neq 0$, $X_{21}(z_o) \neq 0$, and z_o is not a common zero of P_{12} and P_{21}; since z_o may be a zero of one of P_{12} or P_{21}, there are two cases: 1) If $P_{12}(z_o) \neq 0$, $P_{21}(z_o) \neq 0$, then $\operatorname{rank} N_{12}(z_o) = n_{i2} = n_{o2}$, $\operatorname{rank} X_{21}(z_o) = n_{o2}$; hence, $N_{12}(z_o) \in \mathbb{R}^{n_{o1} \times n_{i2}}$ has a left-inverse $\hat{N}_{12}$ and $X_{21}(z_o) \in \mathbb{R}^{n_{o2} \times n_{i1}}$ has a right-inverse $\hat{X}_{21}$. Therefore, $N_{12}\tilde{N}_{C2}X_{21}(z_o) = 0$ implies $\tilde{N}_{C2}(z_o) = 0$, which is a contradiction; and 2) If z_o is a zero of either P_{12} or P_{21}, then $\operatorname{rank}\tilde{N}_{C2}(z_o) = n_{o2}$. Either $\hat{X}_{21} \in \mathbb{R}^{n_{i1} \times n_{o2}}$ exists such that $X_{21}(z_o)\hat{X}_{21} = I$ (if $P_{21}(z_o) \neq 0$) or $\hat{N}_{12} \in \mathbb{R}^{n_{i2} \times n_{o1}}$ exists such that $\hat{N}_{12}N_{12}(z_o) = I$ (if $P_{12}(z_o) \neq 0$). Therefore, $N_{12}\tilde{N}_{C2}X_{21}(z_o) = 0$ implies either $N_{12}(z_o) = 0$, or $X_{21}(z_o) = 0$; again we have a contradiction. Since P_{12} or P_{21} is strictly proper, the sign of $\det\ D_{22}$ is the same at all of these real blocking $\mathcal{U}$-zeros as the sign of $\det\ D_{22}(\infty)$; therefore $\hat{P} := P_{12}D_{22}\tilde{N}_{C2}P_{21}$ is strongly $\mathcal{R}$-stabilizable. By (16) and (19), $\tilde{D}_{C2}$ in (11) is biproper, i.e., C_2 is proper since $\det(V_D - Q_{11}\Lambda)(\infty) \neq 0$. Since $P_{12} \in \mathcal{M}(R_s)$ or $P_{21} \in \mathcal{M}(R_s)$, $Q_1 \in \mathcal{M}(\mathcal{R})$ satisfying (13) can be chosen strictly proper so that C_1 is proper.
2) If $\operatorname{rank} P_{22} = \operatorname{rank}(N_{22}D_{22}^{-1}) = n_{o2} = n_{i2}$, then $\operatorname{rank} N_{22} = n_{o2}$. By (10), $\det\ U_2$ and $\det\ N_{22}$ have the same sign at all real blocking $\mathcal{U}$-zeros of D_{22}. By (16), the real blocking $\mathcal{U}$-zeros of D_{22} are those of its smallest invariant factor ψ_{n_o}. Since the $\mathcal{U}$-zeros of P_{22} are those of $\det\ N_{22}$, the sign of $\det\ N_{22}$ is the same at all real $\mathcal{U}$-zeros of ψ_{n_o}. Therefore, $q \in \mathcal{R}$ exists such that $(\det\ U_2 + q\psi_{n_o})$ is a unit of $\mathcal{R}$ implies $Q_2 \in \mathcal{M}(\mathcal{R})$ exists such that $\tilde{N}_{C2}$ is $\mathcal{R}$-unimodular [9]. We show that since $\tilde{N}_{C2}$ is $\mathcal{R}$-unimodular, the only real blocking $\mathcal{U}$-zeros of $N_{12}\tilde{N}_{C2}X_{21}$ are the blocking $\mathcal{U}$-zeros of P_{12}, or P_{21} and possibly some of the real $\mathcal{U}$-zeros of P_{12} or of P_{21}: If $z_o \in \mathbb{R} \cap \mathcal{U}$ is such that $P_{12}(z_o) \neq 0$, $P_{21}(z_o) \neq 0$, then $\operatorname{rank} N_{12}(z_o) + \operatorname{rank} X_{21}(z_o) - n_{o2} > 0$. Therefore, $\operatorname{rank} N_{12}\tilde{N}_{C2}X_{21}(z_o) \geq \operatorname{rank} N_{12}(z_o) + \operatorname{rank} X_{21}(z_o) - n_{o2} > 0$ implies $N_{12}\tilde{N}_{C2}X_{21}(z_o) \neq 0$. Since the signs of $\det\ D_{22}$ and $\det\ D_{22}(\infty)$ are the same at all real $\mathcal{U}$-zeros of P_{12} and P_{21}, by Lemma 2, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable. Since $P_{22} \in \mathcal{M}(R_s)$, the existence of proper controllers follows from Theorem 2-2)-b).
3) First we show that $P_{12} = N_{12}D_{22}^{-1}$ has an $\mathcal{R}$-stable left inverse if and only if $N_{12} \in \mathcal{R}^{n_{o1} \times n_{i2}}$ has an $\mathcal{R}$-stable left inverse $N_{12}^I \in \mathcal{R}^{n_{i2} \times n_{o1}}$; if $P_{12}^I \in \mathcal{R}^{n_{i2} \times n_{o1}}$ exists such that $P_{12}^I P_{12} = I$, then $P_{12}^I N_{12} = D_{22}$. Since (N_{12}, D_{22}) is right-coprime, $(V_{12}P_{12}^I + U_{12})N_{12} = I$ implies $(V_{12}P_{12}^I + U_{12}) \in \mathcal{M}(\mathcal{R})$ is a left-inverse of N_{12}. Conversely, if $N_{12}^I \in$

$\mathcal{R}^{n_{i2} \times n_{o1}}$ exists such that $N_{12}^I N_{12} = I$, then $N_{12}^I P_{12} = D_{22}^{-1}$ implies $D_{22} N_{12}^I P_{12} = I$; hence, $D_{22} N_{12}^I \in \mathcal{M}(\mathcal{R})$ is a left-inverse of P_{12}. It can be shown similarly that $P_{21} = \tilde{D}_{22}^{-1} \tilde{N}_{21}$ has an $\mathcal{R}$-stable right-inverse if and only if $\tilde{N}_{21} \in \mathcal{R}^{n_{o2} \times n_{i1}}$ has a right-inverse $\tilde{N}_{21}^I \in \mathcal{R}^{n_{i1} \times n_{o2}}$. Construct $Q_{11} \in \mathcal{R}^{r \times r}$ with q_{jj} chosen as above; since $\operatorname{rank} P_{12}(s) = n_{i2}$ and $\operatorname{rank} P_{21}(s) = n_{o2}$ for all $s \in \mathcal{U}$, P_{12} and P_{21} have no $\mathcal{U}$-zeros. Therefore, Q_{11} is chosen so that the nondiagonal entries $q_{ij} \neq 0$ are constants, and $q_{jj} \in \mathcal{R}$ are such that $(v_j - q_{jj}\lambda_j)(\infty) \neq 0$. To guarantee that $\tilde{N}_{C2}$ has no real blocking $\mathcal{U}$-zeros, let

$$Q_2 = -\hat{A} + M^{-1} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tilde{M}^{-1} \in \mathcal{R}^{n_{i2} \times n_{o2}}$$

$Q_{12} \in \mathcal{R}^{r \times (n_{o2} - r)}$, $Q_{21} \in \mathcal{R}^{(n_{i2} - r) \times r}$, $Q_{22} \in \mathcal{R}^{(n_{o2} - r) \times (n_{o2} - r)}$ can be arbitrary if both $n_{o2} > 1$ and $n_{i2} > 1$; if $n_{i2} = 1$, let $Q_{12} \in \mathbb{R}^{1 \times (n_{o2} - 1)}$ be nonzero real; if $n_{o2} = 1$, let $Q_{21} \in \mathbb{R}^{(n_{i2} - 1) \times 1}$ be nonzero real. Let $Q_1 := \tilde{N}_{21}^I \hat{Q}_1 N_{12}^I \in \mathcal{M}(\mathcal{R})$. Since $\tilde{N}_{C2}$ has no real blocking $\mathcal{U}$-zeros, $\hat{Q}_1 \in \mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_{22} + \tilde{N}_{21} \tilde{N}_{21}^I \hat{Q}_1 N_{12}^I N_{12} \tilde{N}_{C2} = \tilde{D}_{22} + \hat{Q}_1 \tilde{N}_{C2}$ is $\mathcal{R}$-unimodular, i.e., (14) holds. Since $\tilde{D}_{C2}$ is biproper, C_2 is proper. There is $\hat{Q}_1 \in \mathcal{M}(\mathcal{R})$ such that $\tilde{D}_{22} + \hat{Q}_1 \tilde{N}_{C2}$ is $\mathcal{R}$-unimodular if and only if $\tilde{Q}_1 \in \mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_{22} + (s + a)^{-1} \tilde{Q}_1 \tilde{N}_{C2}$ is $\mathcal{R}$-unimodular; choosing $Q_1 = (s + a)^{-1} \tilde{Q}_1 \in \mathcal{M}(R_s)$ implies C_1 is proper.

4) Let C_S be any $\mathcal{R}$-stable $\mathcal{R}$-stabilizing controller for P_{22}. Without loss of generality, let the RCF $P_{22} = N_{22} D_{22}^{-1}$ satisfy $D_{22} + C_S N_{22} = I$; hence, $N_{22} D_{22} = (I - N_{22} C_S) N_{22}$. Then $U_2 = C_S + T(I - N_{22} C_S)$, $V_2 = I - T N_{22}$, $\tilde{U}_2 = C_S + D_{22} T$, $\tilde{V}_2 = I - N_{22} T$, $T \in \mathcal{M}(\mathcal{R})$ satisfy (10). By assumption, $P_{21} = \tilde{D}_{22}^{-1} \tilde{N}_{21}$ implies $\tilde{N}_{21}$ has a right-inverse $\tilde{N}_{21}^I \in \mathcal{M}(\mathcal{R})$. Also, $\tilde{L} P_{12} = \tilde{L} N_{12} D_{22}^{-1} = P_{22} = N_{22} D_{22}^{-1}$ implies $\tilde{L} N_{12} = N_{22}$. Let C_1, C_2 be given by (11) and (12), $Q_1 = \tilde{N}_{21}^I \tilde{L} \in \mathcal{R}^{n^{i1} \times n_{o1}}$, $Q_2 = -T$. Then (13) becomes $D_{22} + C_S N_{22} = I$. Since P_{11}, $P_{12} \in \mathcal{M}(R_s)$, $(N_{11} - N_{12} Q_2 \tilde{N}_{21}) \in \mathcal{M}(R_s)$ implies $\tilde{D}_{C1}$ is biproper, i.e., $C_1 \in \mathcal{M}(R_p)$. Since $C_2 = C_S \in \mathcal{M}(\mathcal{R})$, (C_1, C_2) is a reliable decentralized controller pair.

5) Let C_S be as in 4); let $D_{22} + C_S N_{22} = I$. By assumption, $P_{12} = N_{12} D_{22}^{-1}$ implies N_{12} has a left-inverse $N_{12}^I \in \mathcal{M}(\mathcal{R})$. Also, $P_{21} \tilde{R} = \tilde{D}_{22}^{-1} \tilde{N}_{21} \tilde{R} = P_{22} = \tilde{D}^{-1} \tilde{N}_{22}$ implies $\tilde{N}_{21} \tilde{R} = \tilde{N}_{22} = N_{22}$. Let C_1, C_2 be given by (11) and (12), $Q_1 = \tilde{R} N_{12}^I \in \mathcal{R}^{n_{i1} \times n_{o1}}$, $Q_2 = -T$. The conclusion follows as in 4).

6) Choose C_S as in 4). By assumption, $\tilde{L} P_{12} = P_{22}$ implies $\tilde{L} N_{12} = N_{22}$, and $P_{21} \tilde{R} = P_{22}$ implies $\tilde{N}_{21} \tilde{R} = \tilde{N}_{22} = N_{22}$. Let $Q_2 = -T$, $Q_1 = \tilde{R} \hat{Q}_1 \tilde{L}$, $\hat{Q}_1 = \sum_{m=2}^{k} r_m k^{-m} (C_S N_{22})^{m-2} C_S$; k is any integer such that $k > \|C_S N_{22}\|$ and r_m are the binomial coefficients. By (13) $D_{22} + C_S N_{22} \hat{Q}_1 N_{22} = I - C_S N_{22} + \sum_{m=2}^{k} r_m k^{-m} (C_S N_{22})^m = (I - k^{-1} C_S N_{22})^k$ is $\mathcal{R}$-unimodular. Then $C_1 \in \mathcal{M}(R_s)$ since $\hat{Q}_1, Q_1 \in \mathcal{M}(R_s)$. Since $C_2 = C_S$ is proper, (C_1, C_2) is a reliable decentralized controller pair. $\square$

References

[1] A. N. Gündeş and C. A. Desoer, *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators*, Lecture Notes in Control and Information Sciences, vol. 142. Berlin, Germany: Springer-Verlag, 1990.

[2] K. D. Minto and R. Ravi, “New results on the multi-controller scheme for the reliable control of linear plants,” in *Proc. American Control Conf.*, 1991, pp. 615–619.

[3] C. N. Nett, “Algebraic aspects of linear control system stability,” *IEEE Trans. Automat. Contr.*, vol. 31, pp. 941–949, 1986.

[4] K. A. Ünyelioğlu and A. B. Özgüler, “Reliable decentralized stabilization of feedforward and feedback interconnected systems,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1119–1132, 1992.

[5] D. D. Šiljak, “On reliability of control,” in *Proc. 17th IEEE Conf. Decision and Control*, 1978, pp. 687–694.

[6] ——, “Reliable control using multiple control systems,” *Int. J. Contr.*, vol. 31, pp. 303–329, 1980.

[7] X. L. Tan, D. D. Šiljak, and M. Ikeda, “Reliable stabilization via factorization methods,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1786–1791, 1992.

[8] M. Vidyasagar and N. Viswanadham, “Algebraic design techniques for reliable stabilization,” *IEEE Trans. Automat. Contr.*, vol. 27, pp. 1085–1095, 1982.

[9] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: M.I.T. Press, 1985.

Useful Nonlinearities and Global Stabilization of Bifurcations in a Model of Jet Engine Surge and Stall

Miroslav Krstić, Dan Fontaine, Petar V. Kokotović, and James D. Paduano

***Abstract*— Compressor stall and surge are complex nonlinear instabilities that reduce the performance and can cause failure of aircraft engines. We design a feedback controller that globally stabilizes a broad range of possible equilibria in a nonlinear compressor model. With a novel backstepping design we retain the system's useful nonlinearities which would be cancelled in a feedback linearizing design. The design control law is simple and, moreover, it is optimal with respect to a meaningful nonquadratic cost functional. As in a previous bifurcation-theoretic design, we change the character of the bifurcation at the stall inception point from subcritical to supercritical. However, since we do not approach bifurcation control using a normal form but using Lyapunov tools, our controller achieves not only local but also global stability. The controller requires minimal modeling information (bounds on the slope of the stall characteristic and the B-parameter) and simpler sensing (rotating stall is stabilized without measuring its amplitude).**

***Index Terms*— Axial flow compressors, backstepping, bifurcation control, jet engines, rotating stall, surge.**

I. Introduction

In control engineering the importance of qualitative low-order nonlinear models is twofold. First, they can capture the dominant dynamic phenomena; second, they are testbeds which help refine new nonlinear design methods. One such model, the Moore–Greitzer

Manuscript received March 14, 1997. This work was supported in part by the Air Force Office of Scientific Research under Grants F496209610223 and 95-1-0409 and in part by the National Science Foundation under Grant ECS-9624386.

M. Krstić is with the Department of AMES, University of California at San Diego, La Jolla, CA 92093-0411 USA (e-mail: mkrstic@ames.ucsd.edu).

D. Fontaine and P. V. Kokotović are with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 USA.

J. D. Paduano is with the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

Publisher Item Identifier S 0018-9286(98)08423-2.