Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

Simultaneously stabilizing controller design for a class of linear plants[1]

A. N. Gündeş and M. G. Kabuli
Electrical and Computer Engineering, University of California, Davis, CA 95616
gundes@ece.ucdavis.edu kabuli@ece.ucdavis.edu

Abstract

All simultaneously stabilizing controllers are characterized for a class of linear, time-invariant, multi-input multi-output plants. These plants all have poles at zero but they have no other unstable poles.

1. Introduction

The parametrization of all stabilizing controllers for linear time-invariant (LTI), multi-input multi-output (MIMO) plants leads to explicit necessary and sufficient conditions for existence of controllers that simultaneously stabilize two given plants [4]. However, there are no known necessary and sufficient conditions for existence of simultaneously stabilizing controllers for a class of three or more arbitrary plants [1]. Instead of conditions applicable to a completely general class of (three or more) plants, it may be possible to conclusively answer the question of simultaneous stabilizability for some special classes [2], [5], [3]. As a special case, we consider the class $\mathcal{P} = \{P_o, P_1, \ldots, P_n\}$ of LTI MIMO plants that have no other poles in the region of instability except at $s = 0$; furthermore, for $j = 0, \ldots, n$, $(s^m P_j)$ have the same DC-gain and $(s^m P_j)(0)$ is full row-rank (see Assumption 2.1). These plants are simultaneously stabilizable; all simultaneously stabilizing controllers are characterized in the main result (Proposition 2.3). The simultaneously stabilizing controllers are defined in terms of a stable controller-parameter that satisfies a unimodularity condition; the 'central' controller for which this controller-parameter is zero is actually stable.

Due to the algebraic framework described in the following notation, the results apply to continuous-time as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at $s = 0$ should be interpreted at $z = 1$. **Notation:** Let $\mathcal{U}$ be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, rational functions (with real coefficients), proper and strictly-proper rational functions, proper rational functions with no poles in $\mathcal{U}$ are denoted by $\mathbb{R}$, R, $\mathrm{R_p}$, $\mathrm{R_s}$, $\mathcal{R}$, respectively. The set of matrices with entries in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called stable iff $M \in \mathcal{M}(\mathcal{R})$ ($M \in \mathcal{R}^{i\times j}$ is used to indicate the order explicitly); a stable M is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. For $M \in \mathcal{M}(\mathcal{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s\in\partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$.

2. Main Results

Consider the standard LTI, MIMO, well-posed feedback system $\mathcal{S}(P_j, C)$ in Figure 1: $P_j \in \mathrm{R_p}^{n_y\times n_u}$ and $C \in \mathrm{R_p}^{n_u\times n_y}$ represent the transfer-functions of the plant and the controller. It is assumed that P_j and C have no hidden modes corresponding to eigenvalues in the region of instability $\mathcal{U}$.

2.1 Assumption: $P_j \in \mathrm{R_p}^{n_y\times n_u}$ belongs to the class $\mathcal{P} := \{P_o, P_1, \ldots, P_n\}$; for $j \in \{0, \ldots, n\}$, *i)* $s^m P_j$ have no poles in $\mathcal{U}$; *ii)* $\mathrm{rank}(s^m P_j(0)) = \mathrm{rank}(s^m P_o(0)) = n_y \le n_u$; *iii)* $(s^m P_j)(0) = (s^m P_o)(0)$. □

Assumption 2.1(i) implies that the only $\mathcal{U}$-poles of P_j are at $s = 0$; (ii) implies that each P_j in the class $\mathcal{P}$ has at least one entry with exactly m poles at $s = 0$; furthermore, P_j has no (transmission) zeros at $s = 0$.

2.2 Definition: The system $\mathcal{S}(P_j, C)$ is said to be stable iff the transfer-function H from (u, u_P) to (y, y_C) is stable, i.e., $H \in \mathcal{M}(\mathcal{R})$. The controller C is said to be a stabilizing controller for the plant P_j (or C stabilizes P_j) iff $C \in \mathcal{M}(\mathrm{R_p})$ and $\mathcal{S}(P_j, C)$ is stable. The stabilizing controller C is said to simultaneously stabilize all $P_j \in \mathcal{P}$ iff $\mathcal{S}(P_j, C)$ is stable for all $j \in \{0, \ldots, n\}$. □

We give a characterization of all controllers that simultaneously stabilize all $P_j \in \mathcal{P}$ in Proposition 2.3; this characterization is based on all stabilizing controllers for one of the plants, P_o, called the nominal plant.

Proposition 2.3: Let $P_j \in \mathrm{R_p}^{n_y\times n_u}$ belong to the class $\mathcal{P} := \{P_o, P_1, \ldots, P_n\}$ satisfying Assumption 2.1. For $i = 1, \ldots, m$, let $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$. Define $N_j := \left(\prod_{i=1}^{m}(s + \alpha_i) \right)^{-1} s^m P_j \in \mathcal{R}^{n_y\times n_u}$. Let $N_o(0)^I$ be any right-inverse of $N_o(0)\mathbb{R}^{n_y\times n_u}$ (by Assumption 2.1, $N_j(0)N_o(0)^I = I$). Let $k_1 \in \mathbb{R}$ and for $\nu = 2, \ldots, m$, let $k_\nu \in \mathbb{R}$ satisfy

$$0 < k_1 < \min_{j\in\{0,\ldots,n\}} \left\| s^{-1}(I - N_j N_o(0)^I) \right\|^{-1} \quad (1)$$

$$0 < k_\nu < \min_{j\in\{0,\ldots,n\}} \| s^{-1}(I + N_j N_o(0)^I \sum_{i=1}^{\nu-1} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell)^{-1} \times (I + N_j N_o(0)^I \sum_{i=1}^{\nu-2} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell) \|^{-1}. \quad (2)$$

[1] This work was supported by the NSF Grant ECS-9257932.

Define $\hat{K} := N_o(0)^I \sum_{i=1}^{m} \frac{1}{s^i} \prod_{\ell=1}^{i} k_\ell \in \mathrm{R}^{n_u \times n_y}$. The controller C simultaneously stabilizes all $P_j \in \mathcal{P}$ if and only if

$$C = \frac{s^m}{\prod_{i=1}^{m}(s+\alpha_i)}(I - QN_o)^{-1}\left(Q + \hat{K}\right), \qquad (3)$$

where $Q \in \mathcal{R}^{n_u \times n_y}$ is such that, for $j \in \{1, \ldots, n\}$,

$$D_j := I + \; Q\,(N_j - N_o)\left(I + \hat{K}N_j\right)^{-1} \qquad (4)$$

is unimodular, and $(I - QN_o)$ is biproper (holds for all $Q \in \mathcal{M}(\mathcal{R})$ when $P_o \in \mathcal{M}(\mathrm{R_s})$). □

Comments: a) A block-diagram of $\mathcal{S}(P_j, C)$, where C is given by (3), is shown in Figure 2. The stable controller-parameter Q in the simultaneously stabilizing controller characterization (3) must be such that D_j in (4) is unimodular. This condition is obviously satisfied for $Q = 0$. The simultaneously stabilizing controller corresponding to $Q = 0$ is denoted by C_{Q0}; note that C_{Q0} is stable and strictly-proper. A sufficient condition to make D_j in (4) unimodular is to choose $Q \in \mathcal{M}(\mathcal{R})$ 'sufficiently small', i.e., $\|Q\| < \min_{j\in\{0,\ldots,n\}} \|(N_j - N_o)(I + \hat{K}N_j)^{-1}\|^{-1}$. In addition, choosing $Q \in \mathcal{M}(\mathcal{R})$ strictly-proper is sufficient to make $(I - QN_o)$ biproper; the simultaneously-stabilizing controller is strictly-proper if and only if $Q \in \mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\mathrm{R_s})$. **b)** For $j \in \{0, \ldots, n\}$, the (input-output) transfer-function $H_{yj} = PC(I + PC)^{-1}$ from u to y and the (input-error) transfer-function $H_{ej} = I - H_{yj}$ from u to e are achievable using the controllers in (3) if and only if $H_{yj} = N_j(I + \hat{K}N_j + Q(N_j - N_o))^{-1}(Q + \hat{K})$ $= N_j(I + \hat{K}N_j)^{-1}D_j^{-1}(Q + \hat{K})$, $H_{ej} = (I + N_j(I - QN_o)^{-1}(Q + \hat{K}))^{-1}$, where $Q \in \mathcal{R}^{n_u \times n_y}$ is such that D_j in (4) is unimodular and $(I - QN_o)$ is biproper. The expressions for these transfer-functions are simplified for the nominal plant P_o as $H_{yo} = (I + N_o\hat{K})^{-1}N_o(Q + \hat{K})$, $H_{eo} = (I + N_o\hat{K})^{-1}(I - N_oQ)$. Note that $H_{ej}(0) = 0$ due to the poles of P_j at zero. This guarantees that step inputs (and polynomial inputs of order up to $m - 1$) at each channel of u are tracked asymptotically with zero steady-state error due to the m plant poles at zero. Additional design goals may be achievable by appropriately selecting the controller-parameter Q in (3). □

We give two design examples based on Proposition 2.3. The class $\mathcal{P}$ in Example 1 has in fact infinitely many plants that all have one pole at $s = 0$.

Example 1: Consider $\mathcal{P} := \{\, s^{-1}(s+1)^{-1}(a_j s^2 - 2s - 4.5)\,,\, s^{-1}(s+2)^{-1}(b_j s - 9) \mid -3 \le a_j \le 2\,,\, -4 \le b_j \le 2\,\}$. With $m = 1$, $P_j \in \mathcal{P}$ satisfy Assumption 2.1. Choosing an arbitrary member of $\mathcal{P}$ as the nominal plant, let $P_o = -9s^{-1}(s+2)^{-1}$. Choosing $\alpha_1 = 10$, $N_o = -9(s+2)^{-1}(s+10)^{-1}$, $N_j \in \{\,(s+1)^{-1}(s+10)^{-1}(a_j s^2 - 2s - 4.5)\,,\,(s+2)^{-1}(s+10)^{-1}(b_j s - 9) \mid -3 \le a_j \le 2\,,\, -4 \le b_j \le 2\,\}$. Choose $k_1 \in \mathbb{R}$ satisfying (1) as $k_1 = 1.2 < \min_j \|s^{-1}(1 + 9^{-1}20N_j)\|^{-1}$. By (3), the simultaneously stabilizing controllers are $C = s(s+10)^{-1}(1 + 9(s+2)^{-1}(s+10)^{-1}Q\,)^{-1}(Q - 8(3s)^{-1})$, where $Q \in \mathcal{R}$ is such that D_j in (4) is unimodular; $(1 - QN_o)$ is biproper for all $Q \in \mathcal{R}$ since P_o is strictly-proper. The controller $C_{Q0} \in \mathcal{M}(\mathcal{R})$ corresponding to $Q = 0$ is $C_{Q0} = -8(3(s+10))^{-1}$. □

Example 2: Consider $\mathcal{P} := \{\, P_o = -60s^{-2}\,,\, P_1 = s^{-2}(s+7)^{-1}(s+8)^{-1}(5s - 112)(s+5)(s+6)\,,\, P_2 = 3s^{-2}(3s+10)^{-1}(2s+5)^{-1}(2s^2 - 49s - 300)(s+5)(s+6)\,\}$. With $m = 2$, $P_o, P_1, P_2 \in \mathcal{P}$ satisfy Assumption 2.1. Choosing $\alpha_1 = 5$, $\alpha_2 = 6$, $N_o = -60(s+5)^{-1}(s+6)^{-1}$, $N_1 = (s+7)^{-1}(s+8)^{-1}(5s - 112)$, $N_2 = (3(3s+10))^{-1}(2s+5)^{-1}(2s^2 - 49s - 300)$. Choose $k_1 \in \mathbb{R}$ satisfying (1) and $k_2 \in \mathbb{R}$ satisfying (2) as $k_1 = 1.5 < \min_j \|s^{-1}(1 + 0.5N_j)\|^{-1}$, $k_2 = 1 < \min_{j\in\{0,1,2\}} \|s^{-1}(1 - 0.5k_1 s^{-1}N_j\,)^{-1}\|^{-1}$. By (3), the simultaneously stabilizing controllers are $C = (s+5)^{-1}(s+6)^{-1}s^2(1 + 60Q(s+5)^{-1}(s+6)^{-1})^{-1}(Q - 3(4s^2)^{-1}(s+1)\,)$, where $Q \in \mathcal{R}$ is such that D_j in (4) is unimodular; $(1 - QN_o)$ is biproper for all $Q \in \mathcal{R}$ since P_o is strictly-proper. The controller $C_{Q0} \in \mathcal{M}(\mathcal{R})$ corresponding to $Q = 0$ is $C_{Q0} = -3(4(s+5)(s+6))^{-1}(s+1)$.

References

[1] V. Blondel, *Simultaneous Stabilization of Linear Systems*, Lecture Notes in Control and Information Sciences, vol. 191, Springer, Berlin, 1994.

[2] V. Blondel, G. Campion, M. Gevers, "A sufficient condition for simultaneous stabilization," *IEEE Trans. Automatic Control*, vol. 38, no. 8, pp. 1264-1266, 1993.

[3] A. N. Gündeş, "Simultaneous stabilization of multiplicatively perturbed plants," *Systems and Control Letters*, vol. 30, pp. 149-158, 1997.

[4] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, M.I.T. Press, 1985.

[5] K. Wei, "Simultaneous stabilization of single-input single-output discrete time systems," *IEEE Trans. Automatic Control*, vol. 38, no.3, pp. 446-450, 1993.

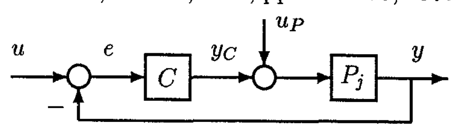

Figure 1: The system $\mathcal{S}(P_j, C)$

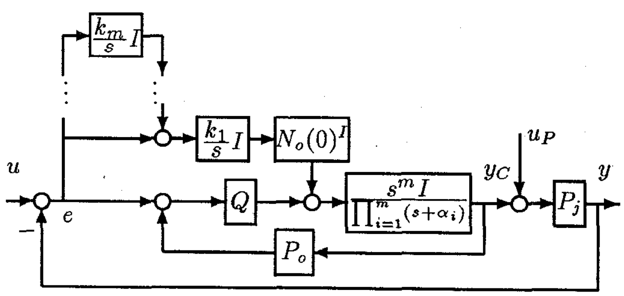

Figure 2: The stable system $\mathcal{S}(P_j, C)$, where $P_j \in \mathcal{P}$