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Pole placement using constant output feedback[1]

M. G. Kabuli, A. N. Gündeş, N. At, M. Koca
Electrical and Computer Engineering, University of California, Davis, CA 95616
Email: kabuli@ece.ucdavis.edu

Abstract
Pole placement by constant output feedback for single-input single-output, linear, time-invariant systems is considered. Root-loci intercepts with domain boundary for a general class of desirable closed-loop pole locations are computed using a single generalized-eigenvalue problem based on state-space representations. Suitable output feedback gain intervals can be determined without graphics.

1 Introduction

Stabilization of finite-dimensional, single-input single-output, linear, time-invariant systems with constant output feedback is a rudimentary control problem that is covered extensively in all introductory control textbooks. The Nyquist stability criterion and root-locus methods can be used to completely characterize the set of such constant output feedback gains. In fact, these standard results can also be used for closed-loop pole placement in desirable domains other than the ones that can be easily obtained by translated half-planes, unit-circles, set complements, and set intersections. In the root-locus method, root-loci are plotted by solving a sufficiently long sequence of eigenvalue problems to capture the curvatures. It is then a simple graphical check to superimpose the loci and the desired domain of closed-loop pole locations to characterize the solution. Determining boundary intercept values require zooming in on the loci and hence solving even more eigenvalue problems. For standard domains such as the closed-right-half-plane, polynomial methods like the Routh-Hurwitz criterion can be used to obtain a long cumbersome array of polynomial inequality constraints on the gain. By using the Hermite-Biehler Theorem [3], one can describe these intercepts as roots of pre-determined polynomials independent of the gains, all of which are based on a polynomial factorization of the plant. In the case of the well-known Nyquist stability criterion, the problem is reduced to winding numbers of a directional graph of the plant along the directional boundary of the domain. Once again, graphics provides this crucial visual feedback. In the root-locus method, the loci are independent of the domains selected, hence a single plotting area is sufficient to solve the problem. In the Nyquist plot case, however, for each boundary plot, the associated Nyquist plot needs to be updated and the number of winding numbers have to be computed. In other words, one now requires *two* plotting areas, where typically a large number of points along the boundary is selected to sufficiently capture the curvatures in each plotting area. Both of these methods, with their current standard implementations, completely rely on graphics, which can be easily discarded by focusing on the crux of the problem: determine the boundary crossings using polynomial factors (see e.g., [2], [5]),or equivalently, determine the real-crossings on the Nyquist plot. As stated in [4], the only important features of the Nyquist plot are the points where it crosses the real axis, and the signs of the crossings.

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2 Preliminaries

2.1 Notation: A finite-dimensional, linear, time-invariant, single-input single-output plant is denoted by p, where a minimal state-space description (A, b, c, d) is used to evaluate $p(s) = d + c(sI - A)^{-1}b$, for $s \in \mathbb{C}$. The state-space and transfer function descriptions of the plant p are used interchangeably. Let $\mathcal{U} \subset \mathbb{C}$ denote a simply connected, closed domain with nonempty interior, where $\mathcal{U}$ is symmetric about the real-axis; $\mathcal{U}$ corresponds to the undesired pole locations. The desired pole locations $\mathbb{C} \setminus \mathcal{U}$ is denoted by the open set $\mathcal{D}$. Let $\partial\mathcal{U}$ denote the directional boundary contour of $\mathcal{U}$, where the direction is adopted according to the right-hand rule, i.e., the interior of $\mathcal{U}$ remains to the right when traversing along the directional contour $\partial\mathcal{U}$. The plant p is assumed to have no hidden modes in $\mathcal{U}$ and no poles on $\partial\mathcal{U}$. The number of poles of p in $\mathcal{U}$ is denoted by $\Pi_{\mathcal{U}}$. For a given directional contour $\partial\mathcal{U}$, let $\partial\mathcal{U}_+ \subset \partial\mathcal{U}$ be the unique directional half-boundary contour that satisfies the following: the directional half-contour $\partial\mathcal{U}_+$ starts on the real-axis at a point in $\partial\mathcal{U} \cap \mathbb{R}$, continues along the direction of $\partial\mathcal{U}$ and traverses that entire half of $\partial\mathcal{U}$. Let $s_u(\cdot)$ be a directional real-parametric representation of the directional half-boundary contour $\partial\mathcal{U}_+ = \{ s_u(\lambda) \mid \lambda \in \mathbb{R}_+ \}$, where $s_u : \mathbb{R}_+ \to \partial\mathcal{U}_+$ is one-to-one and onto, $s_u(0)$ is real, and traversing along the directional half-contour $\partial\mathcal{U}_+$ is equivalent to monotonically increasing $\lambda \in \mathbb{R}_+$. Let $\Lambda(p, \partial\mathcal{U}_+) = \{ \lambda \in \mathbb{R}_+ \mid \mathrm{Imag}\,(p(s_u(\lambda))) = 0 \}$.

2.2 Pole Placement: The plant $p(s) = d + c(sI - A)^{-1}b$ is said to be $\mathcal{U}$-stable iff p has no poles in $\mathcal{U}$, i.e., A has no eigenvalues in $\mathcal{U}$. A constant output feedback $k \in \mathbb{R}$ is said to $\mathcal{U}$-stabilize p iff $(1 + kp)^{-1}$ is $\mathcal{U}$-stable, i.e., $\sigma(A - \frac{k}{1+kd}bc) \subset \mathcal{D}$, where $\sigma(\cdot)$ denotes the spectrum. Let the set of desirable constant output feedback gains be denoted by

$\mathcal{K}(p,\mathcal{U}) = \left\{ k \in \mathbb{R} \ \mid \ \sigma(A - \frac{k}{1+kd}bc) \subset \mathcal{D} \right\}$; $\mathcal{K}(p,\mathcal{U})$ can be completely determined by the contour integral $\frac{1}{2\pi j}\int_{\partial\mathcal{U}} \frac{kp'(s)}{1+kp(s)}ds$, which is equal to the number of clockwise encirclements of the point $-1/k$ by the graph $p(\partial\mathcal{U})$. By the residue theorem (see e.g. [1]) , this contour integral is also equal to the number of $\mathcal{U}$-zeros minus the number of $\mathcal{U}$-poles of $(1 + kp)$. This widely-used application of the residue theorem has the simple graphical interpretation known as the Nyquist stability criterion and requires *only* the graph of $p(s_u(\cdot))$ for $\lambda \in \mathbb{R}_+$ (due to the symmetry about the real-axis) to determine the winding numbers. Two crucial items in this graphical interpretation are: **i)** the characterization of the set $\Lambda(p, \partial\mathcal{U}_+)$, and **ii)** the winding numbers. Both of these items can be easily computed without any graphical means for the case of polynomial boundaries.

3 Main Results

3.1 Proposition (*Undesirable domain with polynomial boundary*)**:** Let the directional half-contour of a simply connected, closed domain $\mathcal{U}$ with nonempty interior be given by $\partial\mathcal{U}_+ = \{s_u(\lambda) \mid \lambda \in \mathbb{R}_+\}$, where $s_u(\lambda) = \sum_{m=0}^{M} s_m\lambda^m = f_R(\lambda)+jf_I(\lambda)$ for fixed real polynomials f_R and f_I, with $f_I(0) = 0$, and $s_M \neq 0$. Let p have no poles on $\partial\mathcal{U}_+$. Let $\Lambda(p, \partial\mathcal{U}_+) = \{ 0, \infty \} \cup \Lambda_+$ be the disjoint partitioning of $\Lambda(p, \partial\mathcal{U}_+)$. The members of Λ_+ are the nonzero finite positive real eigenvalues (including multiplicities) of the generalized eigenvalue problem

$$\det \begin{bmatrix} \lambda I & -I & 0 & \dots & 0 & 0 \\ 0 & \lambda I & -I & \dots & 0 & 0 \\ & & \ddots & & \vdots & \\ 0 & 0 & \dots & \lambda I & -I & 0 \\ A_0 & A_1 & \dots & A_{(M-2)} & \lambda A_M + A_{(M-1)} & \begin{bmatrix} -b \\ 0 \end{bmatrix} \\ 0 & 0 & \dots & 0 & [0 \ \ c] & 0 \end{bmatrix}$$

$= 0$, where $\sum_{m=0}^{M} A_m\lambda^m = \begin{bmatrix} f_R(\lambda)I - A & -f_I(\lambda)I \\ f_I(\lambda)I & f_R(\lambda)I - A \end{bmatrix}$. □

3.2 Corollary (*Undesirable domain with affine boundary*)**:** Consider the piecewise-linear boundary of a translated wedge domain described by $s_u(\lambda) = \alpha_1 + \lambda(\alpha_2 + j\alpha_3)$, where $\alpha_1, \alpha_2, \alpha_3$ are fixed reals with $\alpha_3 \neq 0$. The set of intercepts Λ_+ are obtained by the generalized eigenvalue problem $\det \begin{bmatrix} \lambda\alpha_2 I + (\alpha_1 I - A) & -\lambda\alpha_3 I & -b \\ \lambda\alpha_3 I & \lambda\alpha_2 I + (\alpha_1 I - A) & 0 \\ 0 & c & 0 \end{bmatrix} = 0$.

3.3 Corollary (*Undesirable domain: translated half-plane*)**:** Consider the translated half-plane described by $s_u(\lambda) = \alpha_1 + \lambda(j\alpha_3)$, where α_1 and α_3 are fixed reals with $\alpha_3 \neq 0$. The intercepts $\lambda \in \Lambda_+$ are obtained by $\lambda = \sqrt{\eta}/|\alpha_3|$, where the positive square-root is taken over positive nonzero finite generalized eigenvalues η of $\det \begin{bmatrix} \eta I + (\alpha_1 I - A)^2 & -b \\ c & 0 \end{bmatrix} = 0$.

3.4 Algorithm: The following steps determine the number of $\mathcal{U}$-poles of $(1+kp)^{-1}$ over the disjoint open intervals of suitable k values for the specific class of boundaries in Proposition 3.1 . The pole-placement problem, i.e., deriving $\mathcal{K}(p,\mathcal{U})$, is a special case of returning disjoint open intervals that have zero $\mathcal{U}$-pole counts.

[Step 0] Given minimal (A, b, c, d) and the directional half-contour $\partial\mathcal{U}_+$.

[Step 1] Determine Λ_+ with $N \geq 0$ distinct values (see Proposition 3.1). Evaluate $p(\Lambda(p, \partial\mathcal{U}_+))$. $N + 1$ evaluations ($p(\infty) = d$), determine all real-axis crossings.

[Step 2] Determine the counter-clockwise winding number in each consecutive open interval determined by the points in $p(\Lambda(p, \partial\mathcal{U}_+))$. The simplest approach requires at most one more p evaluation between each consecutive point in Λ_+ . Hence, at most $2(N + 1)$, evaluations of p will suffice to compute the winding numbers. A more refined approach is to use the type of multiplicities (odd or even) in Λ_+ (not required at 0 and ∞) and the fact that the graph is a continuous-curve to weave the directions between the real intercepts.

[Step 3] Determine $\Pi_{\mathcal{U}}$. Express the winding numbers obtained over the $-\frac{1}{k}$ valued intercepts in terms k's.

[Step 4] Return open intervals of k values and the associated number of $\mathcal{U}$-eigenvalues of $(A - \frac{k}{1+kd}bc)$, i.e., the difference between the winding number and $\Pi_{\mathcal{U}}$.

4 Concluding Remarks

A simple algorithm for deriving the set $\mathcal{K}(p,\mathcal{U})$ for polynomial parametrized boundaries is provided. The overall computational effort is one generalized eigenvalue problem (to determine all of the real-axis intercepts), one eigenvalue problem (to determine $\Pi_{\mathcal{U}}$) and at most $2(N + 1)$ evaluations of p ,where N is the number of distinct members of Λ_+. No graphics is required. Since contour direction reversal is equivalent to complementation of the original $\mathcal{D}$, utilizing this polynomial class of boundaries together with the widely used conformal mappings (e.g., half-plane translations, linear fractional transformations), one can easily generate intersections that result in a wider class of domains and easily solve problems of the type $\bigcap_{\ell=1}^{L}\bigcap_{m=1}^{M}\mathcal{K}(p_\ell,\mathcal{U}_m)$ for multiple plants for a union of undesired domains.

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