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Stabilizing controller design with integral action [1]

A. N. Gündeş and M. G. Kabuli
Electrical and Computer Engineering, University of California, Davis, CA 95616
gundes@ece.ucdavis.edu kabuli@ece.ucdavis.edu

Abstract

A parametrization of stabilizing controllers with type-m integral action is obtained in the standard linear time-invariant, multi-input multi-output unity-feedback system.

1 Introduction

We consider the design of stabilizing controllers with integral action in the standard linear time-invariant (LTI), multi-input multi-output (MIMO) unity-feedback system. We extend the parametrization of controllers with (type-1) integral action given in [1] to a similar simple parametrization for type-m integral action for any positive integer m. We also consider decoupling with integral action and parametrize controllers with integral action that also achieve diagonal input-output transfer-function matrices.

Due to the algebraic framework, the results apply to continuous-time and discrete-time systems; for discrete-time systems, all evaluations and poles at $s = 0$ would be interpreted at $z = 1$. Notation: Let $\mathcal{U}$ be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, of rational functions, of proper and strictly-proper rational functions, of proper rational functions that have no poles in $\mathcal{U}$ with real coefficients are denoted by $\mathbb{R}$, R, $\mathrm{R_p}$, $\mathrm{R_s}$, $\mathcal{R}$, respectively. The set of matrices with entries in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; $M \in \mathcal{M}(\mathcal{R})$ is called stable; $M \in \mathcal{M}(\mathcal{R})$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. A right-coprime-factorization (RCF), a left-coprime-factorization (LCF) of $P \in \mathcal{M}(\mathrm{R_p})$ are denoted by $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$; $N, D, \widetilde{N}, \widetilde{D} \in \mathcal{M}(\mathcal{R})$, $D, \widetilde{D}$ are biproper. Let $\mathrm{rank}P = r$; $s_o \in \mathcal{U}$ is a zero (blocking-zero) of P iff $\mathrm{rank}N(s_o) < r$ ($P(s_o) = 0 = N(s_o)$). For $M \in \mathcal{M}(\mathcal{R})$, the norm $\| \cdot \|$ is defined as $\| M \| = \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$; $\bar{\sigma}$ denotes the maximum singular value, $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$.

2 Main Results

Consider the LTI, MIMO system $\mathcal{S}(P, C)$ in Fig. 1: $P \in \mathrm{R_p}^{n_y \times n_u}$ and $C \in \mathrm{R_p}^{n_u \times n_y}$ represent the transfer-functions of the plant and the controller. It is assumed that $\mathcal{S}(P, C)$ is a well-posed system, and P and C have no hidden modes corresponding to eigenvalues in $\mathcal{U}$. Let H_{yu} (H_{eu}) denote the input-output (input-error) transfer-function from u to y (u to e, respectively). The system $\mathcal{S}(P, C)$ is said to be stable iff $H \in \mathcal{M}(\mathcal{R})$, where H is the transfer-function from (u, u_P) to (y, y_C). The stable system $\mathcal{S}(P, C)$ is said to have integral action in each output channel iff $H_{eu}(s)$ has blocking-zeros at $s = 0$ [3]. If H_{eu} has (at least) m blocking-zeros at zero, i.e., $(s^{-(m-1)}H_{eu})(0) = 0$, then $\mathcal{S}(P, C)$ is said to have type-m integral action, where $m \geq 1$ is an integer. The controller C is said to be a stabilizing controller for P (or C is said to stabilize P) iff $C \in \mathcal{M}(\mathrm{R_p})$ and $\mathcal{S}(P, C)$ is stable; C is said to be a ***stabilizing controller with integral action*** iff C stabilizes P and $D_C(s)$ has blocking-zeros at $s = 0$; C is said to be a ***stabilizing controller with type-m integral action*** iff C stabilizes P and $D_C(s)$ has (at least) m blocking-zeros at $s = 0$, where $C = N_C D_C^{-1}$ is any RCF of C, $D_C \in \mathcal{R}^{n_y \times n_y}$. Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N} \in \mathrm{R_p}^{n_y \times n_u}$ be any RCF, LCF. All stabilizing controllers for P are parametrized as $C = (\widetilde{U} + DQ)(\widetilde{V} - NQ)^{-1} = (V - Q\widetilde{N})^{-1}(U + Q\widetilde{D})$; $Q \in \mathcal{R}^{n_u \times n_y}$ satisfies $(\widetilde{V} - NQ)$ biproper (holds for all $Q \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$); $U, V, \widetilde{U}, \widetilde{V} \in \mathcal{M}(\mathcal{R})$ such that $VD + UN = I_{n_u}$, $\widetilde{D}\widetilde{V} + \widetilde{N}\widetilde{U} = I_{n_y}$, $V\widetilde{U} = U\widetilde{V}$ [4]. For any stabilizing controller, $H_{eu} = (I_{n_y} + PC)^{-1} = I_{n_y} - N(U + Q\widetilde{D}) = (\widetilde{V} - NQ)\widetilde{D}$. If $\mathcal{S}(P, C)$ is stable, $H_{eu}(0) = I_{n_y} - PC(I_{n_y} + PC)^{-1}(0) = I_{n_y} - N(0)(U + Q\widetilde{D})(0) = 0$ only if $\mathrm{rank}N(0) = n_y \leq n_u$. Therefore, a necessary condition for integral action in the stable $\mathcal{S}(P, C)$ is $\mathrm{rank}P = n_y \leq n_u$ and P has no zeros at $s = 0$.

Since any RCF of a stabilizing C is $(N_C, D_C) = ((\widetilde{U} + DQ)R, (\widetilde{V} - NQ)R)$ for some unimodular $R \in \mathcal{M}(\mathcal{R})$, $D_C(0) = 0$ is equivalent to $(\widetilde{V} - NQ)(0) = 0$. Therefore, if C is a stabilizing controller with integral action, then $H_{eu}(0) = (\widetilde{V} - NQ)(0)\widetilde{D}(0) = 0$; hence, the stable $\mathcal{S}(P, C)$ has integral action in each output channel based on the definitions above. Designing the stabilizing controllers so that $D_C(0) = 0$ is sufficient (but not necessary) for the stable $\mathcal{S}(P, C)$ to have integral action. When P has no poles at $s = 0$, in particular when $P \in \mathcal{M}(\mathcal{R})$, $H_{eu}(0) = 0$ if and only if $D_C(0) = 0$, i.e., the stable $\mathcal{S}(P, C)$ has integral action if and only if C is a stabilizing controller with integral action. If D_C has m blocking-zeros at zero, i.e., $(s^{-(m-1)}D_C)(0) = 0$, then H_{eu} has m blocking-zeros at zero; this is again sufficient for type-m integral action in the stable $\mathcal{S}(P, C)$.

2.1. Theorem: Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N} \in \mathrm{R_p}^{n_y \times n_u}$

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be any RCF, LCF; rank$N(0) = n_y \le n_u$. Let $K_1 \in \mathbb{R}^{n_u \times n_y}$ stabilize $s^{-1}N$. Then C_m is a stabilizing controller with type-m integral action if and only if

$$C_m = (V - Q_m\widetilde{N})^{-1}(U + Q_m\widetilde{D} + \frac{K_1}{s}\sum_{\ell=0}^{m-1}(I_{n_y} + N\frac{K_1}{s})^{\ell});$$

$Q_m \in \mathcal{R}^{n_u \times n_y}$ satisfies $(V - Q_m\widetilde{N})$ is biproper (which holds for all $Q_m \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$). □

In Theorem 2.1, $K_1 \in \mathbb{R}^{n_u \times n_y}$ is any constant controller that stabilizes $s^{-1}N$. Since rank$N(0) = n_y$ by assumption, constant controllers that stabilize $s^{-1}N$ exist. Let $N(0)^I \in \mathbb{R}^{n_u \times n_y}$ denote any right-inverse of $N(0) \in \mathbb{R}^{n_y \times n_u}$; K_1 can be chosen as $K_1 = \alpha N(0)^I$ for any $\alpha \in \mathbb{R}$ such that $0 < \alpha <\| s^{-1}(NN(0)^I - I_{n_y}) \|^{-1}$. When C_m is a stabilizing controller with type-m integral action as in Theorem 2.1, the corresponding achievable (input-output) transfer-function of the stable $\mathcal{S}(P,C)$ is $H_{yu} = PC(I_{n_y} + PC)^{-1} = I_{n_y} - (I_{n_y} + s^{-1}NK_1\sum_{\ell=0}^{m-1}(I_{n_y} + s^{-1}NK_1)^{\ell})^{-1}(\widetilde{V} - NQ_m)\widetilde{D}$, and $H_{eu} = I_{n_y} - H_{yu} = (I_{n_y} + s^{-1}NK_1\sum_{\ell=0}^{m-1}(I_{n_y} + s^{-1}NK_1)^{\ell})^{-1}(\widetilde{V} - NQ_m)\widetilde{D}$. The parametrization in Theorem 2.1 is simplified for stable plants: Let $P \in \mathcal{R}^{n_y \times n_u}$; rank$P(0) = n_y \le n_u$. Let $K_1 \in \mathbb{R}^{n_u \times n_y}$ be any constant controller that stabilizes $s^{-1}P$. Then C_m is a stabilizing controller with type-m integral action if and only if $C_m = (I_{n_u} - Q_mP)^{-1}(Q_m + s^{-1}K_1\sum_{\ell=0}^{m-1}(I_{n_y} + s^{-1}PK_1)^{\ell})$ for some $Q_m \in \mathcal{R}^{n_u \times n_y}$ such that $(I_{n_u} - Q_mP)$ is biproper (holds for all $Q_m \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_s})$). The corresponding achievable $H_{yu} = I_{n_y} - (I_{n_y} + s^{-1}PK_1\sum_{\ell=0}^{m-1}(I_{n_y} + s^{-1}PK_1)^{\ell})^{-1}(I_{n_y} - PQ_m)$.

By Theorem 2.1, any stabilizing controller with type-m integral action is obtained by adding 'integral terms' to any stabilizing controller. As an example, Fig. 2 shows the block-diagram of C_2 with type-2 integral action.

We now parametrize stabilizing controllers with type-m integral action such that the (input-output) transfer-function H_{yu} of the stable system $\mathcal{S}(P,C)$ is diagonal and nonsingular. We assume that P has no zeros at zero since this is a necessary condition for integral action; a necessary condition to achieve decoupling is that rank$P = n_y$. A sufficient condition to achieve decoupling is that the full row-rank P has no pole-zero coincidences in $\mathcal{U}$. Here we assume that there exist stabilizing controllers that achieve decoupling for the given plant (see [2] for necessary and sufficient conditions to achieve decoupling). Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N} \in \mathrm{R_p}^{n_y \times n_u}$ be any RCF, LCF; rank$P = n_y \le n_u$, rank$N(0) = n_y$. Let $C_d = \widetilde{U}_d\widetilde{V}_d^{-1} = V_d^{-1}U_d$ be any stabilizing controller such that H_{yu} is diagonal and nonsingular; $V_dD + U_dN = I_{n_u}$, $\widetilde{D}\widetilde{V}_d + \widetilde{N}\widetilde{U}_d = I_{n_y}$, $V_d\widetilde{U}_d = U_d\widetilde{V}_d$. Let $\delta_{Lj} \in \mathcal{R}$ be any greatest-common-divisor of all entries in the j-th row of $N \in \mathcal{R}^{n_y \times n_u}$, $j = 1, \ldots, n_y$; define $\Delta_L := \mathrm{diag}\left[\delta_{L1} \cdots \delta_{Ln_y}\right]$, $\Delta_L\hat{N} := N$. Let $\hat{N}^I \in \mathrm{R}^{n_u \times n_y}$ denote any right-inverse of $\hat{N}$. Write the ij-th entry of $\hat{N}^I$ as $a_{ij}b_{ij}^{-1}$, $i = 1, \ldots, n_u$, $j = 1, \ldots, n_y$ ($a_{ij}, b_{ij} \in \mathcal{R}$, $b_{ij} \ne 0$, (a_{ij}, b_{ij}) coprime). Let $\delta_{Rj} \in \mathcal{R}$ be any least-common-multiple of $(b_{1j}, \ldots, b_{n_uj})$ in the j-th column of $\hat{N}^I$; define $\Delta_R := \mathrm{diag}\left[\delta_{R1} \cdots \delta_{Rn_y}\right]$. Let $\phi_{Rj} \in \mathcal{R}$ be any greatest-common-divisor of all entries in the j-th column of $\widetilde{D}$, $j = 1, \ldots, n_y$; define $\Phi_R := \mathrm{diag}\left[\phi_{R1} \cdots \phi_{Rn_y}\right]$, $\hat{D}\Phi_R := \widetilde{D}$. Write the ij-th entry of $\hat{D}^{-1} \in \mathrm{R}^{n_y \times n_y}$, $i,j = 1, \ldots, n_y$, as $c_{ij}d_{ij}^{-1}$ ($c_{ij}, d_{ij} \in \mathcal{R}$, $d_{ij} \ne 0$, (c_{ij}, d_{ij}) coprime). Let $\phi_{Lj} \in \mathcal{R}$ be any least-common-multiple of $(d_{1j}, \ldots, d_{n_yj})$ in the j-th row of $\hat{D}^{-1}$; define $\Phi_L := \mathrm{diag}\left[\phi_{L1} \cdots \phi_{Ln_y}\right]$. Let $K_d \in \mathbb{R}^{n_y \times n_y}$ be any diagonal constant controller that stabilizes $s^{-1}\Delta_L\Delta_R$ (note rank$N(0) = n_y$ implies rank$(\Delta_L\Delta_R)(0) = n_y$). Then C_m is a stabilizing controller with type-m integral action with H_{yu} diagonal and nonsingular if and only if $C_m = (V_d - Q_m\widetilde{N})^{-1}(U_d + Q_m\widetilde{D} + s^{-1}\hat{N}^I\Delta_RK_d\sum_{\ell=0}^{m-1}(I_{n_y} + s^{-1}\Delta_L\Delta_RK_d)^{\ell})$; $Q_m = \hat{N}^I\Delta_RQ_d\Phi_L\hat{D}^{-1}$, $Q_d \in \mathcal{R}^{n_y \times n_y}$ diagonal, nonsingular, $Q_d(\infty) \ne (\widetilde{V}_d\widetilde{D}(\Delta_L\Delta_R\Phi_L\Phi_R)^{-1})(\infty)$. The corresponding achievable $H_{yu} = I_{n_y} - (I_{n_y} + s^{-1}\Delta_L\Delta_RK_d\sum_{\ell=0}^{m-1}(I_{n_y} + s^{-1}\Delta_L\Delta_RK_d)^{\ell})^{-1}(\widetilde{V}_d\widetilde{D} - \Delta_L\Delta_RQ_d\Phi_L\Phi_R)$ is diagonal.

References

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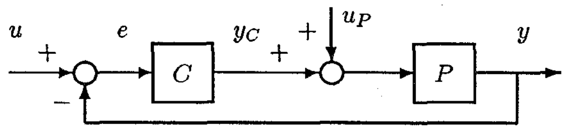

Figure 1: The system $\mathcal{S}(P,C)$

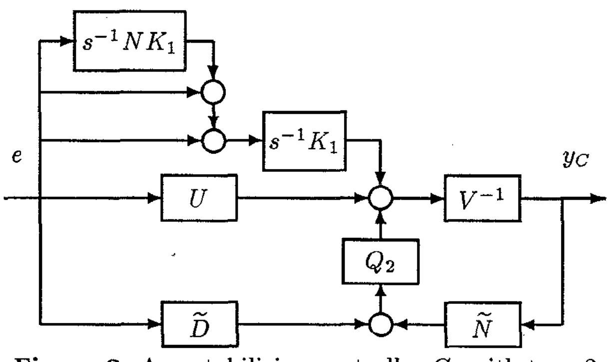

Figure 2: Any stabilizing controller C_2 with type-2 integral action