Proceedings of the American Control Conference
Albuquerque, New Mexico June 1997
0-7803-3832-4/97/$10.00 © 1997 AACC

Simple parametrization of controllers with integral action [1]

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Abstract

A parametrization of stabilizing controllers with integral action is obtained in the the standard linear time-invariant, multi-input-multi-output unity-feedback system. These controllers guarantee asymptotic tracking of step reference inputs at each output channel with zero steady-state error.

1. Introduction

Stabilizing controllers that achieve robust asymptotic tracking of general reference signals can be designed by using the well-known parametrization of all controllers that stabilize a given plant [4]. For asymptotic tracking of step reference signals, controllers are designed to have integral action (see for example [3], [1]).

In this paper we parametrize controllers with integral action in the standard linear time-invariant (LTI), multi-input-multi-output (MIMO) unity-feedback system. The given plant is not necessarily stable. We show that any stabilizing controller with integral action (as defined here) is expressed as the sum of two controllers, any arbitrary stabilizing controller plus an 'integral controller' as stated in the main result (Theorem 2.1).

Notation: Let $\mathcal{U}$ be the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The set of real numbers, the set of proper rational functions which have no poles in the region of instability $\mathcal{U}$, the sets of proper and strictly-proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathcal{R}$, $\mathrm{R_p}$, $\mathrm{R_{sp}}$, respectively. The set of matrices whose entries are in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called stable iff $M \in \mathcal{M}(\mathcal{R})$; a stable M is called unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. Let $\mathrm{rank}P = r$; $s_o \in \mathcal{U}$ is called a (transmission) $\mathcal{U}$-zero of P iff $\mathrm{rank}P(s_o) < r$, equivalently, $\mathrm{rank}N(s_o) = \mathrm{rank}\widetilde{N}(s_o) < r$; s_o is called a blocking-zero of P iff $P(s_o) = 0$; $s_o \in \mathcal{U}$ is a blocking-zero of P if and only if $N(s_o) = \widetilde{N}(s_o) = 0$.

Due to the algebraic framework described in this notation, the results apply to continuous-time and as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at $s = 0$ would be interpreted at $z = 1$.

[1] This work was supported by the National Science Foundation Grant ECS-9257932.

2. Main results

Consider the standard LTI, MIMO, unity-feedback system $\mathcal{S}(P,C)$ (Fig. 1): $\mathcal{S}(P,C)$ is a well-posed system, where $P \in \mathrm{R_p}^{n_o \times n_i}$ and $C \in \mathrm{R_p}^{n_i \times n_o}$ are the transfer-functions of the plant and the controller; P and C have no hidden modes corresponding to eigenvalues in $\mathcal{U}$.

Let H_{eu} (H_{yu}) denote the input-error (input-output) transfer-function from u to e (u to y, respectively). The system $\mathcal{S}(P,C)$ is said to be stable iff the transfer-function H from (u, u_P) to (y, y_C) is stable. The stable system $\mathcal{S}(P,C)$ is said to have integral action in each output channel iff $H_{eu}(s) = I - H_{yu}(s)$ has blocking-zeros at $s = 0$. If $\mathcal{S}(P,C)$ is stable, $H_{eu}(0) = (I_{n_o} + PC)^{-1}(0) = I_{n_o} - PC(I_{n_o} + PC)^{-1}(0) = 0$ only if $\mathrm{rank}P(0) = n_o$; therefore, a necessary condition for the stable system $\mathcal{S}(P,C)$ to have integral action is that the plant P has no (transmission) zeros at zero.

Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ be any right-coprime-factorization (RCF) and left-coprime-factorization (LCF) of $P \in \mathrm{R_p}^{n_o \times n_i}$, where $N, D, \widetilde{N}, \widetilde{D} \in \mathcal{M}(\mathcal{R})$, D, $\widetilde{D}$ are biproper. The controller C is said to be a stabilizing controller for the plant P (or C is said to stabilize P) iff $C \in \mathcal{M}(\mathrm{R_p})$ and the system $\mathcal{S}(P,C)$ is stable. Let $C = N_C D_C^{-1}$ be any RCF of C. The controller C is said to be a *stabilizing controller with integral action* iff C stabilizes P and $D_C(s)$ has blocking-zeros at $s = 0$, where $D_C \in \mathcal{R}^{n_o \times n_o}$ is the denominator-matrix of any RCF $N_C D_C^{-1}$ of C. The controller C stabilizes P if and only if $\widetilde{D}D_C + \widetilde{N}N_C$ is unimodular for any RCF $N_C D_C^{-1}$ of C (where D_C is biproper) [4], [2]. All stabilizing controllers for P are given by $C = (\widetilde{U} + DQ)(\widetilde{V} - NQ)^{-1} = (V - Q\widetilde{N})^{-1}(U + Q\widetilde{D})$, where $Q \in \mathcal{R}^{n_i \times n_o}$ is such that $(\widetilde{V} - NQ)$ is biproper (holds for all $Q \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_{sp}})$); $U, V, \widetilde{U}, \widetilde{V} \in \mathcal{M}(\mathcal{R})$ are such that $VD + UN = I_{n_i}$, $\widetilde{D}\widetilde{V} + \widetilde{N}\widetilde{U} = I_{n_o}$, $V\widetilde{U} = U\widetilde{V}$.

Using the parametrization of all stabilizing controllers, for any stabilizing controller C, $H_{eu} = (I_{n_o} + PC)^{-1} = I_{n_o} - N(U + Q\widetilde{D}) = (\widetilde{V} - NQ)\widetilde{D}$. Since (N_C, D_C) in any arbitrary RCF of a stabilizing controller is given by $(N_C, D_C) = ((\widetilde{U} + DQ)R, (\widetilde{V} - NQ)R)$ for some unimodular $R \in \mathcal{M}(\mathcal{R})$, $D_C(0) = 0$ is equivalent to $(\widetilde{V} - NQ)(0) = 0$. Therefore, if C is a stabilizing controller with integral action, then $H_{eu}(0) = (\widetilde{V} - NQ)(0)\widetilde{D}(0) = 0$ and hence, the stable system $\mathcal{S}(P,C)$ has integral in each output channel as defined above. Although designing the stabilizing controllers so that $D_C(0) = 0$ is a sufficient condition for the stable system $\mathcal{S}(P, C)$ to have integral action, it is clearly not necessary. However, when P has no poles at $s = 0$, and in particular when P is stable, $H_{eu}(0) = 0$ if and only if $D_C(0) = 0$, i.e., the stable system $\mathcal{S}(P, C)$ has integral action if and only if the controller C is a stabilizing controller with integral action.

2.1. Theorem *(Parametrization of controllers)*:

Let $P \in \mathrm{R_p}^{n_o \times n_i}$, where $\mathrm{rank}P(0) = n_o \leq n_i$. Let $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ be any RCF and LCF of P. Let $K_i \in \mathrm{I\!R}^{n_i \times n_o}$ be any constant controller that stabilizes $\frac{N}{s}$. The controller C is a stabilizing controller with integral action if and only if

$$C = (V - Q_1\widetilde{N})^{-1}(U + Q_1\widetilde{D}) + (V - Q_1\widetilde{N})^{-1}\frac{K_i}{s} \quad (1)$$

equivalently, $C = (\widetilde{U} + DQ_1)(\widetilde{V} - NQ_1)^{-1}(I_{n_o} + N\frac{K_i}{s}) + D\frac{K_i}{s}$ for some $Q_1 \in \mathcal{R}^{n_i \times n_o}$ such that $(V - Q_1\widetilde{N})$ is biproper (holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_{sp}})$). The corresponding H_{yu} of the stable system $\mathcal{S}(P, C)$ is $H_{yu} = (I_{n_o} + N\frac{K_i}{s})^{-1}(N(U + Q_1\widetilde{D}) + N\frac{K_i}{s})$. □

For the special case of stable plants, the parametrization (1) of all controllers with integral action is simplified as follows: Let $P \in \mathcal{R}^{n_o \times n_i}$, where $\mathrm{rank}P(0) = n_o \leq n_i$. Let $K_i \in \mathrm{I\!R}^{n_i \times n_o}$ be any constant controller that stabilizes $\frac{P}{s}$. The controller C is a stabilizing controller with integral action if and only if $C = (I_{n_i} - Q_1 P)^{-1}Q_1 + (I_{n_i} - Q_1 P)^{-1}\frac{K_i}{s}$ for some $Q_1 \in \mathcal{R}^{n_i \times n_o}$ such that $(I_{n_i} - Q_1 P)$ is biproper (holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_{sp}})$). The corresponding (input-output) transfer-function H_{yu} of the stable system $\mathcal{S}(P, C)$ is $H_{yu} = (I_{n_o} + P\frac{K_i}{s})^{-1}(PQ_1 + P\frac{K_i}{s})$.

In Theorem 2.1, K_i is chosen as any constant controller that stabilizes $\frac{N}{s}$. Since $\mathrm{rank}N(0) = n_o$ by assumption, existence of constant controllers that stabilize $\frac{N}{s}$ is guaranteed; in fact, K_i can be chosen as $aN(0)^I$ for some $a \in \mathrm{I\!R}$, $a > 0$, where $N(0)^I$ denotes a right-inverse of $N(0)$. Theorem 2.1 states that any stabilizing controller with integral action is expressed as the sum of two controllers, any stabilizing controller $(V - Q_1\widetilde{N})^{-1}(U + Q_1\widetilde{D})$ plus an 'integral controller' $(V - Q_1\widetilde{N})^{-1}\frac{K_i}{s}$. The block-diagram of the system $\mathcal{S}(P, C)$ with the stabilizing controller C as in (1) is shown in Figure 2.

The parametrization in (1) of all stabilizing controllers with integral action can also be obtained starting with a state-space representation of P as follows: Let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be a state-space representation of $P = \bar{C}(sI_n - \bar{A})^{-1}\bar{B} + \bar{D}$, where $(\bar{A}, \bar{B})$ is stabilizable and $(\bar{C}, \bar{A})$ is detectable. Let $K \in \mathrm{I\!R}^{n_i \times n}$ and $L \in \mathrm{I\!R}^{n \times n_o}$ be such that $A_K := (sI_n - \bar{A} + \bar{B}K)^{-1} \in \mathcal{M}(\mathcal{R})$, $A_L := (sI_n - \bar{A} + L\bar{C})^{-1} \in \mathcal{M}(\mathcal{R})$. The controller C is a stabilizing controller with integral action if and only if $C = (I_{n_i} - KA_L(\bar{B} - L\bar{D}) - Q_1(\bar{C}A_L(\bar{B} - L\bar{D}) + \bar{D}))^{-1} \times (KA_L L + Q_1(I - \bar{C}A_L L) + \frac{K_i}{s})$ for some $Q_1 \in \mathcal{R}^{n_i \times n_o}$ such that $\det(I_{n_i} - Q_1(\infty)\bar{D})) \neq 0$ (holds for all $Q_1 \in \mathcal{M}(\mathcal{R})$ when $P \in \mathcal{M}(\mathrm{R_{sp}})$). This controller is shown in Figure 3.

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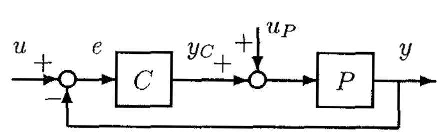

Figure 1: The system $\mathcal{S}(P, C)$

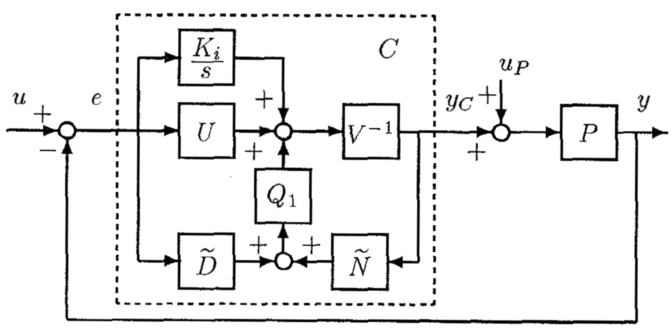

Figure 2: The system $\mathcal{S}(P, C)$, where C is a stabilizing controller with integral action

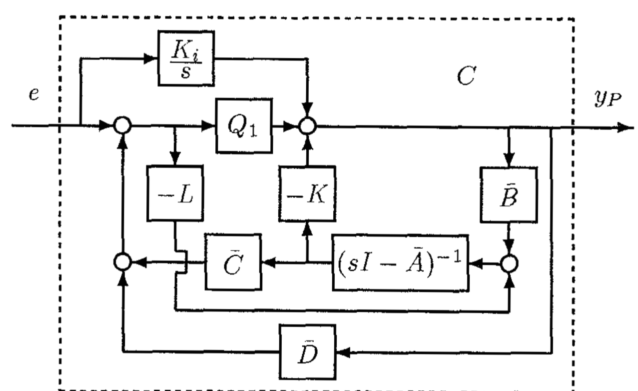

Figure 3: Stabilizing controller C with integral action