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Simultaneous stabilization of multiplicatively perturbed plants[1]

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Abstract

Necessary and sufficient conditions are obtained for simultaneous stabilizability of a given linear, time-invariant, multi-input multi-output nominal plant and a multiplicatively perturbed plant, where the multiplicative perturbation is stable. These conditions are derived from the general parity-interlacing-property applicable to any two arbitrary plants and are expressed explicitly in terms of the real-axis poles and zeros of the nominal plant and the perturbation. The class of perturbations for which simultaneous stabilization is achievable are characterized using these conditions. A special class of unknown diagonal perturbations is also considered and simultaneously stabilizing controllers are designed for the nominal plant under any such unknown perturbation. © 1997 Elsevier Science B.V.

Keywords: Simultaneous stabilization; Stabilizing controller; Multiplicative perturbation

1. Introduction

In the standard linear, time-invariant (LTI), multi-input multi-output (MIMO) unity-feedback system, the parametrization of all stabilizing controllers is now well-known [7]. This characterization has led to increased interest in the problem of simultaneous stabilization of a set of plants using a single controller. In the case of two given plants, necessary and sufficient conditions for existence of simultaneously stabilizing controllers are available based on the parity-interlacing property of a pair of transfer functions associated with these two plants [7, 8, 1]. For the case of three or more plants, there are no necessary and sufficient conditions available to check simultaneous stabilizability and this challenging topic is of active research interest [2, 3, 6].

An important special case of simultaneous stabilization is encountered when considering a class of plants generated by a given nominal plant in different modes of operation or under sensor or actuator failures. Before controller design is attempted, it is crucial to know if this class of plants is in fact simultaneously stabilizable and to characterize the perturbations for which simultaneously stabilizing controllers exist. In this paper we consider simultaneous stabilization of a given nominal (unstable) plant P and a given perturbed plant $(I - \Delta)P$, where the known multiplicative perturbation Δ is stable. The perturbed plant $(I - \Delta)P$ represents the nominal plant under various perturbations and changes, such as sensor failures. The parity-interlacing property, which applies to the general case of any two plants, now has specific implications for the special case of the second plant being a multiplicatively perturbed version of the first one. These implications lead to

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explicit interpretations in terms of the poles and zeros of the nominal plant and put restrictions on perturbations Δ for which simultaneous stabilization is in fact possible to achieve.

The stable perturbation Δ is assumed to be known in Section 2, where general conditions are developed for simultaneous stabilizability of this special set of two plants (Lemma 2.2 and Corollaries 2.3, 2.4). For the special case of single-input single-output (SISO) plants, these conditions lead to a characterization of all perturbations such that P and $(I-\Delta)P$ are simultaneously stabilizable (Corollary 2.5). Structural constraints on Δ are imposed in Section 3, where two cases of diagonal perturbations are considered. In the first case, necessary and sufficient conditions for simultaneous stabilizability are obtained assuming that every output channel of P is multiplied by the same (scalar) stable transfer function $(1-\delta)$; in this case, δ is assumed to be known a priori as in Section 2 (Proposition 3.1 and Corollary 3.2). In the second case, the diagonal matrix Δ is assumed to have exactly one entry δ_j, which may be non-zero. This important case represents possible unknown sensor failures at a pre-specified output channel. Although the location of the failure is known to be the jth output, δ_j may be any arbitrary stable transfer function and the simultaneous stabilizability conditions take into account all possible values including $\delta_j = 1$, which corresponds to the disconnection of the jth sensor. Since δ_j is unknown, the necessary and sufficient conditions presented in this case are in terms of the nominal plant P (Proposition 3.3). A controller design method is proposed for plants satisfying these conditions (Proposition 3.4). These simultaneous stabilizability conditions and the proposed class of simultaneously stabilizing controllers are therefore for a special class of infinitely many plants since δ_j is arbitrary.

The main results of this paper are given in Sections 2 and 3; the proofs are collected in Section 4; brief concluding remarks are given in Section 5. Due to the algebraic framework described in the following notation, the results apply to continuous-time systems as well as discrete-time systems as in the case of all similar work based on factorization methods.

Notation. Let $\mathscr{U}$ contain the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). Let $\mathbb{R}$, R_{p}, $\mathscr{R}$, and $\mathscr{M}(\mathscr{R})$ be the set of real numbers, proper rational functions with real coefficients, proper rational functions with no poles in the region of instability $\mathscr{U}$, and the set of matrices whose entries are in $\mathscr{R}$. A matrix M is called $\mathscr{R}$-stable iff $M \in \mathscr{M}(\mathscr{R})$; $M \in \mathscr{M}(\mathscr{R})$ is called $\mathscr{R}$-unimodular iff M^{-1} is also $\mathscr{R}$-stable. For $M \in \mathscr{M}(\mathscr{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s \in \partial\mathscr{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$, $\partial\mathscr{U}$ denote the maximum singular value of $M(s)$ and the boundary of $\mathscr{U}$, respectively. A right-coprime factorization (RCF) and a left-coprime factorization (LCF) of $P \in \mathsf{R}_{\mathrm{p}}^{n_o \times n_i}$ are denoted by $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$; $N, D, \tilde{N}, \tilde{D} \in \mathscr{M}(\mathscr{R})$; D, $\tilde{D}$ are biproper. Let $\operatorname{rank} P = r$; $s_o \in \mathscr{U}$ is called a (transmission) $\mathscr{U}$-zero of P iff $\operatorname{rank} P(s_o) < r$, i.e., $\operatorname{rank} N(s_o) = \operatorname{rank} \tilde{N}(s_o) < r$; $s_o \in \mathscr{U}$ is called a blocking $\mathscr{U}$-zero of P iff $P(s_o) = 0$, i.e., $N(s_o) = 0 = \tilde{N}(s_o)$; $s_o \in \mathscr{U}$ is called a $\mathscr{U}$-pole of P iff it is a pole of some entry of P, i.e., $\det D(s_o) = 0 = \det \tilde{D}(s_o)$. The identity map is denoted by I; the jth column of I is denoted by e_j. The notation $a := b$ means a is defined as b.

2. Simultaneous stabilizability conditions

Consider the LTI, MIMO system $\mathscr{S}(I-\Delta, P, C)$ shown in Fig. 1, where $P \in \mathsf{R}_{\mathrm{p}}^{n_o \times n_i}$, $C \in \mathsf{R}_{\mathrm{p}}^{n_i \times n_o}$, $\Delta \in \mathscr{R}^{n_o \times n_o}$ represent the given plant, the controller and the known multiplicative perturbation. The nominal plant and the perturbed plant are denoted by P and $(I-\Delta)P$, respectively. The nominal plant P is not necessarily $\mathscr{R}$-stable. The perturbed plant has the same $\mathscr{U}$-poles as P. If $\Delta = 0$, then $\mathscr{S}(I-\Delta, P, C)$ becomes the standard unity-feedback system $\mathscr{S}(P, C)$, called the nominal system. It is assumed that P and C do not have any hidden modes corresponding to eigenvalues in $\mathscr{U}$, and that the system $\mathscr{S}(I-\Delta, P, C)$ is well-posed.

Definition 2.1 (*$\mathscr{R}$-stability, $\mathscr{R}$-stabilizing controller*). The system $\mathscr{S}(I-\Delta, P, C)$ is said to be $\mathscr{R}$-stable iff the closed-loop transfer function H from $u := [\,u_P^{\mathrm{T}}\ u_C^{\mathrm{T}}\,]^{\mathrm{T}}$ to $y := [\,y_P^{\mathrm{T}}\ y_C^{\mathrm{T}}\,]^{\mathrm{T}}$ is $\mathscr{R}$-stable. Similarly, the nominal system $\mathscr{S}(P, C)$ is said to be $\mathscr{R}$-stable iff $H : u \mapsto y$ is $\mathscr{R}$-stable when $\Delta = 0$. The controller C is said to be an $\mathscr{R}$-stabilizing controller for $P \in \mathsf{R}_{\mathrm{p}}^{n_o \times n_i}$ iff $C \in \mathsf{R}_{\mathrm{p}}^{n_i \times n_o}$ and $\mathscr{S}(P, C)$ is $\mathscr{R}$-stable. The controller C is said to

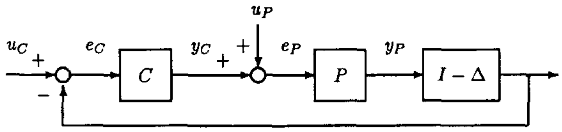

Fig. 1. The system $\mathscr{S}(I-\Delta,P,C)$.

be a simultaneously $\mathscr{R}$-stabilizing controller for P and $(I-\Delta)P$ iff $C \in \mathsf{R}_{\mathrm{p}}^{n_i \times n_o}$ and the systems $\mathscr{S}(P,C)$ and $\mathscr{S}(I-\Delta,P,C)$ are both $\mathscr{R}$-stable. The plants P and $(I-\Delta)P$ are said to be simultaneously $\mathscr{R}$-stabilizable iff there exists a simultaneously $\mathscr{R}$-stabilizing controller C.

When $\Delta = I$, the system $\mathscr{S}(I-\Delta,P,C)$ becomes open loop; by Definition 2.1, it is clear that the system $\mathscr{S}(I-\Delta,P,C)$ is $\mathscr{R}$-stable when $\Delta = I$ if and only if both P and C are $\mathscr{R}$-stable.
It is well-known (see for example [7, 5]) that the controller $C \in \mathscr{M}(\mathsf{R}_{\mathrm{p}})$ is an $\mathscr{R}$-stabilizing controller for P if and only if C is given by (1):

$$C = (V - Q\tilde{N})^{-1}(U + Q\tilde{D}) = (\tilde{U} + DQ)(\tilde{V} - NQ)^{-1}\ , \tag{1}$$

where $Q \in \mathscr{M}(\mathscr{R})$ is such that $(V - Q\tilde{N})$ is biproper (which holds for all $Q \in \mathscr{M}(\mathscr{R})$ when P is strictly proper), where $U, V, \tilde{U}, \tilde{V} \in \mathscr{M}(\mathscr{R})$ satisfy (2):

$$\begin{bmatrix} V & U \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{U} \\ N & \tilde{V} \end{bmatrix} = I. \tag{2}$$

Lemma 2.2 (Closed-loop stability of $\mathscr{S}(I-\Delta,P,C)$). *Let ND^{-1}, $\tilde{D}^{-1}\tilde{N}$ be any RCF and any LCF of $P \in \mathsf{R}_{\mathrm{p}}^{n_o \times n_i}$. Let $\Delta \in \mathscr{M}(\mathscr{R})$.*
(a) *Let $N_C D_C^{-1}$, $\tilde{D}_C^{-1}\tilde{N}_C$ be any RCF and any LCF of $C \in \mathsf{R}_{\mathrm{p}}^{n_i \times n_o}$. The system $\mathscr{S}(I-\Delta,P,C)$ is $\mathscr{R}$-stable if and only if*

$$\begin{bmatrix} \tilde{D} & -\tilde{N}N_C \\ (I-\Delta) & D_C \end{bmatrix} \text{ is } \mathscr{R}\text{-unimodular}, \tag{3}$$

equivalently,

$$[\tilde{D}_C D + \tilde{N}_C (I-\Delta)N] \text{ is } \mathscr{R}\text{-unimodular}. \tag{4}$$

(b) *Let $U, V \in \mathscr{M}(\mathscr{R})$ be as in* (2). *The controller C is a simultaneously $\mathscr{R}$-stabilizing controller for P and $(I-\Delta)P$ if and only if C is given by* (1), *where $Q \in \mathscr{M}(\mathscr{R})$ is such that $(V - Q\tilde{N})$ is biproper and*

$$I_{n_i} - (U + Q\tilde{D})\Delta N \text{ is } \mathscr{R}\text{-unimodular}. \tag{5}$$

(c) *Let $U \in \mathscr{M}(\mathscr{R})$ be as in* (2). *The plants P and $(I-\Delta)P$ are simultaneously $\mathscr{R}$-stabilizable if and only if*
(i) *the pair $(I-\Delta,\tilde{D})$ is right-coprime, and*
(ii) $\det(I_{n_i} - U\Delta N)$ *has the same sign at all real blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$.*

By Lemma 2.2(b), the plants P and $(I-\Delta)P$ are simultaneously $\mathscr{R}$-stabilizable if and only if there exists $Q \in \mathscr{M}(\mathscr{R})$ such that (5) holds. It follows from standard strong $\mathscr{R}$-stabilizability results that condition (5) holds if and only if the pair $(I - U\Delta N, \tilde{D}\Delta N)$ satisfies the well-known parity-interlacing property [7, 8, 1], which is equivalent to Lemma 2.2(c). However, condition (5) is not explicit, and the equivalent conditions in terms of the blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$ require the entire list of such zeros on the real axis. We now search for explicit conditions in terms of the poles of the given plant P and the known multiplicative perturbation Δ for simultaneous $\mathscr{R}$-stabilizability of the plants P and $(I-\Delta)P$. Lemma 2.2(c) immediately leads to the following necessary conditions.

Corollary 2.3 (Necessary conditions for simultaneous stabilizability). *Let $\tilde{D}^{-1}\tilde{N}$ be any LCF of $P \in \mathrm{R}_{\mathrm{p}}^{n_o \times n_i}$. Let $\Delta \in \mathcal{M}(\mathcal{R})$. If the plants P and $(I-\Delta)P$ are simultaneously $\mathcal{R}$-stabilizable, then*

(i) *$(I-\Delta, \tilde{D})$ is right-coprime, equivalently, $(I-\Delta, \tilde{D}\Delta)$ is right-coprime, and*

(ii) *$\det\,(I-\Delta)$ has the same sign at all real-axis blocking $\mathcal{U}$-zeros of $\tilde{D}\Delta$; furthermore, this sign is positive whenever ΔP has real-axis blocking $\mathcal{U}$-zeros.*

The sign test in Corollary 2.3 is performed at the real-axis blocking $\mathcal{U}$-zeros of $\tilde{D}\Delta$, which are in fact a subset of the real-axis $\mathcal{U}$-poles of P. To see that P has poles (equivalently, that $\det \tilde{D}$ is zero) at the real-axis blocking $\mathcal{U}$-zeros of $\tilde{D}\Delta$, observe that the set of blocking $\mathcal{U}$-zeros of the product $\tilde{D}\Delta$ is the union of the sets of blocking $\mathcal{U}$-zeros of $\tilde{D}$, of Δ and in addition, possibly some of the $\mathcal{U}$-zeros that are common to both $\tilde{D}$ and Δ; clearly, $(\tilde{D}\Delta)(s_o)=0$ at $s_o \in \mathcal{U}$ only if both rank $\tilde{D}(s_o) < n_o$ and rank $\Delta(s_o) < n_o$. To check the positivity of the sign when ΔP has real-axis blocking $\mathcal{U}$-zeros, note that the sets of the blocking $\mathcal{U}$-zeros of Δ and of P are subsets of the blocking $\mathcal{U}$-zeros of the product ΔP. If P is strictly-proper, then P has a real-axis blocking $\mathcal{U}$-zero at infinity. For full (normal) row-rank P, if there are any additional blocking $\mathcal{U}$-zeros of ΔP, then these would be some of the transmission $\mathcal{U}$-zeros of P that are also $\mathcal{U}$-zeros of Δ; for $\operatorname{rank} P < n_o$, ΔP may have additional blocking $\mathcal{U}$-zeros at points other than the transmission $\mathcal{U}$-zeros of P. Based on these observations, condition (ii) of Corollary 2.3 can be stated more explicitly as follows: If the plants P and $(I-\Delta)P$ are simultaneously $\mathcal{R}$-stabilizable, then $\det(I-\Delta)$ has the same sign at all real-axis blocking $\mathcal{U}$-zeros of $\tilde{D}$ and at all other real-axis $\mathcal{U}$-poles of P for which $\tilde{D}\Delta = 0$; furthermore, this sign is positive whenever Δ, or P, or the product ΔP have real-axis blocking $\mathcal{U}$-zeros.

We now state that the necessary conditions for simultaneous $\mathcal{R}$-stabilizability of P and $(I-\Delta)P$ given in Corollary 2.3 are also sufficient when P is full (normal) row-rank and has no real-axis pole-zero coincidences in the region of instability $\mathcal{U}$.

Corollary 2.4 (Necessary sufficient conditions for simultaneous stabilizability). *Let $\tilde{D}^{-1}\tilde{N}$ be any LCF of $P \in \mathrm{R}_{\mathrm{p}}^{n_o \times n_i}$. Let $\operatorname{rank} P = n_o \leqslant n_i$ and let P have no coinciding poles and zeros in $\mathbb{R} \cap \mathcal{U}$. Let $\Delta \in \mathcal{M}(\mathcal{R})$. Under these assumptions, the plants P and $(I-\Delta)P$ are simultaneously $\mathcal{R}$-stabilizable if and only if the two conditions in Corollary 2.3 hold.*

A special case of plants which have no pole-zero coincidences on the real-axis portion of the region of instability $\mathcal{U}$ is the case of SISO plants. Corollary 2.4 leads to the following conditions, which characterize all Δ such that P and $(I-\Delta)P$ are $\mathcal{R}$-stabilizable.

Corollary 2.5 (Necessary and sufficient conditions when P is SISO). *Let $P \in \mathrm{R}_{\mathrm{p}}$. Let $\Delta \in \mathcal{R}$. The plants P and $(I-\Delta)P$ are simultaneously $\mathcal{R}$-stabilizable if and only if*

(i) *$\Delta \neq 1$ at all $\mathcal{U}$-poles of P, and*

(ii) *$(1-\Delta)$ has the same sign at all real-axis $\mathcal{U}$-poles of P; furthermore, $\Delta < 1$ at all real-axis $\mathcal{U}$-poles of P whenever P or Δ have real-axis $\mathcal{U}$-zeros.*

Corollary 2.5 defines the class of all perturbations $\Delta \in \mathcal{R}$ such that P and $(I-\Delta)P$ are simultaneously $\mathcal{R}$-stabilizable. To illustrate the advantages of a necessary and sufficient condition, consider the following example as a comparison with the standard small-gain sufficient condition: Let $\mathcal{U} = \mathbb{C}_+ \cup \{\infty\}$,

$$P = \frac{(-3s^2+10s+1)}{s(s-1)(s+7)}.$$

A coprime-factorization of P is

$$P = ND^{-1} = \tilde{D}^{-1}\tilde{N} = \left(\frac{-3s^2+10s+1}{(s+1)^3}\right)\left(\frac{s(s-1)(s+7)}{(s+1)^3}\right)^{-1}.$$

Using $U=1$, $V=1$ as a solution for (2), all $\mathcal{R}$-stabilizing controllers are given by (1) as $C = (1-QN)^{-1}(1+QD)$, where $Q \in \mathcal{R}$. By (5), C also $\mathcal{R}$-stabilizes $(I-\Delta)P$ if and only if $Q \in \mathcal{R}$ is such that

$I - (U + Q\tilde{D})\Delta N$ is $\mathscr{R}$-unimodular. A sufficient condition to satisfy this would be to seek Q so that $\|(U+Q\tilde{D})\Delta N\| < 1$. Since P has a pole at $s=0$, $((U+Q\tilde{D})\Delta N)(0)=\Delta(0)$; therefore $\|(U+Q\tilde{D})\Delta N\| \geqslant |\Delta(0)|$ for any $Q \in \mathscr{R}$. The existence of $Q \in \mathscr{M}(\mathscr{R})$ satisfying (5) cannot be concluded using this small-gain condition. However, the necessary and sufficient conditions of Corollary 2.5 define the class of perturbations explicitly: Since P is strictly proper, P has a $\mathscr{U}$-zero at infinity; the only $\mathscr{U}$-poles of P are on the real-axis at 0 and 1; therefore, P and $(I - \Delta)P$ are simultaneously $\mathscr{R}$-stabilizable for any $\Delta \in \mathscr{R}$ such that $\Delta(0) < 1$, and $\Delta(1) < 1$.

3. Conditions for diagonal perturbations

In this section we consider two special cases of diagonal perturbations. The first case assumes that each output channel of the plant is multiplied by the same $\mathscr{R}$-stable transfer function $(1 - \delta)$, i.e., $\Delta = \delta I$. This case is a generalization of constant output perturbations considered in [4]. The second case assumes that all but the jth entry of the diagonal matrix Δ is zero; the nonzero entry δ_j can be any arbitrary $\mathscr{R}$-stable transfer function. The significance of this case is that $\Delta = e_j \delta_j e_j^{\mathrm{T}}$ can be thought of as an *unknown* sensor failure in the jth output channel.

3.1. Diagonal perturbation case $\Delta = \delta I$

As shown in Lemma 2.2, necessary and sufficient conditions for simultaneous $\mathscr{R}$-stabilizability of P and $(I - \Delta)P$ are derived based on real-axis blocking zeros $(\tilde{D}\Delta N)$. When $\Delta = \delta I$, it is possible to explicitly characterize the exact set of such $\mathscr{U}$-zeros based on the invariant factors of the numerator and denominator matrices; we briefly summarize the Smith–McMillan form here for this purpose [7]: Let $P \in \mathrm{R}_{\mathrm{p}}^{n_o \times n_i}$, where rank $P =: r \leqslant \min\{n_o, n_i\}$. There exist $\mathscr{R}$-unimodular matrices $L \in \mathscr{R}^{n_o \times n_o}$, $R \in \mathscr{R}^{n_i \times n_i}$ such that

$$P = L \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o - r)\times(n_i - r)} \end{bmatrix} \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I_{(n_i - r)} \end{bmatrix} R = L \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I_{(n_o - r)} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o - r)\times(n_i - r)} \end{bmatrix} R, \tag{6}$$

$$\Lambda := \mathrm{diag}\,[\lambda_1 \cdots \lambda_r] \ , \quad \Psi := \mathrm{diag}\,[\psi_1 \cdots \psi_r],$$

where, for $j = 1, \ldots, r$, the (numerator and denominator) invariant factors $\lambda_j \in \mathscr{R}$, $\psi_j \in \mathscr{R}$, and ψ_j is biproper; for $j = 1, \ldots, r-1$, λ_j divides λ_{j+1}, and ψ_{j+1} divides ψ_j; for $j = 1, \ldots, r$, the pair (λ_j, ψ_j) is coprime, equivalently, there exist $u_j \in \mathscr{R}$, $v_j \in \mathscr{R}$ such that

$$v_j \psi_j + u_j \lambda_j = 1. \tag{7}$$

Proposition 3.1 (Necessary and sufficient conditions when $\Delta = \delta I$). *Let $P \in \mathrm{R}_{\mathrm{p}}^{n_o \times n_i}$. Let $\Delta = \delta I_{n_o}$, $\delta \in \mathscr{R}$. Consider the Smith–McMillan form (6) of P. For $i \in \{1, \ldots, m\}$, let $s_i \in \mathbb{R} \cap \mathscr{U}$ denote a $\mathscr{U}$-pole of P, which satisfies the following*:

- *$\psi_{\ell_i}(s_i) = 0$ for some $\ell_i \in \{1, \ldots, r\}$, and*
- *$\psi_j(s_i) \neq 0$ for all $j \in \{1, \ldots, r\}$ such that $j \geqslant (\ell_i + 1)$, and*
- *$\lambda_j(s_i) = 0$ for all $j \in \{1, \ldots, r\}$ such that $j \geqslant (\ell_i + 1)$.*

The plants P and $(1 - \delta)P$ are simultaneously $\mathscr{R}$-stabilizable if and only if

(i) *$\delta \neq 1$ at all $\mathscr{U}$-poles of P, and*

(ii) *$(1 - \delta(s_i))^{\ell_i}$ has the same sign for all $i \in \{1, \ldots, m\}$; furthermore, this sign is positive whenever P has real-axis blocking $\mathscr{U}$-zeros or δ has real-axis $\mathscr{U}$-zeros.*

Corollary 3.2 (Sufficient conditions when $\Delta = \delta I$). *Suppose that the assumptions of Proposition* 3.1 *hold.*

(a) *The plants P and $(1 - \delta)P$ are simultaneously $\mathscr{R}$-stabilizable if $\delta \neq 1$ at all $\mathscr{U}$-poles of P and $\delta < 1$ at all real-axis $\mathscr{U}$-poles of P.*

(b) *The plants P and $(1 - \delta)P$ are simultaneously $\mathscr{R}$-stabilizable if $\delta \neq 1$ at all $\mathscr{U}$-poles of P and $\delta < 1$ at all real-axis $\mathscr{U}$-zeros ψ_r and at all coinciding real-axis $\mathscr{U}$-poles and $\mathscr{U}$-zeros of P.*

(c) *The plants P and $(1-\delta)P$ are simultaneously $\mathscr{R}$-stabilizable if $\delta \neq 1$ at all $\mathscr{U}$-poles of P and $\delta(s_i) < 1$ for all $i \in \{1,\dots,m\}$.*

The three sufficient conditions given in Corollary 3.2 for simultaneous $\mathscr{R}$-stabilizability of P and $(1-\delta)P$ are listed in decreasing conservatism.

3.2. Diagonal perturbation case $\Delta = e_j \delta_j e_j^{\mathrm{T}}$

Suppose that one of the n_o sensor channels is multiplicatively perturbed by an unknown $\mathscr{R}$-stable failure and that the location of this failure is known. This failure can be modeled as a diagonal matrix $\Delta = e_j \delta_j e_j^{\mathrm{T}}$, where $j \in \{1,\dots,n_o\}$ is the location of the failure and $\delta_j \in \mathscr{R}$ is arbitrary. Under normal operation of the jth channel, $\delta_j = 0$; all other values of the $\mathscr{R}$-stable δ_j imply a failure and, in particular, $\delta_j = 1$ corresponds to a disconnection failure. For this class of diagonal perturbations, we give necessary and sufficient conditions for simultaneous $\mathscr{R}$-stabilizability of P and $(I-\Delta)P = (I - e_j \delta_j e_j^{\mathrm{T}})P$ in Proposition 3.3. For plants satisfying these conditions, we propose a class of simultaneously $\mathscr{R}$-stabilizing controllers in Proposition 3.4. Note that since $\delta_j \in \mathscr{R}$ is unknown, these conditions are in fact for simultaneous $\mathscr{R}$-stabilizability of a special class of infinitely many plants, and the controllers proposed actually $\mathscr{R}$-stabilize all plants in this class simultaneously.

Proposition 3.3 (Necessary and sufficient conditions when $\Delta = e_j\delta_j e_j^{\mathrm{T}}$). *Let $P \in \mathrm{R}_{\mathrm{p}}^{n_o \times n_i}$. Let $\Delta = e_j\delta_j e_j^{\mathrm{T}}$, for some $j \in \{1,\dots,n_o\}$, where $\delta_j \in \mathscr{R}$. Let $\tilde{D}^{-1}\tilde{N}$ be any LCF of P, where $d_j \in \mathscr{R}^{n_o \times 1}$ denotes the jth column of $\tilde{D}$. The plants P and $(I - e_j\delta_j e_j^{\mathrm{T}})P$ are simultaneously $\mathscr{R}$-stabilizable for all $\delta_j \in \mathscr{R}$ if and only if* $\operatorname{rank} d_j(s) = 1$ *for all $s \in \mathscr{U}$.*

Proposition 3.4 (A class of $\mathscr{R}$-stabilizing controllers for P and $(I - e_j\delta_j e_j^{\mathrm{T}})P$). *Let $P \in \mathrm{R}_{\mathrm{p}}^{n_o \times n_i}$. Let $\Delta = e_j\delta_j e_j^{\mathrm{T}}$, for some $j \in \{1,\dots,n_o\}$, where $\delta_j \in \mathscr{R}$. Let $\tilde{D}^{-1}\tilde{N}$ be any LCF of P, where $d_j \in \mathscr{R}^{n_o \times 1}$ denotes the jth column of $\tilde{D}$. Let* $\operatorname{rank} d_j(s) = 1$ *for all $s \in \mathscr{U}$, equivalently, let d_j have a left-inverse denoted by $y_j^{\mathrm{T}} \in \mathscr{R}^{1 \times n_o}$. Let $U, V, \tilde{U}, \tilde{V} \in \mathscr{M}(\mathscr{R})$ satisfy (2). The controller C given by*

$$C = (V - \hat{Q}\tilde{N} + (U + \hat{Q}\tilde{D})e_j y_j^{\mathrm{T}}\tilde{N})^{-1}(U + \hat{Q}\tilde{D})(I - e_j y_j^{\mathrm{T}}\tilde{D}) \tag{8}$$

simultaneously $\mathscr{R}$-stabilizes P and $(I - e_j\delta_j e_j^{\mathrm{T}})P$ for all $\delta_j \in \mathscr{R}$, where $\hat{Q} \in \mathscr{M}(\mathscr{R})$ satisfies

$$\det(V - \hat{Q}\tilde{N})(\infty) \neq 0 \quad \textit{and} \quad y_j^{\mathrm{T}}(\tilde{V} - N\hat{Q})^{-1}e_j(\infty) = 1 + y_j^{\mathrm{T}}\tilde{N}(V - \hat{Q}\tilde{N})^{-1}(U + \hat{Q}\tilde{D})e_j(\infty) \neq 0. \tag{9}$$

Condition (9) *holds for all $\hat{Q} \in \mathscr{M}(\mathscr{R})$ when P is strictly proper. The controller C given by* (8) *is strictly proper when $\hat{Q} \in \mathscr{M}(\mathscr{R})$ satisfies $\hat{Q}(\infty) = -U(\infty)\tilde{D}^{-1}(\infty)$.*

The expression for the proposed controllers in (8) can be simplified by observing that any $\mathscr{R}$-stabilizing controller $\hat{C}$ for P is given by (1) as $\hat{C} = (V - \hat{Q}\tilde{N})^{-1}(U + Q\tilde{D})$ for some $\hat{Q} \in \mathscr{M}(\mathscr{R})$. Therefore, the controller design suggested in Proposition 3.4 starts with any arbitrary $\mathscr{R}$-stabilizing controller $\hat{C}$ for P and derives the controller C to simultaneously $\mathscr{R}$-stabilize P and $(I - e_j\delta_j e_j^{\mathrm{T}})P$ as

$$C = \hat{C}(I + e_j y_j^{\mathrm{T}}\tilde{D}P\hat{C})^{-1}(I - e_j y_j^{\mathrm{T}}\tilde{D}). \tag{10}$$

Since $\hat{C}$ is any controller that $\mathscr{R}$-stabilizes P, it is proper and hence, $\det(V - \hat{Q}\tilde{N})(\infty) \neq 0$. Condition (9) is then equivalent to

$$1 + y_j^{\mathrm{T}}\tilde{D}P\hat{C}e_j(\infty) \neq 0, \tag{11}$$

which ensures the properness of the proposed controller C.

4. Proofs

Proof of Lemma 2.2. (a) Using an LCF $P=\tilde{D}^{-1}\tilde{N}$ and an RCF $C=N_C D_C^{-1}$, with $\xi_C := D_C^{-1}e_C$, the system $\mathscr{S}(I-\Delta,P,C)$ is described in the bicoprime factorized form

$$\begin{bmatrix} \tilde{D} & -\tilde{N}N_C \\ (I-\Delta) & D_C \end{bmatrix}\begin{bmatrix} y_P \\ \xi_C \end{bmatrix} = \begin{bmatrix} \tilde{N} & 0 \\ 0 & I_{n_o} \end{bmatrix}\begin{bmatrix} u_P \\ u_C \end{bmatrix}, \quad \begin{bmatrix} I_{n_o} & 0 \\ 0 & N_C \end{bmatrix}\begin{bmatrix} y_P \\ \xi_C \end{bmatrix} = \begin{bmatrix} y_P \\ y_C \end{bmatrix}.$$

Similarly, using an RCF $P=ND^{-1}$ and an LCF $C=\tilde{D}_C^{-1}\tilde{N}_C$, with $\xi_P := D^{-1}e_P$, the system $\mathscr{S}(I-\Delta,P,C)$ is described in the bicoprime factorized form

$$\left[\tilde{D}_C D + \tilde{N}_C(I-\Delta)N\right]\xi_P = \left[\tilde{D}_C \;\; \tilde{N}_C\right]\begin{bmatrix} u_P \\ u_C \end{bmatrix}, \quad \begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix}\xi_P + \begin{bmatrix} 0 \\ -u_P \end{bmatrix} = \begin{bmatrix} y_P \\ y_C \end{bmatrix}.$$

It follows using standard arguments that the transfer function $H: u \mapsto y$ is $\mathscr{R}$-stable if and only if (3) holds, equivalently, (4) holds.

(b) The nominal system $\mathscr{S}(P,C)$ and the system $\mathscr{S}(I-\Delta,P,C)$ are both $\mathscr{R}$-stable if and only if C is an $\mathscr{R}$-stabilizing controller for P and (4) holds; condition (5) follows by using the LCF $C=\tilde{D}_C^{-1}\tilde{N}_C = (V-Q\tilde{N})^{-1}(U+Q\tilde{D})$ in (4).

(c) If $\mathscr{S}(I-\Delta,P,C)$ is $\mathscr{R}$-stable, then $\mathscr{R}$-unimodularity of the matrix in (3) implies that the pair $(I-\Delta,\tilde{D})$ is right coprime; therefore this condition is necessary. By (5), P and $(I-\Delta)P$ can be simultaneously $\mathscr{R}$-stabilized if and only if there exists $Q\in\mathscr{M}(\mathscr{R})$ such that $(I-U\Delta N)-Q\tilde{D}\Delta N$ is $\mathscr{R}$-unimodular; it follows by standard strong $\mathscr{R}$-stabilizability results that this $\mathscr{R}$-unimodularity is satisfied for some $Q\in\mathscr{M}(\mathscr{R})$ if and only if $\det(I-U\Delta N)$ has the same sign at all real blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$, provided that the pair $((I-U\Delta N),-\tilde{D}\Delta N)$ is right coprime [7]. The right-coprimeness claim is shown as follows: Since $(I-\Delta,\tilde{D})$ is right coprime, there exist $A,B\in\mathscr{M}(\mathscr{R})$ such that $A(I-\Delta)+B\tilde{D}=I$; therefore, $(I+UA\Delta N)(I-U\Delta N)+(-UB+UA\Delta\tilde{V})(-\tilde{D}\Delta N)=I$ implies right coprimeness. □

Proof of Corollary 2.3. By Lemma 2.2(c), the right coprimeness of $(I-\Delta,\tilde{D})$ is a necessary condition for simultaneous $\mathscr{R}$-stabilizability of P and $(I-\Delta)P$. If $(I-\Delta,\tilde{D})$ is right coprime, then there exist $A,B\in\mathscr{M}(\mathscr{R})$ such that $A(I-\Delta)+B\tilde{D}=I$; therefore, $(A+B\tilde{D})(I-\Delta)+B\tilde{D}\Delta=I$ implies $(I-\Delta,\tilde{D}\Delta)$ is right coprime. Conversely, if $(I-\Delta,\tilde{D}\Delta)$ is right coprime, then there exist $\hat{A},\hat{B}\in\mathscr{M}(\mathscr{R})$ such that $\hat{A}(I-\Delta)+\hat{B}\tilde{D}\Delta=I$; therefore, $(\hat{A}-\hat{B}\tilde{D})(I-\Delta)+\hat{B}\tilde{D}=I$ implies $(I-\Delta,\tilde{D})$ is right coprime. This proves condition (i). If P and $(I-\Delta)P$ are simultaneous $\mathscr{R}$-stabilizable, then $\det(I-U\Delta N)(s_o)$ has the same sign for all $s_o\in\mathbb{R}\cap\mathscr{U}$ such that $\tilde{D}\Delta(s_o)=0$ since these blocking $\mathscr{U}$-zeros are included in the set of real-axis blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$. By (2), $\det(I-U\Delta N)(s_o)=\det(I-NU\Delta)(s_o)=\det(I-(I-\tilde{V}\tilde{D})\Delta)(s_o)=\det(I-\Delta)(s_o)$. Furthermore, if there exist $z_o\in\mathbb{R}\cap\mathscr{U}$ such that $(\Delta P)(z_o)=(\Delta ND^{-1})(z_o)=0$, then $(\tilde{D}\Delta N)(z_o)=0$ implies $\det(I-U\Delta N)(z_o)=1$; since the sign is positive at these real-axis blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$, the sign of $\det(I-\Delta)(s_o)$ must also be positive and the necessity of condition (ii) thus follows. □

Proof of Corollary 2.4. The necessity of the two conditions in Corollary 2.3 was shown for any arbitrary P. The sufficiency of these two conditions for this class of plants follows by Lemma 2.2(c), from the fact that the only real-axis blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$ are those of $(\tilde{D}\Delta)$ and of (ΔN): For all $s_o\in\mathbb{R}\cap\mathscr{U}$ such that $\tilde{D}\Delta(s_o)=0$, by (2), $\det(I-\Delta)(s_o)=\det(I-\Delta+\tilde{V}\tilde{D}\Delta)(s_o)=\det(I-NU\Delta)(s_o)=\det(I-U\Delta N)(s_o)$. Therefore, when $(\tilde{D}\Delta N)$ has no real-axis blocking $\mathscr{U}$-zeros other than those of $(\tilde{D}\Delta)$ and of (ΔN), Corollary 2.3(ii) implies condition (ii) of Lemma 2.2(c). To show that $(\tilde{D}\Delta N)$ has no additional blocking $\mathscr{U}$-zeros, suppose that $(\tilde{D}\Delta N)(s)=0$ for some $s\in\mathbb{R}\cap\mathscr{U}$. If rank $\tilde{D}(s)=n_o$ and rank $N(s)=n_o$, then $\tilde{D}^{-1}(s)$ exists and $N(s)$ has a right-inverse $\hat{N}(s)$; hence $\tilde{D}^{-1}(s)(\tilde{D}\Delta N)(s)\hat{N}(s)=\Delta(s)=0$ implies $(\Delta N)=0$. Since P has no coinciding poles and zeros in $\mathbb{R}\cap\mathscr{U}$, either rank $\tilde{D}(s)<n_o$ or rank $N(s)<n_o$ but not both. If rank $\tilde{D}(s)=n_o$, then $\tilde{D}^{-1}(s)$ exists and hence, $\tilde{D}^{-1}(s)(\tilde{D}\Delta N)(s)=(\Delta N)(s)=0$. If rank $N(s)=n_o$, then $N(s)$ has a right-inverse $\hat{N}(s)$ and hence, $(\tilde{D}\Delta N)(s)\hat{N}(s)=(\tilde{D}\Delta)(s)=0$. Note that the case of rank $\tilde{D}(s)=n_o$ and rank $N(s)=n_o$ implies $\Delta(s)=0$, which is a blocking $\mathscr{U}$-zero of both $(\tilde{D}\Delta)$ and (ΔN). □

Proof of Corollary 2.5. Since scalar plants have no pole-zero coincidences, this is a special case of the class of plants in Corollary 2.4. For scalar $P=\tilde{D}^{-1}\tilde{N}$ and Δ, the pair $(I-\Delta,\tilde{D})$ is coprime if and only if $(I-\Delta)(s)\neq 0$ for all $s\in\mathscr{U}$ such that $\tilde{D}(s)=0$, which is equivalent to condition (i). The set of $\mathscr{U}$-zeros of the product $(\tilde{D}\Delta)$ is simply the union of the corresponding sets for $\tilde{D}$ and for Δ. Therefore, $(I-\Delta)(s)$ must have the same sign for all $s\in\mathbb{R}\cap\mathscr{U}$ such that $\tilde{D}(s)=0$ and this sign must be positive whenever (ΔP) has real-axis $\mathscr{U}$-zeros. Again, the $\mathscr{U}$-zeros of the product (ΔP) are equivalent to the $\mathscr{U}$-zeros of the individual scalar transfer functions. □

Proof of Proposition 3.1. The plants P and $(1-\delta)P$ are simultaneously $\mathscr{R}$-stabilizable if and only if Lemma 2.2(c) holds; now $((1-\delta),\tilde{D})$ is right coprime if and only if $(1-\delta(s))\neq 0$ for all $s\in\mathscr{U}$ such that rank $\tilde{D}(s)<n_o$, equivalently, condition (i) of Proposition 3.1 holds. Next we investigate the sign of $\det(I_{n_i}-U\delta N)$ at the real blocking $\mathscr{U}$-zeros of $\tilde{D}\delta N$. Consider the Smith–McMillan form (6) of P. Any RCF ND^{-1} and any LCF $\tilde{D}^{-1}\tilde{N}$ of P is given in terms of this Smith–McMillan form as

$$(N,D)=\left(L\begin{bmatrix}\Lambda & 0\\ 0 & 0_{(n_o-r)\times(n_i-r)}\end{bmatrix}M,\ R^{-1}\begin{bmatrix}\Psi & 0\\ 0 & I_{(n_i-r)}\end{bmatrix}M\right), \tag{12}$$

$$(\tilde{D},\tilde{N})=\left(\tilde{M}\begin{bmatrix}\Psi & 0\\ 0 & I_{(n_o-r)}\end{bmatrix}L^{-1},\ \tilde{M}\begin{bmatrix}\Lambda & 0\\ 0 & 0_{(n_o-r)\times(n_i-r)}\end{bmatrix}R\right) \tag{13}$$

for some $\mathscr{R}$-unimodular $M\in\mathscr{M}(\mathscr{R})$ and for some $\mathscr{R}$-unimodular $\tilde{M}\in\mathscr{M}(\mathscr{R})$. Let $U_D:=\operatorname{diag}[u_1\ \cdots\ u_r]$, $V_D:=\operatorname{diag}[v_1\ \cdots\ v_r]$; then by (7), $(V_D\Psi+U_D\Lambda)=I_r$. A solution for $U,V,\tilde{U},\tilde{V}$ satisfying (2) is

$$U:=M^{-1}\begin{bmatrix}U_D & 0\\ 0 & 0_{(n_i-r)\times(n_o-r)}\end{bmatrix}L^{-1},\quad V:=M^{-1}\begin{bmatrix}V_D & 0\\ 0 & I_{(n_i-r)}\end{bmatrix}R,$$

$$\tilde{U}:=R^{-1}\begin{bmatrix}U_D & 0\\ 0 & 0_{(n_i-r)\times(n_o-r)}\end{bmatrix}\tilde{M}^{-1},\quad \tilde{V}:=L\begin{bmatrix}V_D & 0\\ 0 & I_{(n_o-r)}\end{bmatrix}\tilde{M}^{-1}. \tag{14}$$

By (14),

$$\det(I_{n_i}-U\delta N)=\det(I_{n_i}-\delta UN)=\det\left(I_{n_i}-\delta\begin{bmatrix}U_D\Lambda & 0\\ 0 & 0_{(n_i-r)\times(n_i-r)}\end{bmatrix}\right)=\prod_{j=1}^{r}(1-\delta u_j\lambda_j). \tag{15}$$

Since δ is scalar, $s_o\in\mathscr{U}$ is a blocking $\mathscr{U}$-zero of $(\delta\tilde{D}N)$ if and only if s_o is a $\mathscr{U}$-zero of δ or a blocking $\mathscr{U}$-zero of $(\tilde{D}N)$. By (12) and (13), since $M,\ \tilde{M}\in\mathscr{M}(\mathscr{R})$ are $\mathscr{R}$-unimodular, $(\delta\tilde{D}N)(s_o)=0$ if and only if $(\Psi\Lambda)(s_o)=0$, equivalently, $(\psi_j\lambda_j)(s_o)=0$ for $j=1,\ldots,r$. Since (λ_j,ψ_j) is coprime, $(\psi_j\lambda_j)(s_o)=0$ means that for each j, either $\psi_j(s_o)=0$ or $\lambda_j(s_o)=0$, but not both. For any $s_o\in\mathbb{R}\cap\mathscr{U}$ such that $(\tilde{D}N)(s_o)=0$, there are two cases to consider:

Case 1: Suppose that $\psi_j(s_o)\neq 0$ for all $j\in\{1,\ldots,r\}$, equivalently, $\det\tilde{D}(s_o)\neq 0$; then $(\tilde{D}N)(s_o)=0$ implies $N(s_o)=0$, i.e., s_o is a blocking $\mathscr{U}$-zero of P. Therefore $\det(I-\delta UN)(s_o)=1$.

Case 2: Suppose that $\psi_\ell(s_o)=0$ for some $\ell\in\{1,\ldots,r\}$, but $\psi_j(s_o)\neq 0$ for $j>\ell$; then $\psi_j(s_o)=0$ for all $j\leqslant\ell$ because ψ_{j+1} divides ψ_j. Since (λ_j,ψ_j) is coprime, $\lambda_\ell(s_o)\neq 0$. But $(\psi_j\lambda_j)(s_o)=0$ for $j=1,\ldots,r$ implies therefore $\lambda_j(s_o)=0$ for all $j\geqslant(\ell+1)$. By (7), $(u_j\lambda_j)(s_o)=1$ for all $j\leqslant\ell$ and $(u_j\lambda_j)(s_o)=0$ for all $j\geqslant(\ell+1)$. By (15),

$$\det(I-\delta UN)(s_o)=\prod_{j=1}^{r}(1-\delta(u_j\lambda_j)(s_o))=(1-\delta(s_o))^{\ell}; \tag{16}$$

Note that if $\ell=r$, then the smallest invariant-factor $\psi_r(s_o)=0$ and hence, $\lambda_j(s_o)\neq 0$ for $j=1,\ldots,r$.

Suppose that $\tilde{D}N$ has m real blocking $\mathscr{U}$-zeros $s_1,\ldots,s_m$ as described in Case 2 and that the corresponding indices are $\ell_1,\ldots,\ell_m$, i.e., $\psi_{\ell_i}(s_i)=0$ for $i\in\{1,\ldots,m\}$. By (16), the sign of $\det(I-\delta UN)$ remains the same at all s_i if and only if the sign of $(1-\delta)^{\ell}$ remains the same for all $i\in\{1,\ldots,m\}$. Furthermore, if P has any blocking real-axis $\mathscr{U}$-zeros as in Case 1, or if δ has any zeros in $\mathbb{R}\cap\mathscr{U}$, then this sign must match the positive sign of $\det(I-\delta UN)=1$. □

Proof of Corollary 3.2. The plants P and $(1-\delta)P$ are simultaneously $\mathscr{R}$-stabilizable if and only if the two conditions of Proposition 3.1 hold at the $\mathscr{U}$-poles s_i of P. If the sign of $(1-\delta(s_i))>0$ at all these poles, then $(1-\delta(s_i))^{\ell_i}$ obviously has the same sign regardless of the indices ℓ_i. Therefore, part (c) of Corollary 3.2 is a sufficient condition. But since the $\mathscr{U}$-poles described as s_i are a subset of those described in part (b), and these in turn are a subset of all real-axis $\mathscr{U}$-poles of P, the sufficiency of part (a) and part (b) are also obvious. □

Proof of Proposition 3.3. By Lemma 2.2 (c), the plants P and $(I-e_j\delta_j e_j^{\mathrm{T}})P$ are simultaneously $\mathscr{R}$-stabilizable if and only if $((I-e_j\delta_j e_j^{\mathrm{T}}),\tilde{D})$ is right-coprime and $\det(I-Ue_j\delta_j e_j^{\mathrm{T}}N)=1-\delta_j e_j^{\mathrm{T}}NUe_j$ has the same sign at all $s\in\mathbb{R}\cap\mathscr{U}$ such that $(\tilde{D}e_j\delta_j e_j^{\mathrm{T}}N)(s)=(\delta_j d_j e_j^{\mathrm{T}}N)(s)=0$. If $((I-\delta_j e_j e_j^{\mathrm{T}}),\tilde{D})$ is right-coprime for all $\delta_j\in\mathscr{R}$, then it is right coprime for $\delta_j=1$, i.e., the entries of the jth column of $\tilde{D}$ cannot all become zero for some $s\in\mathscr{U}$. This proves necessity of rank $d_j(s)=1$ for all $s\in\mathscr{U}$. To prove sufficiency, observe that if all entries of d_j do not become zero for some $s\in\mathscr{U}$, then $(\delta_j d_j e_j^{\mathrm{T}}N)(s)=0$ if and only if either $\delta_j(s)=0$ or the jth row of $N(s)$, $e_j^{\mathrm{T}}N(s)=0$. Since $1-(\delta_j e_j^{\mathrm{T}}NUe_j)(s)=1$ for all such s, the sign is always positive for all real-axis blocking $\mathscr{U}$-zeros of $(\tilde{D}\Delta N)$ as required. □

Proof of Proposition 3.4. Let $\hat{Q}\in\mathscr{M}(\mathscr{R})$ be such that (9) holds, i.e., $\det(V-\hat{Q}\tilde{N})(\infty)\neq 0$, equivalently, $(\tilde{V}-N\hat{Q})(\infty)\neq 0$. If P is strictly-proper, then $\det(V-\hat{Q}\tilde{N})(\infty)\neq 0$ for all $\hat{Q}\in\mathscr{M}(\mathscr{R})$. By (9), $\hat{Q}$ also satisfies $(y_j^{\mathrm{T}}(\tilde{V}-N\hat{Q})^{-1}e_j)(\infty)\neq 0$; by (2), $(\tilde{V}-N\hat{Q})^{-1}=\tilde{D}+\tilde{N}(\tilde{V}-N\hat{Q})^{-1}(U+\hat{Q}\tilde{D})$ and hence, since $y_j^{\mathrm{T}}\tilde{D}e_j=y_j^{\mathrm{T}}d_j=1$, this condition is equivalent to $(y_j^{\mathrm{T}}\tilde{N}(\tilde{V}-N\hat{Q})^{-1}(U+\hat{Q}\tilde{D})e_j)(\infty)\neq -1$. If P is strictly-proper, equivalently, $\tilde{N}\in\mathscr{M}(\mathscr{R})$, then this condition holds for all $\hat{Q}\in\mathscr{M}(\mathscr{R})$. To show that $\hat{Q}\in\mathscr{M}(\mathscr{R})$ satisfying condition (9) exists for any P, one solution would be to restrict $\hat{Q}(\infty)=-U(\infty)\tilde{D}^{-1}(\infty)$, which implies $(U+\hat{Q}\tilde{D})(\infty)=0$.

Define $\tilde{N}_C:=(U+\hat{Q}\tilde{D})$, $\tilde{D}_C:=(V-\hat{Q}\tilde{N})$, $N_C:=(\tilde{U}+D\hat{Q})$, $D_C:=(\tilde{V}-N\hat{Q})$, $\hat{C}:=\tilde{D}_C^{-1}\tilde{N}_C=N_C D_C^{-1}$. Let $Q\in\mathscr{M}(\mathscr{R})$ be defined as

$$Q=\hat{Q}-(U+\hat{Q}\tilde{D})e_j y_j^{\mathrm{T}}=\hat{Q}-\tilde{N}_C e_j y_j^{\mathrm{T}}. \tag{17}$$

Let $C:=(V-Q\tilde{N})^{-1}(U+Q\tilde{D})=(\tilde{D}_C+\tilde{N}_C e_j y_j^{\mathrm{T}}\tilde{N})^{-1}(\tilde{N}_C-\tilde{N}_C e_j y_j^{\mathrm{T}}\tilde{D})$, as proposed in (8); note that C is in the form given by (1). Now apply Lemma 2.2(b); using Q given by (17) in (5), $I-(U+Q\tilde{D})\delta_j e_j e_j^{\mathrm{T}}N=I-\delta_j(U+\hat{Q}\tilde{D})(I-e_j y_j^{\mathrm{T}}\tilde{D})e_j e_j^{\mathrm{T}}N=I-\delta_j\tilde{N}_C e_j(1-y_j^{\mathrm{T}}\tilde{D}e_j)e_j^{\mathrm{T}}N=I$ since $y_j^{\mathrm{T}}\tilde{D}e_j=y_j^{\mathrm{T}}d_j=1$. Therefore, (5) holds for all $\delta_j\in\mathscr{R}$. It remains to show that $(V-Q\tilde{N})$ is biproper to establish that C is a simultaneously $\mathscr{R}$-stabilizing controller for the plants P and $(I-\delta_j e_j e_j^{\mathrm{T}})P$ for all $\delta_j\in\mathscr{R}$: Since $\tilde{D}_C=(V-\hat{Q}\tilde{N})$ is biproper by (9), $(V-Q\tilde{N})=(\tilde{D}_C+\tilde{N}_C e_j y_j^{\mathrm{T}}\tilde{N})=(I+\tilde{N}_C e_j y_j^{\mathrm{T}}\tilde{N}\tilde{D}_C^{-1})\tilde{D}_C$ is biproper if and only if $\det(I+\tilde{N}_C e_j y_j^{\mathrm{T}}\tilde{N}\tilde{D}_C^{-1})(\infty)\neq 0$, equivalently, $1+(y_j^{\mathrm{T}}\tilde{N}\tilde{D}_C^{-1}\tilde{N}_C e_j)(\infty)\neq 0$. But as shown above, this is equivalent to $(y_j^{\mathrm{T}}(\tilde{V}-N\hat{Q})^{-1}e_j)(\infty)\neq 0$, which holds since $\hat{Q}$ satisfies (9). Note that $\tilde{N}\tilde{D}_C^{-1}\tilde{N}_C=\tilde{N}\hat{C}=\tilde{D}P\hat{C}$. □

5. Conclusions

There exists a single controller that simultaneously $\mathscr{R}$-stabilizes two given plants P_1 and P_2 if and only if a pair of matrices associated with these two plants satisfy the parity-interlacing property [7]. While the parity-interlacing property test requires calculation of this pair of matrices to answer the question of simultaneous $\mathscr{R}$-stabilizability for two arbitrarily selected plants, it may be possible to arrive at a conclusion without this calculation for special cases, such as the one considered in this paper. Motivated by the importance of designing controllers for a plant under different modes of operation, necessary and sufficient conditions are obtained for the existence of simultaneously $\mathscr{R}$-stabilizing controllers in the case that the second plant $(I-\Delta)P$

represents the nominal plant P under a known $\mathscr{R}$-stable multiplicative perturbation Δ. These conditions are important for determining the class of perturbations for which a single controller can be designed to achieve simultaneous $\mathscr{R}$-stabilization. A special class of diagonal unknown perturbations, $\Delta = e_j \delta_j e_j^{\mathrm{T}}$, is also considered and controllers $\mathscr{R}$-stabilizing the entire class $(1 - e_j \delta_j e_j^{\mathrm{T}})P$ simultaneously with the nominal P are proposed. Possible extensions of this work include developing similar explicit design methodologies for the general cases of Δ using the simultaneous $\mathscr{R}$-stabilizability conditions given here.

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