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Two-stage Controller Design with Integral Action and Decoupling

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Abstract

Two-stage stabilizing controller design methods that achieve type-m integral action and diagonal input-output transfer-functions are developed for linear, time-invariant, multi-input multi-output feedback systems. Integral action can be achieved with the proposed configurations if and only if the given plant is full row-rank and has no transmission zeros at zero. All stabilizing controllers with at least m integrators in each output channel are explicitly characterized. A parametrization of all decoupling controllers which also achieve type-m integral action is obtained.

1 Introduction

The well-known parametrization of all stabilizing controllers in the standard linear, time-invariant (LTI) multi-input multi-output (MIMO) feedback system makes it possible to characterize all achievable transfer-functions for a given plant [6]. Two-stage controller configurations (Fig. 1, Fig. 2) are considered here instead of the standard system configuration with one controller. The advantage is that the given plant P is stabilized by the first controller C_f in each of the subsystem configurations $\mathcal{S}(P, C_f)$ and $\widehat{\mathcal{S}}(P, C_f)$; the second controller is implemented with m integrators in each channel and is used to achieve integral action and decoupling. In order to achieve integral action with the second controller, the first controller C_f is chosen so that the transfer-function from e_1 to y has no (transmission) zeros at zero; C_f can be designed to satisfy this condition if and only if $P(0)$ is full row-rank. Decoupling can be achieved with these two-stage designs if and only if P is full row-rank; therefore, for any plant that satisfies the necessary condition for integral action, it is possible to achieve decoupling as well.

Using the standard one-degree-of-freedom design, it is possible to achieve decoupling only for a restricted set of plants; a sufficient condition to achieve decoupling is that P is full row-rank and has no pole-zero coincidences in the region of instability [5]. Various parametrizations of all decoupling controllers for plants satisfying this sufficient condition are available [2], as well as results on necessary and sufficient conditions to achieve decoupling in the standard feedback configuration [3]. Although decoupling is achievable for any full row-rank P using two-degrees-of-freedom design as in [1], the two-stage design here can also achieve robust asymptotic tracking, and type-m integral action [4].

Due to the algebraic framework, the results apply to continuous-time as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at $s = 0$ would be interpreted at 1.

Notation: Let $\mathcal{U}$ contain the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, polynomials, rational functions, proper rational functions with no poles in the region of instability $\mathcal{U}$, proper and strictly proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathbb{R}[s]$, $\mathcal{F}$, $\mathcal{R}$, $\mathrm{R_p}$, $\mathrm{R_{sp}}$, respectively. The set of matrices with entries in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called $\mathcal{R}$-stable iff $M \in \mathcal{M}(\mathcal{R})$; $M \in \mathcal{M}(\mathcal{R})$ is called $\mathcal{R}$-unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{R})$. A right-coprime-factorization (RCF) and a left-coprime-factorization (LCF) of $P \in \mathrm{R_p}^{n_o \times n_i}$ are denoted by ND^{-1} and $\widetilde{D}^{-1}\widetilde{N}$, where $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$, $N, D, \widetilde{N}, \widetilde{D} \in \mathcal{M}(\mathcal{R})$, D and $\widetilde{D}$ are biproper. Let $\mathrm{rank}P = r$; $s_o \in \mathcal{U}$ is called a (transmission) $\mathcal{U}$-zero of P iff $\mathrm{rank}P(z_o) < r$, equivalently, $\mathrm{rank}N(z_o) = \mathrm{rank}\widetilde{N}(z_o) < r$.

2 Stability

Consider the LTI, MIMO control systems $\mathcal{S}(P, C_f, C)$, $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ in Fig. 1, Fig. 2; $\mathcal{S}(P, C_f, C)$ and $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ are well-posed, where the subsystems $\mathcal{S}(P, C_f)$ and $\widehat{\mathcal{S}}(P, C_f)$ are also well-posed. The plant and the controllers are represented by their transfer-functions $P \in \mathrm{R_p}^{n_o \times n_i}$, $C_f \in \mathrm{R_p}^{n_i \times n_o}$, $C \in \mathrm{R_p}^{n_o \times n_o}$,

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$\widehat{C} \in \mathrm{R_p}^{n_o \times n_o}$, respectively; P, C_f, C, $\widehat{C}$ have no hidden modes associated with eigenvalues in $\mathcal{U}$.

2.1 Definitions (*$\mathcal{R}$-stability*):
The system $\mathcal{S}(P, C_f, C)$ is said to be $\mathcal{R}$-stable iff the transfer-function H from $[u^T \quad u_1^T \quad u_2^T]^T$ to $[y^T \quad y_1^T \quad y_2^T]^T$ in $\mathcal{S}(P, C_f, C)$ is $\mathcal{R}$-stable. The subsystem $\mathcal{S}(P, C_f)$ is said to be $\mathcal{R}$-stable iff the transfer-function H_f from $[e_1^T \quad u_2^T]^T$ to $[y^T \quad y_2^T]^T$ is $\mathcal{R}$-stable. The system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is said to be $\mathcal{R}$-stable iff the transfer-function $\widehat{H}$ from $[u^T \ u_1^T \ u_2^T]^T$ to $[y^T \ y_1^T \ y_2^T]^T$ is $\mathcal{R}$-stable. The subsystem $\widehat{\mathcal{S}}(P, C_f)$ is said to be $\mathcal{R}$-stable iff the transfer-function $\widehat{H}_f$ from $[e_1^T \ u_2^T]^T$ to $[y^T \ y_2^T]^T$ is $\mathcal{R}$-stable. The controller C_f is called an $\mathcal{R}$-stabilizing controller for the subsystem $\mathcal{S}(P, C_f)$ iff $C_f \in \mathcal{M}(\mathrm{R_p})$ $H_f \in \mathcal{M}(\mathcal{R})$. Similarly, C_f is called an $\mathcal{R}$-stabilizing controller for the subsystem $\widehat{\mathcal{S}}(P, C_f)$ iff $C_f \in \mathcal{M}(\mathrm{R_p})$ and $\widehat{H}_f \in \mathcal{M}(\mathcal{R})$. Let C_f be an $\mathcal{R}$-stabilizing controller for the subsystem $\mathcal{S}(P, C_f)$. The controller C is said to be an $\mathcal{R}$-stabilizing controller for the system $\mathcal{S}(P, C_f, C)$ iff $C \in \mathcal{M}(\mathrm{R_p})$ and $H \in \mathcal{M}(\mathcal{R})$. The $\mathcal{R}$-stabilizing controller C is said to achieve type-m integral action iff the input-error transfer-function H_{eu} from u to e has m blocking zeros at $s = 0$; it is said to achieve decoupling iff the input-output transfer-function H_{yu} from u to y is diagonal and nonsingular. Similarly, let C_f be an $\mathcal{R}$-stabilizing controller for the subsystem $\mathcal{S}(P, C_f)$. The controller $\widehat{C}$ is said to be an $\mathcal{R}$-stabilizing controller for the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ iff $\widehat{C} \in \mathcal{M}(\mathrm{R_p})$ and $\widehat{H} \in \mathcal{M}(\mathcal{R})$. The $\mathcal{R}$-stabilizing controller $\widehat{C}$ is said to achieve type-m integral action iff the input-error transfer-function $\widehat{H}_{eu}$ from u to e has m blocking zeros at $s = 0$; it is said to achieve decoupling iff the input-output transfer-function $\widehat{H}_{yu}$ from u to y is diagonal and nonsingular.

2.2 Lemma (*Conditions for $\mathcal{R}$-stability*):
Let ND^{-1} be any RCF, $\widetilde{D}^{-1}\widetilde{N}$ be any LCF of $P \in \mathrm{R_p}^{n_o \times n_i}$; let $\widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ be any LCF of C_f; $N_c D_c^{-1}$ be any RCF of C, $\widehat{N}_c \widehat{D}_c^{-1}$ be any RCF of $\widehat{C}$.
a) The system $\mathcal{S}(P, C_f, C)$ is $\mathcal{R}$-stable if and only if

$$\begin{bmatrix} \widetilde{D}_{cf} D + \widetilde{N}_{cf} N & -\widetilde{N}_{cf} N_c \\ N & D_c \end{bmatrix} \text{ is } \mathcal{R}\text{-unimodular.} \quad (1)$$

The subsystem $\mathcal{S}(P, C_f)$ is $\mathcal{R}$-stable if and only if

$$M_f := \left(\widetilde{D}_{cf} D + \widetilde{N}_{cf} N \right) \text{ is } \mathcal{R}\text{-unimodular.} \quad (2)$$

$\mathcal{S}(P, C_f, C)$ and subsystem $\mathcal{S}(P, C_f)$ are both $\mathcal{R}$-stable if and only if M_f is $\mathcal{R}$-unimodular and

$$D_H := \left(D_c + N M_f^{-1} \widetilde{N}_{cf} N_c \right) \text{ is } \mathcal{R}\text{-unimodular.} \quad (3)$$

b) The system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is $\mathcal{R}$-stable if and only if

$$\begin{bmatrix} \widetilde{D}_{cf} D + \widetilde{N}_{cf} N & -\widetilde{D}_{cf} \widehat{N}_c \\ N & \widehat{D}_c \end{bmatrix} \text{ is } \mathcal{R}\text{-unimodular.} \quad (4)$$

The subsystem $\widehat{\mathcal{S}}(P, C_f)$ is $\mathcal{R}$-stable if and only if (2) holds. $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ and $\widehat{\mathcal{S}}(P, C_f)$ are both $\mathcal{R}$-stable if and only if M_f is $\mathcal{R}$-unimodular and

$$\widehat{D}_H := \left(\widehat{D}_c + N M_f^{-1} \widetilde{D}_{cf} \widehat{N}_c \right) \text{ is } \mathcal{R}\text{-unimodular.} \quad (5)$$

c) The controller $C_f \in \mathrm{R_p}^{n_i \times n_o}$ is an $\mathcal{R}$-stabilizing controller for the subsystem $\mathcal{S}(P, C_f)$ (equivalently, for the subsystem $\widehat{\mathcal{S}}(P, C_f)$) if and only if

$$C_f = (V - Q_f \widetilde{N})^{-1}(U + Q_f \widetilde{D}) = (\widetilde{U} + D Q_f)(\widetilde{V} - N Q_f)^{-1} \quad (6)$$

where $V, U, \widetilde{V}, \widetilde{U} \in \mathcal{M}(\mathcal{R})$ satisfy

$$VD + UN = I_{n_i}, \ \widetilde{D}\widetilde{V} + \widetilde{U}\widetilde{N} = I_{n_o}, \ V\widetilde{U} = U\widetilde{V}, \quad (7)$$

and $Q_f \in \mathcal{R}^{n_i \times n_o}$ is such that $C_f \in \mathcal{M}(\mathrm{R_p})$, equivalently, $(V - Q_f \widetilde{N})$ is biproper. If $P \in \mathcal{M}(\mathrm{R_{sp}})$, then $C_f \in \mathcal{M}(\mathrm{R_p})$ is proper for all $Q_f \in \mathcal{R}^{n_i \times n_o}$.
d) The pair (C_f, C) is an $\mathcal{R}$-stabilizing controller pair for the subsystem $\mathcal{S}(P, C_f)$ and the system $\mathcal{S}(P, C_f, C)$ if and only if C_f is given by (6) and C is given by

$$C = (I_{n_o} - QN\widetilde{N}_{cf})^{-1} Q = Q(I_{n_o} - N\widetilde{N}_{cf} Q)^{-1}, \quad (8)$$

where $Q \in \mathcal{R}^{n_o \times n_o}$ is such that $C \in \mathcal{M}(\mathrm{R_p})$, equivalently, $(I_{n_o} - QN\widetilde{N}_{cf})$ is biproper. If P or C_f is strictly-proper, then $C \in \mathcal{M}(\mathrm{R_p})$ for all $Q \in \mathcal{R}^{n_o \times n_o}$. In (8), $\widetilde{N}_{cf} = (U + Q_f \widetilde{D})$ for some $Q_f \in \mathcal{R}^{n_i \times n_o}$ such that $(V - Q_f \widetilde{N})$ is biproper.
Similarly, $(C_f, \widehat{C})$ is an $\mathcal{R}$-stabilizing controller pair for the subsystem $\widehat{\mathcal{S}}(P, C_f)$ and the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ if and only if C_f is given by (6) and $\widehat{C}$ is given by

$$\widehat{C} = (I_{n_i} - \widehat{Q}N\widetilde{D}_{cf})^{-1}\widehat{Q} = \widehat{Q}(I_{n_o} - N\widetilde{D}_{cf}\widehat{Q})^{-1}, \quad (9)$$

where $\widehat{Q} \in \mathcal{M}(\mathcal{R})$ is such that $(I_{n_i} - \widehat{Q}N\widetilde{N}_{cf})$ is biproper. If $P \in \mathcal{M}(\mathrm{R_{sp}})$, then $\widehat{C} \in \mathcal{M}(\mathrm{R_p})$ for all $\widehat{Q} \in \mathcal{M}(\mathcal{R})$. In (9), $\widetilde{D}_{cf} = (V - Q_f \widetilde{D})$ for some $Q_f \in \mathcal{M}(\mathcal{R})$ such that $\widetilde{D}_{cf}$ is biproper. □

In $\mathcal{S}(P, C_f, C)$, the transfer-function from e_1 to y, $H_{ye_1} = PC_f(I + PC_f)^{-1} \in \mathcal{R}^{n_o \times n_o}$ is achievable using an $\mathcal{R}$-stabilizing controller C_f if and only if $H_{ye_1} = N\widetilde{N}_{cf} = N(U + Q_f \widetilde{D})$. The input-output transfer-function from u to y, $H_{yu} = (I + PC_f + PC_f C)^{-1} PC_f C = H_{ye_1} C (I + H_{ye_1} C)^{-1} \in \mathcal{R}^{n_o \times n_o}$ is achievable using an $\mathcal{R}$-stabilizing controller pair (C_f, C) if and only if

$$H_{yu} = N\widetilde{N}_{cf} N_c = N(U + Q_f \widetilde{D})\, Q, \quad (10)$$

where $Q_f \in \mathcal{R}^{n_i \times n_o}$, $Q \in \mathcal{R}^{n_o \times n_o}$ are such that $(V - Q_f \widetilde{N})$ is biproper and $(I_{n_o} - QN\widetilde{N}_{cf})$ is biproper.

In $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$, the transfer-function from e_1 to y, $\widehat{H}_{ye_1} = P(I + C_f P)^{-1} \in \mathcal{R}^{n_o \times n_i}$ is achievable using an $\mathcal{R}$-stabilizing controller C_f if and only if $\widehat{H}_{ye_1} = N\widetilde{D}_{cf} = N(V - Q_f\widetilde{N})$. The input-output transfer-function from u to y, $\widehat{H}_{yu} = (I + PC_f + PC)^{-1} PC = \widehat{H}_{ye_1}C(I + \widehat{H}_{ye_1}C)^{-1} \in \mathcal{R}^{n_o \times n_o}$ is achievable using an $\mathcal{R}$-stabilizing controller pair $(C_f, \widehat{C})$ if and only if

$$\widehat{H}_{yu} = N\widetilde{D}_{cf}\widehat{N}_c = N(V - Q_f\widetilde{D})\widehat{Q}, \qquad (11)$$

where $Q_f \in \mathcal{R}^{n_i \times n_o}$, $\widehat{Q} \in \mathcal{R}^{n_i \times n_o}$ are such that $(V - Q_f\widetilde{N})$ is biproper and $(I_{n_i} - \widehat{Q}N\widetilde{D}_{cf})$ is biproper.

3 Type-m integral action

In this section it is required that the system $\mathcal{S}(P, C_f, C)$ is $\mathcal{R}$-stable and has (at least) type-m integral action in each channel, while the subsystem $\mathcal{S}(P, C_f)$ is $\mathcal{R}$-stable; equivalently, the input-error transfer-function $H_{eu} = I - H_{yu}$ is required to have (at least) m blocking zeros at $s = 0$, where $m \geq 1$ is a given integer. Similarly, the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is required to be $\mathcal{R}$-stable and have (at least) type-m integral action in each channel, while the subsystem $\widehat{\mathcal{S}}(P, C_f)$ is $\mathcal{R}$-stable; equivalently, $\widehat{H}_{eu} = I - \widehat{H}_{yu}$ is required to have (at least) m blocking zeros at $s = 0$.

Suppose that $\mathcal{S}(P, C_f, C)$ and $\mathcal{S}(P, C_f)$ are $\mathcal{R}$-stable. The input-error transfer-function $H_{eu} = I - N\widetilde{N}_{cf}N_c = D_c$ has m zeros at $s = 0$ if and only if $D_c = \frac{s^m}{(s+a)^m}D_s$ for some biproper $D_s \in \mathcal{R}^{n_o \times n_o}$, and some $-a \in \mathbb{R} \setminus \mathcal{U}$. Then the controller $C = N_cD_c^{-1} = \frac{(s+a)^m}{s^m}N_cD_s^{-1} =: \frac{(s+a)^m}{s^m}C_s$ has no hidden modes associated with eigenvalues in $\mathcal{U}$ only if C_s has no $\mathcal{U}$-zeros at $s = 0$, i.e., rank$N_c(0) = n_o$. By Lemma 2.2(a), the system $\mathcal{S}(P, C_f, C)$ is $\mathcal{R}$-stable if and only if (1) holds, which implies rank$N(0) = n_o$ and rank$\widetilde{N}_{cf}(0) = n_o$; therefore a necessary condition for $\mathcal{R}$-stability of $\mathcal{S}(P, C_f, C)$ is that P and C_f have no (transmission) zeros at $s = 0$ and $n_o \leq n_i$. Furthermore, $\mathcal{S}(P, C_f, C)$ and $\mathcal{S}(P, C_f)$ are both $\mathcal{R}$-stable if and only if (2) and (3) hold, i.e., with the controller C_f as in (6), $D_c + N\widetilde{N}_{cf}N_c = \frac{s^m}{(s+a)^m}D_s + N\widetilde{N}_{cf}N_c$ is $\mathcal{R}$-unimodular. Therefore, another necessary condition for $\mathcal{R}$-stability of $\mathcal{S}(P, C_f, C)$ and $\mathcal{S}(P, C_f)$ is rank$(N\widetilde{N}_{cf})(0) = $ rank$(N(U + Q_f\widetilde{D}))(0) = n_o$. It is shown in Lemma 3.1(a) that there exists $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ such that the subsystem $\mathcal{S}(P, C_f)$ is $\mathcal{R}$-stable and rank$(N\widetilde{N}_{cf})(0) = n_o$ whenever P has no (transmission) zeros at $s = 0$ and $n_o \leq n_i$.

Similarly, when $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ and $\widehat{\mathcal{S}}(P, C_f)$ are $\mathcal{R}$-stable, $\widehat{H}_{eu} = I - N\widetilde{D}_{cf}\widehat{N}_c = \widehat{D}_c$ has m zeros at $s = 0$ if and only if $\widehat{D}_c = \frac{s^m}{(s+a)^m}\widehat{D}_s$ for some biproper $\widehat{D}_s \in \mathcal{R}^{n_o \times n_o}$. Since $\widehat{C} = \widehat{N}_c\widehat{D}_c^{-1} = \frac{(s+a)^m}{s^m}\widehat{N}_c\widehat{D}_s^{-1} =: \frac{(s+a)^m}{s^m}\widehat{C}_s$ has no hidden modes associated with eigenvalues in $\mathcal{U}$ if and only if $\widehat{C}_s$ has no $\mathcal{U}$-zeros at $s = 0$, i.e., rank$\widehat{N}_c(0) = n_o$. By Lemma 2.2(b), the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is $\mathcal{R}$-stable if and only if (4) holds, which implies rank$N(0) = n_o \leq n_i$ and rank$\widetilde{D}_{cf}(0) = n_i$; therefore, a necessary condition for $\mathcal{R}$-stability of $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is that P has no (transmission) zeros and C_f has no poles at $s = 0$. Furthermore, $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ and $\widehat{\mathcal{S}}(P, C_f)$ are both $\mathcal{R}$-stable if and only if (2) and (5) hold, i.e., with C_f as in (6), $\widehat{D}_c + N\widetilde{D}_{cf}\widehat{N}_c = \frac{s^m}{(s+a)^m}\widehat{D}_s + N\widetilde{D}_{cf}\widehat{N}_c$ is $\mathcal{R}$-unimodular. Therefore, another necessary condition for $\mathcal{R}$-stability of $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ and $\widehat{\mathcal{S}}(P, C_f)$ is rank$(N\widetilde{D}_{cf})(0) = $ rank$(N(V - Q_f\widetilde{N}))(0) = n_o$. It is shown in Lemma 3.1(b) that there exists $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ such that the subsystem $\widehat{\mathcal{S}}(P, C_f)$ is $\mathcal{R}$-stable and rank$(N\widetilde{D}_{cf})(0) = n_o$ whenever P has no (transmission) zeros at $s = 0$ and $n_o \leq n_i$.

3.1 Lemma (*Full rank H_{ye_1} and $\widehat{H}_{ye_1}$*):
Let ND^{-1} be any RCF, $\widetilde{D}^{-1}\widetilde{N}$ be any RCF of $P \in \mathrm{R_p}^{n_o \times n_i}$; let $V, U, \widetilde{V}, \widetilde{U} \in \mathcal{M}(\mathcal{R})$ be as in (7).
a) For each $s_o \in \mathcal{U}$, there exists $Q_f \in \mathcal{M}(\mathcal{R})$ such that rank$(U + Q_f\widetilde{D})(s_o) = \min\{n_o, n_i\}$. Let rank$P = n_o \leq n_i$. Let $s_o \in \mathcal{U}$ be such that rank$P(s_o) = n_o$. There exists $Q_f \in \mathcal{M}(\mathcal{R})$ such that rank $\Big(N(U + Q_f\widetilde{D})(s_o)\Big) = n_o$ and $(V - Q_f\widetilde{N})$ is biproper.
b) For each $s_o \in \mathcal{U}$, there exists $Q_f \in \mathcal{M}(\mathcal{R})$ such that rank$(V - Q_f\widetilde{N})(s_o) = n_i$. Let rank$P = n_o \leq n_i$. Let $s_o \in \mathcal{U}$ be such that rank$P(s_o) = n_o$. There exists $Q_f \in \mathcal{M}(\mathcal{R})$ such that rank $\Big(N(V - Q_f\widetilde{N})(s_o)\Big) = n_o$ and $(V - Q_f\widetilde{N})$ is biproper. □

Since this condition is necessary for integral action, it is assumed that $n_o \leq n_i$ and P has no (transmission) zeros at $s = 0$, i.e., rank$P(0) = n_o$.

3.2 Proposition (*Integral action for fixed C_f*):
Let $P \in \mathrm{R_p}^{n_o \times n_i}$, rank$P(0) = n_o$. Let ND^{-1} be any RCF, $\widetilde{D}^{-1}\widetilde{N}$ be any LCF of P. Let $V, U, \widetilde{V}, \widetilde{U} \in \mathcal{M}(\mathcal{R})$ satisfy (7).
a) Let $C_f = (V - Q_f\widetilde{N})^{-1}(U + Q_f\widetilde{D})$, where $Q_f \in \mathcal{R}^{n_i \times n_o}$ is such that rank$(N\widetilde{N}_{cf})(0) = $ rank$(N(U + Q_f\widetilde{D}))(0) = n_o$ and $(V - Q_f\widetilde{N})$ is biproper. Then the set $\mathbf{S}_m$ of all $\mathcal{R}$-stabilizing controllers C that achieve type-m integral action for the system $\mathcal{S}(P, C_f, C)$ is

$$\mathbf{S}_m = \left\{ C = \frac{(s+a)^m}{s^m}C_s = Q(I - N\widetilde{N}_{cf}Q)^{-1} \;\middle|\right.$$

$$Q = \frac{\Phi}{(s+a)^m} + \frac{s^m}{(s+a)^m}Q_s, \; -a \in \mathbb{R} \setminus \mathcal{U},$$

$$Q_s \in \mathcal{R}^{n_o \times n_o}, \; \det(I - N\widetilde{N}_{cf}Q_s)(\infty) \neq 0,$$

$$\Phi = \Phi_{m-1}s^{m-1} + \Phi_{m-2}s^{m-2} + \ldots + \Phi_0 \in \mathbb{R}[s]^{n_o \times n_o},$$

$$\frac{d^\ell}{ds^\ell}(I - N\widetilde{N}_{cf}\frac{\Phi}{(s+a)^m})\bigg|_{s=0} = 0, \ell = 0, \ldots, m-1 \bigg\}. \tag{12}$$

b) Let $n_o = n_i$. Let $C_f = (V - Q_f\widetilde{N})^{-1}(U + Q_f\widetilde{D})$, where $Q_f \in \mathcal{R}^{n_i \times n_o}$ is such that $\text{rank}(N\widetilde{D}_{cf})(0) = \text{rank}(N(V - Q_f\widetilde{N}))(0) = n_o$ and $(V - Q_f\widetilde{N})$ is biproper. Then the set $\widehat{\mathbf{S}}_m$ of all $\mathcal{R}$-stabilizing controllers $\widehat{C}$ that achieve type-m integral action for the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is

$$\widehat{\mathbf{S}}_m = \bigg\{ \widehat{C} = \frac{(s+a)^m}{s^m}\widehat{C}_s = \widehat{Q}(I - N\widetilde{D}_{cf}\widehat{Q})^{-1} \ \bigg|$$

$$\widehat{Q} = \frac{\widehat{\Phi}}{(s+a)^m} + \frac{s^m}{(s+a)^m}\widehat{Q}_s, \ -a \in \mathbb{R} \setminus \mathcal{U},$$

$$\widehat{Q}_s \in \mathcal{R}^{n_o \times n_o}, \ \det(I - N\widetilde{D}_{cf}\widehat{Q}_s)(\infty) \neq 0,$$

$$\widehat{\Phi} = \widehat{\Phi}_{m-1}s^{m-1} + \widehat{\Phi}_{m-2}s^{m-2} + \ldots + \widehat{\Phi}_0 \in \mathbb{R}[s]^{n_o \times n_o},$$

$$\frac{d^\ell}{ds^\ell}(I - N\widetilde{D}_{cf}\frac{\widehat{\Phi}}{(s+a)^m})\bigg|_{s=0} = 0, \ell = 0, \ldots, m-1 \bigg\}. \tag{13}$$

□

In Proposition 3.2(a), $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ is designed first as any $\mathcal{R}$-stabilizing controller for $\mathcal{S}(P, C_f)$ such that $N\widetilde{N}_{cf} = N(U + Q_f\widetilde{D})$ has no (transmission) zeros at $s = 0$. Existence of $Q_f \in \mathcal{M}(\mathcal{R})$ satisfying this condition is guaranteed by Lemma 3.1(a). The controller C is implemented as $C = \frac{(s+a)^m}{s^m}C_s$, where $-a \in \mathbb{R} \setminus \mathcal{U}$ is arbitrary. For $\ell = 0, 1, \ldots, m-1$, the ℓ-th coefficient-matrix $\Phi_\ell \in \mathbb{R}^{n_o \times n_o}$ of the polynomial-matrix $\Phi \in \mathbb{R}[s]^{n_o \times n_o}$ is defined by $\frac{d^\ell}{ds^\ell}(I - N\widetilde{N}_{cf}\frac{\Phi}{(s+a)^m})\Big|_{s=0} = 0$. Therefore, $I - N\widetilde{N}_{cf}\frac{\Phi}{(s+a)^m} = \frac{s^m}{(s+a)^m}Y$ for some biproper $Y \in \mathcal{R}^{n_o \times n_o}$. Therefore, for all controllers $C \in \mathbf{S}_m$ in (12), the input-error transfer-function $H_{eu} = I - N\widetilde{N}_{cf}\left(\frac{\Phi}{(s+a)^m} + \frac{s^m}{(s+a)^m}Q_s\right) = \frac{s^m}{(s+a)^m}(Y - N\widetilde{N}_{cf}Q_s)$ has m zeros at $s = 0$ due to the term $\frac{s^m}{(s+a)^m}$. Note that the integral action achieved at all outputs is more than type-m whenever $Q_s \in \mathcal{M}(\mathcal{R})$ is such that $Q_s(0) = (N\widetilde{N}_{cf})(0)^{-1}Y(0)$ since $D_s = (Y - N\widetilde{N}_{cf}Q_s)$ has additional blocking-zeros at $s = 0$.

Similarly, in Proposition 3.2(b), where it is assumed that P is square ($n_o = n_i$) for simplicity so that $\widehat{H}_{ye_1}$ is square, $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ is designed first as any $\mathcal{R}$-stabilizing controller for $\widehat{\mathcal{S}}(P, C_f)$ such that $N\widetilde{D}_{cf} = N(V - Q_f\widetilde{N})$ has no (transmission) zeros at $s = 0$. Existence of $Q_f \in \mathcal{M}(\mathcal{R})$ satisfying this condition is guaranteed by Lemma 3.1(b). The controller $\widehat{C}$ is implemented as $\widehat{C} = \frac{(s+a)^m}{s^m}\widehat{C}_s$. For $\ell = 0, 1, \ldots, m-1$, $\widehat{\Phi}_\ell \in \mathbb{R}^{n_o \times n_o}$ of $\widehat{\Phi} \in \mathbb{R}[s]^{n_o \times n_o}$ is defined by $\frac{d^\ell}{ds^\ell}(I - N\widetilde{D}_{cf}\frac{\widehat{\Phi}}{(s+a)^m})\Big|_{s=0} = 0$. Therefore, $I - N\widetilde{D}_{cf}\frac{\widehat{\Phi}}{(s+a)^m} = \frac{s^m}{(s+a)^m}\widehat{Y}$ for some biproper $\widehat{Y} \in \mathcal{R}^{n_o \times n_o}$. Therefore, for all controllers $\widehat{C} \in \widehat{\mathbf{S}}_m$ in (13), the input-error transfer-function $\widehat{H}_{eu} = I - N\widetilde{D}_{cf}\left(\frac{\widehat{\Phi}}{(s+a)^m} + \frac{s^m}{(s+a)^m}\widehat{Q}_s\right) = \frac{s^m}{(s+a)^m}(\widehat{Y} - N\widetilde{D}_{cf}\widehat{Q}_s)$ has m zeros at $s = 0$ due to the term $\frac{s^m}{(s+a)^m}$. The integral action achieved at all outputs is more than type-m whenever $\widehat{Q}_s \in \mathcal{M}(\mathcal{R})$ is such that $\widehat{Q}_s(0) = (N\widetilde{D}_{cf})(0)^{-1}\widehat{Y}(0)$ $\widehat{D}_s = (\widehat{Y} - N\widetilde{D}_{cf}\widehat{Q}_s)$ has additional blocking-zeros at $s = 0$.

In the case that $m = 1$, since $\Phi = a(N\widetilde{N}_{cf})(0)^{-1}$, the expression in (12) is simplified as follows: With $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ a fixed $\mathcal{R}$-stabilizing controller for $\mathcal{S}(P, C_f)$ such that $\text{rank}(N\widetilde{N}_{cf})(0) = n_o$, the set $\mathbf{S}_1$ of all $\mathcal{R}$-stabilizing controllers C which achieve type-1 integral-action in the system $\mathcal{S}(P, C_f, C)$ is

$$\mathbf{S}_1 = \bigg\{ C = \frac{(s+a)}{s}C_s = Q(I - N\widetilde{N}_{cf}Q)^{-1} \ \bigg|$$

$$Q = \frac{a(N\widetilde{N}_{cf})(0)^{-1}}{(s+a)} + \frac{s}{(s+a)}Q_s, \ -a \in \mathbb{R} \setminus \mathcal{U},$$

$$Q_s \in \mathcal{R}^{n_o \times n_o}, \ \det(I - N\widetilde{N}_{cf}Q_s)(\infty) \neq 0 \bigg\}. \tag{14}$$

Similarly, since $\widehat{\Phi} = a(N\widetilde{D}_{cf})(0)^{-1}$, the expression in (13) is simplified as follows: With $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ a fixed $\mathcal{R}$-stabilizing controller for $\widehat{\mathcal{S}}(P, C_f)$ such that $\text{rank}(N\widetilde{D}_{cf})(0) = n_o$, the set $\widehat{\mathbf{S}}_1$ of all $\mathcal{R}$-stabilizing controllers $\widehat{C}$ which achieve type-1 integral-action in the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is

$$\widehat{\mathbf{S}}_1 = \bigg\{ \widehat{C} = \frac{(s+a)}{s}\widehat{C}_s = \widehat{Q}(I - N\widetilde{D}_{cf}\widehat{Q})^{-1} \ \bigg|$$

$$\widehat{Q} = \frac{a(N\widetilde{D}_{cf})(0)^{-1}}{(s+a)} + \frac{s}{(s+a)}\widehat{Q}_s, \ -a \in \mathbb{R} \setminus \mathcal{U},$$

$$\widehat{Q}_s \in \mathcal{R}^{n_o \times n_o}, \ \det(I - N\widetilde{D}_{cf}\widehat{Q}_s)(\infty) \neq 0 \bigg\}. \tag{15}$$

4 Integral action and decoupling

In this section it is required that $\mathcal{S}(P, C_f)$ is $\mathcal{R}$-stable and the system $\mathcal{S}(P, C_f, C)$ is $\mathcal{R}$-stable, has (at least) type-m integral action in each channel, and is decoupled, i.e., the input-output transfer-function H_{yu} is diagonal and nonsingular. Similarly, the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ is required to have the same properties.

Consider the system $\mathcal{S}(P, C_f, C)$: If there exists an $\mathcal{R}$-stabilizing controller pair (C_f, C) such that $H_{yu} \in$

$\mathcal{R}^{n_o\times n_o}$ is nonsingular, then by (10), $N \in \mathcal{R}^{n_o\times n_i}$, $\widetilde{N}_{cf} \in \mathcal{R}^{n_i\times n_o}$, $N_c \in \mathcal{R}^{n_o\times n_o}$ must all have (normal) rank equal to n_o, equivalently, rank$P = n_o \le n_i$, rank$C_f = n_o$, rank$C = n_o$. Furthermore, $(N\widetilde{N}_{cf}) \in \mathcal{R}^{n_o\times n_o}$ must also have full rank. Therefore a necessary condition for decoupling is that rank$P = n_o \le n_i$, which is also sufficient with this two-stage design. By Lemma 3.1(a), there exists $Q_f \in \mathcal{R}^{n_i\times n_o}$ such that rank$(N\widetilde{N}_{cf}) = $ rank$(U + Q_f\widetilde{D}) = n_o$. For each fixed $\mathcal{R}$-stabilizing controller $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ for $\mathcal{S}(P, C_f)$ such that rank$(N\widetilde{N}_{cf}) = n_o$, all $\mathcal{R}$-stabilizing controllers C for the system $\mathcal{S}(P, C_f, C)$ such that H_{yu} is diagonal and nonsingular and all corresponding achievable H_{yu} are parametrized as follows:

Let $P \in \mathrm{R_p}^{n_o\times n_i}$, rank$P = n_o \le n_i$. Let ND^{-1} be any RCF, $\widetilde{D}^{-1}\widetilde{N}$ be any LCF of P. Choose any $\mathcal{R}$-stabilizing controller $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf} \in \mathrm{R_p}^{n_i\times n_o}$ for the subsystem $\mathcal{S}(P, C_f)$, such that rank$(N\widetilde{N}_{cf}) = n_o$. For $j = 1, \ldots, n_o$, let $\delta_{Lj} \in \mathcal{R}$ be any greatest-common-divisor of all entries in the j-th row of $(N\widetilde{N}_{cf}) \in \mathcal{R}^{n_o\times n_o}$; define

$$\Delta_L := \mathrm{diag}\,[\,\delta_{L1} \cdots \delta_{Ln_o}\,]\,, \quad \Delta_L\mathcal{H} := N\widetilde{N}_{cf}. \qquad (16)$$

For $i, j = 1, \ldots, n_o$, write the ij-th entry of $\mathcal{H}^{-1} \in \mathcal{F}^{n_o\times n_o}$ as $a_{ij}b_{ij}^{-1}$, where a_{ij}, $b_{ij} \in \mathcal{R}$, $b_{ij} \ne 0$ and the pair (a_{ij}, b_{ij}) is coprime. Let $\delta_{Rj} \in \mathcal{R}$ be any least-common-multiple of the denominators $(b_{1j}, \ldots, b_{n_oj})$ of the j-th column of $\mathcal{H}^{-1}$; define

$$\Delta_R := \mathrm{diag}\,[\,\delta_{R1} \cdots \delta_{Rn_o}\,]\,. \qquad (17)$$

With the chosen C_f, the controller C is an $\mathcal{R}$-stabilizing controller such that H_{yu} is diagonal and nonsingular if and only if

$$C = \mathcal{H}^{-1}\Delta_R Q_D(I_{n_o} - \Delta_L\Delta_R Q_D)^{-1}, \qquad (18)$$

and the input-output transfer-function H_{yu} is achievable with (C_f, C) if and only if

$$H_{yu} = \Delta_L\Delta_R Q_D, \qquad (19)$$

where $Q_D \in \mathcal{R}^{n_o\times n_o}$ is diagonal, nonsingular, and satisfies $Q_D(\infty) \ne (\Delta_L(\infty)\Delta_R(\infty)\,)^{-1}$.

The controller C_f in this parametrization is any $\mathcal{R}$-stabilizing controller given by (6) such that rank$\mathcal{H} = n_o$, where $Q_f \in \mathcal{R}^{n_i\times n_o}$ is such that $N\widetilde{N}_{cf} = N(U + Q_f\widetilde{D})$ is full normal rank and $(V - Q_f\widetilde{N})$ is biproper. The diagonal matrix $\Delta_L \in \mathcal{R}^{n_o\times n_o}$ in (16) is nonsingular since rank$(N\widetilde{N}_{cf}) = n_o$ implies $\delta_{Lj} \ne 0$; this matrix extracts from each row of $(N\widetilde{N}_{cf})$ factors common to every entry in that row. The remaining matrix $\mathcal{H}$ is invertible since rank$(N\widetilde{N}_{cf}) = n_o$ but $\mathcal{H}^{-1}$ may not be proper. The diagonal matrix $\Delta_R \in \mathcal{R}^{n_o\times n_o}$ in (17) is nonsingular since $b_{ij} \ne 0$ implies $\delta_{Rj} \ne 0$. By construction $\mathcal{H}^{-1}\Delta_R \in \mathcal{R}^{n_o\times n_o}$ and hence, $\mathcal{H}^{-1}\Delta_R Q_D \in \mathcal{R}^{n_o\times n_o}$ is an $\mathcal{R}$-stable matrix for any $Q_D \in \mathcal{M}(\mathcal{R})$. The controller C in (18) is proper if and only if $(I - \Delta_L\Delta_R Q_D)$ is biproper; if P or C_f is strictly proper, equivalently $N \in \mathcal{M}(\mathrm{R_{sp}})$ or $\widetilde{N}_{cf} \in \mathcal{M}(\mathrm{R_{sp}})$, then $\Delta_L \in \mathcal{M}(\mathcal{R})$ and hence, C is proper for all $Q_D \in \mathcal{R}^{n_o\times n_o}$. The controller C is strictly-proper if $Q_D \in \mathcal{M}(\mathcal{R})$ is strictly-proper.

Similarly, consider the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$: If there exists an $\mathcal{R}$-stabilizing controller pair (C_f, C) such that $\widehat{H}_{yu} \in \mathcal{R}^{n_o\times n_o}$ is nonsingular, then by (11), $N \in \mathcal{R}^{n_o\times n_i}$, $\widetilde{D}_{cf} \in \mathcal{R}^{n_i\times n_o}$, and $\widehat{N}_c \in \mathcal{R}^{n_o\times n_o}$ all are rank $n_o \le n_i$. Furthermore, $(N\widetilde{D}_{cf}) \in \mathcal{R}^{n_o\times n_i}$ is also full row-rank. By Lemma 3.1(b), there exists $Q_f \in \mathcal{M}(\mathcal{R})$ such that rank$(N\widetilde{D}_{cf}) = $ rank$(V - Q_f\widetilde{N}) = n_o$. We assume for simplicity that $n_o = n_i$ so that $(N\widetilde{D}_{cf})$ is square. For each fixed $\mathcal{R}$-stabilizing controller $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$ for $\widehat{\mathcal{S}}(P, C_f)$ such that rank$(N\widetilde{D}_{cf}) = n_o$, all $\mathcal{R}$-stabilizing controllers $\widehat{C}$ for the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ such that $\widehat{H}_yu$ is diagonal and nonsingular are parametrized following similar steps as in the case of $\mathcal{S}(P, C_f, C)$ above: Let $P \in \mathrm{R_p}^{n_o\times n_o}$, rank$P = n_o$. Choose any $\mathcal{R}$-stabilizing controller $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf} \in \mathrm{R_p}^{n_o\times n_o}$ for the subsystem $\widehat{\mathcal{S}}(P, C_f)$, such that rank$(N\widetilde{D}_{cf}) = n_o$. Define $\widehat{\Delta}_L$, $\widehat{\Delta}_R$, $\widehat{H}_f$ for $(N\widetilde{D}_{cf})$ similarly as in (16)-(17). With the chosen C_f, the controller $\widehat{C}$ is an $\mathcal{R}$-stabilizing controller such that $\widehat{H}_{yu}$ is diagonal and nonsingular if and only if $\widehat{C} = \widehat{H}_f^{\ -1}\widehat{\Delta}_R\widehat{Q}_D(I_{n_o} - \widehat{\Delta}_L\widehat{\Delta}_R\widehat{Q}_D)^{-1}$, and the input-output transfer-function $\widehat{H}_{yu}$ is achievable with $(C_f, \widehat{C})$ if and only if $\widehat{H}_{yu} = \widehat{\Delta}_L\widehat{\Delta}_R\widehat{Q}_D$, where $\widehat{Q}_D \in \mathcal{R}^{n_o\times n_o}$ is diagonal, nonsingular, and satisfies $\widehat{Q}_D(\infty) \ne (\widehat{\Delta}_L(\infty)\widehat{\Delta}_R(\infty)\,)^{-1}$.

In Proposition 4.1, all $\mathcal{R}$-stabilizing controllers that achieve type-m integral action and decoupling are parametrized for fixed $C_f = \widetilde{D}_{cf}^{-1}\widetilde{N}_{cf}$, which is designed so that rank$(N\widetilde{N}_{cf})(0) = n_o$. The controller C is designed next to achieve type-m integral action and diagonal, nonsingular H_{yu}. The parametrization is based on Δ_L, $\mathcal{H}$, Δ_R defined by (16)-(17) associated with $N\widetilde{N}_{cf}$. The dual parametrization for the system $\widehat{\mathcal{S}}(P, C_f, \widehat{C})$ would follow entirely similarly based on $\widehat{\Delta}_L$, $\widehat{H}_f$, $\widehat{\Delta}_R$ associated with $N\widetilde{D}_{cf}$, where C_f is chosen as any $\mathcal{R}$-stabilizing controller for the subsystem $\widehat{\mathcal{S}}(P, C_f)$ such that rank$(N\widetilde{D}_{cf}) = n_o$.

4.1 Proposition (*Integral action, decoupling*): Let $P \in \mathrm{R_p}^{n_o\times n_i}$, rank$P(0) = n_o$. Let ND^{-1} be any RCF, $\widetilde{D}^{-1}\widetilde{N}$ be any LCF of P. Let $V, U, \widetilde{V}, \widetilde{U} \in \mathcal{M}(\mathcal{R})$ satisfy (7). Let $C_f = (V - Q_f\widetilde{N})^{-1}(U + Q_f\widetilde{D})$, where $Q_f \in \mathcal{R}^{n_i\times n_o}$ is such that rank$(N\widetilde{N}_{cf})(0) = $ rank$(N(U + Q_f\widetilde{D}))(0) = n_o$ and $(V - Q_f\widetilde{N})$ is biproper. Let Δ_L, $\mathcal{H}$, Δ_R be defined by (16)-(17). Then $C = \frac{(s+a)^m}{s^m} C_s$ is an $\mathcal{R}$-stabilizing controller for the system $\mathcal{S}(P, C_f, C)$ that achieves type-m integral action and is such that H_{yu} is diagonal and nonsingular if and only if $C = Q(I - N\tilde{N}_{cf}Q)^{-1}$, where

$$Q = \mathcal{H}^{-1}\Delta_R \left(\frac{\Phi_D}{(s+a)^m} + \frac{s^m}{(s+a)^m} Q_D \right); \qquad (20)$$

in (20), $-a \in \mathbb{R} \setminus \mathcal{U}$, $Q_D = \operatorname{diag}[q_1 \ldots q_{n_o}] \in \mathcal{R}^{n_o \times n_o}$, $q_j(\infty) \neq (\delta_L j(\infty) \delta_R j(\infty))^{-1}$, $\Phi_D = \operatorname{diag}[\phi_1 \ldots \phi_{n_o}]$; for $j = 1, \ldots, n_o$ $\phi_j = \phi_{j,m-1}s^{m-1} + \phi_{j,m-2}s^{m-2} + \ldots + \phi_{j,0} \in \mathbb{R}[s]$, where, for $\ell = 0, \ldots, m-1$,

$$\frac{d^\ell}{ds^\ell}\left(1 - \delta_L j \delta_R j \frac{\phi_j}{(s+a)^m}\right)\Bigg|_{s=0} = 0. \qquad (21)$$

□

If $C = Q(I - N\tilde{N}_{cf}Q)^{-1}$, with Q as in (20), $H_{yu} = N\tilde{N}_{cf}Q = \Delta_L \Delta_R \left(\frac{\Phi_D}{(s+a)^m} + \frac{s^m}{(s+a)^m} Q_D \right)$ is diagonal and nonsingular for all controllers in Proposition 4.1. For $\ell = 0, \ldots, m-1$, the ℓ-th coefficient $\phi_{j,\ell}$ of the polynomial ϕ_j is defined by (21) as $\frac{d^\ell}{ds^\ell}((s+a)^m y_j)\Big|_{s=0} = \frac{d^\ell}{ds^\ell}(\phi_j x_j)\Big|_{s=0}$, where x_j / y_j is the polynomial factorization of $(\delta_L j \delta_R j)$. The corresponding $H_{eu} = I - \Delta_L \Delta_R \frac{\Phi_D}{(s+a)^m} + \frac{s^m}{(s+a)^m} \Delta_L \Delta_R Q_D$ has m zeros at $s = 0$. The integral action achieved at all outputs may be more than type-m for appropriate choices of the diagonal matrix $Q_D \in \mathcal{M}(\mathcal{R})$.

When $m = 1$, since $\phi_j = a(\Delta_L j \Delta_R j(0))^{-1}$, with $C_f = \tilde{D}_{cf}^{-1}\tilde{N}_{cf}$ a fixed $\mathcal{R}$-stabilizing controller for $\mathcal{S}(P, C_f)$ such that $\operatorname{rank}(N\tilde{N}_{cf})(0) = n_o$, Q in (20) becomes

$$Q = \mathcal{H}^{-1}\Delta_R \left(\frac{a(\Delta_L \Delta_R(0))^{-1}}{(s+a)} + \frac{s}{(s+a)} Q_D \right), \qquad (22)$$

where $Q_D = \operatorname{diag}[q_1 \ldots q_{n_o}] \in \mathcal{R}^{n_o \times n_o}$, $q_j \in \mathcal{R}$ satisfies $q_j(\infty) \neq (\delta_L j(\infty) \delta_R j(\infty))^{-1}$ for $j = 1, \ldots n_o$.

5 Conclusions

We considered two-stage $\mathcal{R}$-stabilizing controller design methods that achieve integral action and decoupling for any full row-rank plant with no (transmission) zeros at zero. The purpose of the first controller C_f is to $\mathcal{R}$-stabilize the given plant P; the parametrization of all C_f in (6) follows from well-known factorization methods. The $\mathcal{R}$-stabilizing controller C_f should be chosen so that decoupling and integral action can be accomplished in the second stage. Therefore, C_f is chosen so that the closed-loop map transfer-function from e_1 to y has no (transmission) zeros at zero; existence of such C_f is guaranteed by Lemma 3.1. The second controller is designed to have m poles at zero in order to have type-m integral action. The parametrization of all $\mathcal{R}$-stabilizing controllers that achieve type-m integral action is given in Proposition 3.2 and the parametrization of the controllers which also achieve decoupling is given in Proposition 4.1. The proposed two-stage designs achieve decoupling for any full row-rank plant, whereas the standard one-degree-of-freedom design can achieve decoupling for a subset of such plants. Design for integral action is also simplified by stabilizing the given plant in the first stage and applying the second controller to a stabilized system.

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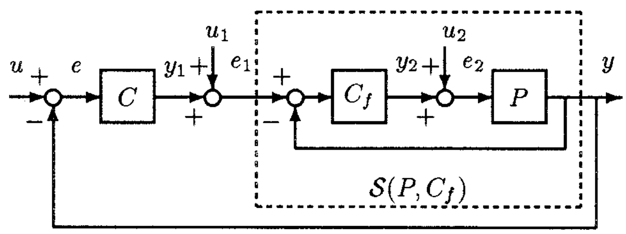

Figure 1: The system $\mathcal{S}(P, C_f, C)$

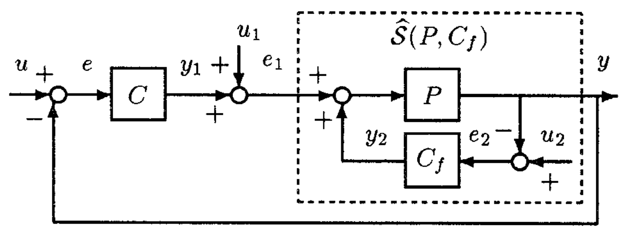

Figure 2: The system $\hat{\mathcal{S}}(P, C_f, \hat{C})$