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Reliable Decentralized Control with Integral Action

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Abstract

Reliable stability with type-m integral action is considered in the linear, time-invariant, multi-input multi-output two-channel decentralized feedback system. The objective is to achieve closed-loop stability under normal operation of both controllers as well as the possible failure of either one of the two controllers. Necessary and sufficient conditions on the given plant are obtained for existence of reliable decentralized controllers achieving integral action. All reliable decentralized controllers with integral action are characterized using algebraic design methods.

1 Introduction

Factorization methods have made it is possible to characterize all controllers that stabilize a given plant in the standard unity-feedback system [9]. We apply these methods to the reliable stabilization problem using a multi-controller configuration. Since the introduction of multi-controller systems in [5], [6], the problem of reliable stabilization has been studied in a factorization setting using full-feedback controllers ([8], [4], [3]) and decentralized controllers [7]. In [1], integral action in the decentralized configuration was considered with scalar channels assuming that the two controllers are stable and conditions on the steady-state gain of the plant were developed for the case of scalar channels.

We study reliable stabilization with type-m integral action in the linear, time-invariant (LTI), multi-input multi-output (MIMO), two-channel decentralized system $\mathcal{S}(P,\widehat{C})$, shown in Figure 1. The controllers are implemented with m integrators in each channel, i.e., $\widehat{C}=\frac{\phi}{s^m}C=\frac{\phi}{s^m}\begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}$, where $\phi=(s+a)^m$, with $-a$ an arbitrary real number in the region of stability. The objective is to develop necessary and sufficient conditions on the plant P for existence of block-diagonal decentralized controllers that ensure reliable stabilization. Once such conditions are established, all decentralized controller pairs $(\widehat{C}_1,\widehat{C}_2)=(\frac{\phi}{s^m}C_1,\frac{\phi}{s^m}C_2)$ are characterized such that the closed-loop system $\mathcal{S}(P,\widehat{C})$ is stable when C_1 and C_2 act together (normal operation) as well as when each controller acts alone (failure mode). Given a plant P, it is possible to characterize all controllers that stabilize P in the standard unity-feedback system configuration. Consider now a pair of controllers that stabilize the given plant P when both controllers act together and when each of the controllers acts alone.

The controller failure model used here is that a controller is replaced by zero if it fails. It is assumed that the failure of a controller is recognized and the controller is taken out of service (i.e., the states in the controller implementation are all set to zero, the initial conditions and the outputs of the channel that failed are set to zero for all inputs). Clearly, stability would be maintained when both controllers are set to zero if and only if the open-loop plant is stable. The integrators in the controller $\frac{\phi}{s^m}C$ provide type-m closed-loop response. These integrators in each of the control channels guarantee that the closed-loop system achieves asymptotic tracking of $(m-1)$-th order polynomial inputs; in particular, when $m=1$, the steady-state error due to step inputs is zero. If one controller fails, this integral action is still present in the outputs of the channel with the active controller.

Due to the algebraic framework described in the following notation, the results apply to continuous-time and as well as discrete-time systems; for the case of discrete-time systems, all evaluations and poles at $s=0$ would be interpreted at 1.

Notation: Let $\mathcal{U}$ contain the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The set of real numbers, the ring of proper rational functions which have no poles in the region of instability $\mathcal{U}$, the sets of proper and strictly-proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathcal{R}$, $\mathrm{R_p}$, $\mathrm{R_{sp}}$, respectively. The set of matrices whose entries are in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$; M is called $\mathcal{R}$-stable iff $M\in\mathcal{M}(\mathcal{R})$; an $\mathcal{R}$-stable M is called $\mathcal{R}$-unimodular iff $M^{-1}\in\mathcal{M}(\mathcal{R})$. A right-coprime-factorization (RCF) and a left-coprime-factorization (LCF) of $P\in{\mathrm{R_p}}^{n_o\times n_i}$ are denoted by ND^{-1} and $\widetilde{D}^{-1}\widetilde{N}$, where $P=ND^{-1}=\widetilde{D}^{-1}\widetilde{N}$, $N,D,\widetilde{N},\widetilde{D}\in\mathcal{M}(\mathcal{R})$, D and $\widetilde{D}$ are biproper. Let $\mathrm{rank}P=r$; $z_o\in\mathcal{U}$

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is called a (transmission) $\mathcal{U}$-zero of P iff $\operatorname{rank}P(z_o) < r$, equivalently, $\operatorname{rank}N(z_o) = \operatorname{rank}\widetilde{N}(z_o) < r$; $z_o \in \mathcal{U}$ is called a blocking $\mathcal{U}$-zero of P iff $P(z_o) = 0$, equivalently, $N(z_o) = 0 = \widetilde{N}(z_o)$; $s_o \in \mathcal{U}$ is called a $\mathcal{U}$-pole of P iff it is a pole of some entry of P, equivalently, $\det D(s_o) = 0 = \det \widetilde{D}(s_o)$.

2 System description

Consider the LTI, MIMO, two-channel decentralized control system $\mathcal{S}(P,\widehat{C})$ shown in Figure 1: $\mathcal{S}(P,\widehat{C})$ is a well-posed system, where P and $\widehat{C}$ represent the transfer-functions of the plant and the decentralized controller. It is assumed that P and $\widehat{C}$ have no hidden modes associated with eigenvalues in $\mathcal{U}$.

Let $u := [u_1^T \ \ u_2^T]^T$, $y = [y_1^T \ \ y_2^T]^T$, $e = [e_1^T \ \ e_2^T]^T$, $u_P = [u_{P1}^T \ \ u_{P2}^T]^T$, $y_C = [y_{C1}^T \ \ y_{C2}^T]^T$. Let H_{eu} denote the input-error transfer-function from u to e, H_{yu} denote the input-output transfer-function from u to y.

It is required that the system $\mathcal{S}(P,\widehat{C})$ is $\mathcal{R}$-stable, i.e., the transfer-function H from (u, u_P) to (y, y_C) is $\mathcal{R}$-stable, and has (at least) type-m integral action in each channel, equivalently, $H_{eu} = I - H_{yu}$ has (at least) m blocking zeros at $s = 0$, where $m \geq 1$ is a given integer. Suppose that $\mathcal{S}(P,\widehat{C})$ is $\mathcal{R}$-stable. Let $\widehat{N}\widehat{D}^{-1}$ be any RCF of $\widehat{C}$. The input-error transfer-function H_{eu} has m zeros at $s = 0$ if $\widehat{D} = \frac{s^m}{\phi}\widehat{D}_s$ for some biproper $\widehat{D}_s \in \mathcal{R}^{n_o \times n_o}$, where

$$\phi := (s+a)^m \ , \ -a \in \mathbb{R} \setminus \mathcal{U}. \tag{1}$$

Therefore, $\mathcal{S}(P,\widehat{C})$ has type-m integral action if $\widehat{C} = \frac{\phi}{s^m}\widehat{N}\widehat{D}_s^{-1} =: \frac{\phi}{s^m}C$ for some $C \in \mathrm{R_p}^{n_i \times n_o}$.

Partition P and $\widehat{C} =: \frac{\phi}{s^m}C$ as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathrm{R_p}^{n_o \times n_i}, \ P_{jj} \in \mathrm{R_p}^{n_{oj} \times n_{ij}}, \tag{2}$$

$$\widehat{C} = \begin{bmatrix} \widehat{C}_1 & 0 \\ 0 & \widehat{C}_2 \end{bmatrix} = \begin{bmatrix} \frac{\phi}{s^m}C_1 & 0 \\ 0 & \frac{\phi}{s^m}C_2 \end{bmatrix} \in \mathrm{R_p}^{n_i \times n_o},$$

$$C_j \in \mathrm{R_p}^{n_{ij} \times n_{oj}}, \ n_o = n_{o1} + n_{o2}, \ n_i = n_{i1} + n_{i2}, \ j = 1,2. \tag{3}$$

Let $N_{Cj}D_{Cj}^{-1}$ be an RCF and $\widetilde{D}_{Cj}^{-1}\widetilde{N}_{Cj}$ be an LCF of C_j, $j = 1,2$. Let $D_C = \operatorname{diag}[D_{C1}, D_{C2}]$, $N_C = \operatorname{diag}[N_{C1}, N_{C2}]$, $\widetilde{D}_C = \operatorname{diag}\left[\widetilde{D}_{C1}, \widetilde{D}_{C2}\right]$, $\widetilde{N}_C = \operatorname{diag}\left[\widetilde{N}_{C1}, \widetilde{N}_{C2}\right]$; then $N_C D_C^{-1}$ is an RCF and $\widetilde{D}_C^{-1}\widetilde{N}_C$ is an LCF of C. Then the controller $\widehat{C} = \frac{\phi}{s^m}C = \frac{\phi}{s^m}N_C D_C^{-1}$ has no hidden modes associated with eigenvalues in $\mathcal{U}$ only if C has no (transmission) zeros at zero, i.e., C_1 and C_2 have no (transmission) zeros at zero; therefore it is assumed that $\operatorname{rank}N_C(0) = n_o \leq n_i$, equivalently, $\operatorname{rank}N_{Cj}(0) = n_{oj} \leq n_{ij}$ for $j = 1,2$.

The failure considered here is the complete failure of the j-th channel. When the second channel fails, $\widehat{C}_2$ is set equal to zero and the corresponding system is called $\mathcal{S}(P,\widehat{C}_1)$; when the first channel fails, $\widehat{C}_1$ is set equal to zero and the corresponding system is called $\mathcal{S}(P,\widehat{C}_2)$.

2.1 Definitions ($\mathcal{R}$-*stability*):

i) The system $\mathcal{S}(P,\widehat{C})$ is said to be $\mathcal{R}$-stable iff the transfer-function H from (u, u_P) to (y, y_C) is $\mathcal{R}$-stable. Similarly, for $j = 1,2$, the system $\mathcal{S}(P, \frac{\phi}{s^m}C_j)$ is said to be $\mathcal{R}$-stable iff the transfer-function H_j from (u_j, u_P) to (y, y_{Cj}) is $\mathcal{R}$-stable.
ii) The decentralized controller $\widehat{C}$ is said to be an $\mathcal{R}$-stabilizing controller for P iff $\widehat{C}$ is proper and $\mathcal{S}(P,\widehat{C})$ is $\mathcal{R}$-stable.
iii) The pair $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$ is called a *reliable decentralized controller pair with type-m integral action* iff $C_1 \in \mathcal{M}(\mathrm{R_p})$, $C_2 \in \mathcal{M}(\mathrm{R_p})$, and the systems $\mathcal{S}(P,\widehat{C})$, $\mathcal{S}(P,\widehat{C}_1)$, $\mathcal{S}(P,\widehat{C}_2)$ are all $\mathcal{R}$-stable. □

In Lemma 2.2, we give necessary and sufficient conditions for $\mathcal{R}$-stability of the system $\mathcal{S}(P,\widehat{C})$ under normal operation and under the complete failure of one of the controllers. The failure model used here assumes that the j-th controller is replaced by zero if it fails; if the j-th channel fails, then the output y_{Cj} of the j-th channel is not measured. We assume that the coprime factorizations used are in special canonical forms. Given any RCF of P, the denominator-matrix can be put into an upper-triangular (Hermite) form by elementary-column-operations; similarly, the denominator-matrix of any given LCF of P can be put into a lower-triangular (Hermite) form by elementary-row-operations [9], [2]. Therefore, without loss of generality, it can be assumed that the RCF ND^{-1} and the LCF $\widetilde{D}^{-1}\widetilde{N}$ of P are given by

$$P = ND^{-1} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}^{-1}$$

$$= \widetilde{D}^{-1}\widetilde{N} = \begin{bmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \\ 0 & \widetilde{D}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{N}_{11} & \widetilde{N}_{12} \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{bmatrix}. \tag{4}$$

2.2 Lemma (*Decentralized stability*):

i) Let $\widetilde{D}^{-1}\widetilde{N}$ be any LCF of $P \in \mathrm{R_p}^{n_o \times n_i}$; let $N_C D_C^{-1}$ be any RCF of C, where $\widehat{C} = \frac{\phi}{s^m}C \in \mathrm{R_p}^{n_i \times n_o}$. Then the system $\mathcal{S}(P,\widehat{C})$ is $\mathcal{R}$-stable if and only if

$$\left[\tfrac{s^m}{\phi}\widetilde{D}D_C + \widetilde{N}N_C\right] \text{ is } \mathcal{R}\text{-unimodular}. \tag{5}$$

In the case that $n_o = n_i$, let ND^{-1} be any RCF of $P \in \mathrm{R_p}^{n_o \times n_o}$; let $\widetilde{D}_C^{-1}\widetilde{N}_C$ be any LCF of C. Then the system $\mathcal{S}(P,\widehat{C})$ is $\mathcal{R}$-stable if and only if

$$\left[\frac{s^m}{\phi}\widetilde{D}_C D + \widetilde{N}_C N\right] \text{ is } \mathcal{R}\text{-unimodular.} \qquad (6)$$

ii) Let the LCF $\widetilde{D}^{-1}\widetilde{N}$ of P be as in (4); let $N_{C2}D_{C2}^{-1}$ be any RCF of C_2, where $\widehat{C}_2 = \frac{\phi}{s^m}C_2$. The system $\mathcal{S}(P,\widehat{C}_2)$ is $\mathcal{R}$-stable if and only if

$$\widetilde{D}_{11} \text{ is } \mathcal{R}\text{-unimodular} \qquad (7)$$

$$\text{and } \frac{s^m}{\phi}\widetilde{D}_{22}D_{C2} + \widetilde{N}_{22}N_{C2} \text{ is } \mathcal{R}\text{-unimodular.} \qquad (8)$$

In the case that $n_{o2} = n_{i2}$, let the RCF ND^{-1} of P be as in (4); let $\widetilde{D}_{C2}^{-1}\widetilde{N}_{C2}$ be any LCF of C_2. The system $\mathcal{S}(P,\widehat{C}_2)$ is $\mathcal{R}$-stable if and only if

$$D_{11} \text{ is } \mathcal{R}\text{-unimodular} \qquad (9)$$

$$\text{and } \frac{s^m}{\phi}\widetilde{D}_{C2}D_{22} + \widetilde{N}_{C2}N_{22} \text{ is } \mathcal{R}\text{-unimodular.} \qquad (10)$$

iv) Let the LCF $\widetilde{D}^{-1}\widetilde{N}$ of P be as in (4); let $N_{C1}D_{C1}^{-1}$ be any RCF of C_1, where $\widehat{C}_1 = \frac{\phi}{s^m}C_1$. The system $\mathcal{S}(P,\widehat{C}_1)$ is $\mathcal{R}$-stable if and only if

$$\begin{bmatrix} \frac{s^m}{\phi}\widetilde{D}_{11}D_{C1} + \widetilde{N}_{11}N_{C1} & \widetilde{D}_{12} \\ \widetilde{N}_{21}N_{C1} & \widetilde{D}_{22} \end{bmatrix} \text{ is } \mathcal{R}\text{-unimodular.} \qquad (11)$$

In the case that $n_{o1} = n_{i1}$, let the RCF ND^{-1} of P be as in (4); let $\widetilde{D}_{C1}^{-1}\widetilde{N}_{C1}$ be any LCF of C_1. The system $\mathcal{S}(P,\widehat{C}_1)$ is $\mathcal{R}$-stable if and only if

$$\begin{bmatrix} \frac{s^m}{\phi}\widetilde{D}_{C1}D_{11} + \widetilde{N}_{C1}N_{11} & \widetilde{N}_{C1}N_{12} \\ D_{21} & D_{22} \end{bmatrix} \text{ is } \mathcal{R}\text{-unimodular.} \qquad (12)$$

□

2.3 Lemma (*Admissible plants*):
Let $P \in \mathrm{R_p}^{n_o \times n_i}$ be partitioned as in (2). Let $n_o = n_i$ and $n_{oj} = n_{ij}$, $j = 1,2$. Let $\widehat{C} = \frac{\phi}{s^m}C \in \mathrm{R_p}^{n_i \times n_o}$ be partitioned as in (3).

a) The following conditions hold at $s = 0$.

i) If the system $\mathcal{S}(P,\widehat{C})$ is $\mathcal{R}$-stable, then P has no (transmission) zeros at $s = 0$;

ii) if the system $\mathcal{S}(P,\widehat{C}_2)$ is $\mathcal{R}$-stable, then P_{22} has no (transmission) zeros at $s = 0$;

iii) if the system $\mathcal{S}(P,\widehat{C}_1)$ is $\mathcal{R}$-stable, then P_{11} has no (transmission) zeros at $s = 0$.

b) For the system $\mathcal{S}(P,\widehat{C}_2)$, the following three conditions are equivalent:

i) There exists a controller $\widehat{C}_2 = \frac{\phi}{s^m}C_2$ such that the system $\mathcal{S}(P,\widehat{C}_2)$ is $\mathcal{R}$-stable;

ii) P_{22} has no (transmission) zeros at $s = 0$ and P has an RCF ND^{-1} and an LCF $\widetilde{D}^{-1}\widetilde{N}$ of the form

$$P = \begin{bmatrix} N_{11} & N_{12} \\ \frac{s^m}{\phi}\widetilde{V}_2\widetilde{N}_{21} & N_{22} \end{bmatrix} \begin{bmatrix} I_{n_{i1}} & 0 \\ -\widetilde{U}_2\widetilde{N}_{21} & D_{22} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I_{n_{o1}} & -N_{12}U_2 \\ 0 & \widetilde{D}_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & \frac{s^m}{\phi}N_{12}V_2 \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{bmatrix}, \qquad (13)$$

where $N_{11} \in \mathcal{R}^{n_{o1}\times n_{i1}}$, $N_{12} \in \mathcal{R}^{n_{o1}\times n_{i2}}$, $\widetilde{N}_{21} \in \mathcal{R}^{n_{o2}\times n_{i1}}$ are arbitrary $\mathcal{R}$-stable matrices, (N_{22}, D_{22}) is right-coprime, $(\widetilde{D}_{22}, \widetilde{N}_{22})$ is left-coprime, and $\widetilde{U}_2, \widetilde{V}_2, U_2, V_2 \in \mathcal{M}(\mathcal{R})$ satisfy the identity

$$\begin{bmatrix} V_2 & U_2 \\ -\widetilde{N}_{22} & \frac{s^m}{\phi}\widetilde{D}_{22} \end{bmatrix} \begin{bmatrix} \frac{s^m}{\phi}D_{22} & -\widetilde{U}_2 \\ N_{22} & \widetilde{V}_2 \end{bmatrix} = I; \qquad (14)$$

iii) P_{22} has no (transmission) zeros at $s = 0$ and $(P_{11} - P_{12}D_{22}U_2P_{21}) \in \mathcal{M}(\mathcal{R})$, $P_{12}D_{22} \in \mathcal{M}(\mathcal{R})$, $\widetilde{D}_{22}P_{21} \in \mathcal{M}(\mathcal{R})$, where $N_{22}D_{22}^{-1}$ is an RCF and $\widetilde{D}_{22}^{-1}\widetilde{N}_{22}$ is an LCF of P_{22} and U_2 satisfies (14). □

2.4 Theorem (*Stabilizing controllers*):
Let $P \in \mathcal{R}^{n_o \times n_i}$, partitioned as in (2), have no (transmission) zeros at $s = 0$, and let P_{11} and P_{22} have no (transmission) zeros at $s = 0$. Let P have an RCF ND^{-1} and an LCF $\widetilde{D}^{-1}\widetilde{N}$ of the form given by (13). Let $n_o = n_i$ and $n_{oj} = n_{ij}$, $j = 1,2$. Let $\widehat{C} = \frac{\phi}{s^m}C \in \mathrm{R_p}^{n_i \times n_o}$ be as in (3).

a) The system $\mathcal{S}(P,\widehat{C}_2)$ is $\mathcal{R}$-stable if and only if for some LCF $\widetilde{D}_{C2}^{-1}\widetilde{N}_{C2}$ and some RCF $N_{C2}D_{C2}^{-1}$ of C_2,

$$\frac{s^m}{\phi}\widetilde{D}_{C2}D_{22} + \widetilde{N}_{C2}N_{22} = I,$$

$$\frac{s^m}{\phi}\widetilde{D}_{22}D_{C2} + \widetilde{N}_{22}N_{C2} = I. \qquad (15)$$

The controller $\widehat{C}_2 = \frac{\phi}{s^m}C_2$ $\mathcal{R}$-stabilizes the system $\mathcal{S}(P,\widehat{C}_2)$ if and only if

$$C_2 = \widetilde{D}_{C2}^{-1}\widetilde{N}_{C2} = (V_2 - Q_2\widetilde{N}_{22})^{-1}(U_2 + \frac{s^m}{\phi}Q_2\widetilde{D}_{22})$$

$$= N_{C2}D_{C2}^{-1} = (\widetilde{U}_2 + \frac{s^m}{\phi}D_{22}Q_2)(\widetilde{V}_2 - N_{22}Q_2)^{-1} \qquad (16)$$

for some $Q_2 \in \mathcal{R}^{n_{i2}\times n_{o2}}$ such that $(V_2 - Q_2\widetilde{N}_{22})$ is biproper.

b) The system $\mathcal{S}(P,\widehat{C})$ and the system $\mathcal{S}(P,\widehat{C}_2)$ are both $\mathcal{R}$-stable if and only if $\widehat{C}_2 = \frac{\phi}{s^m}C_2$ is given by (16) and some LCF $\widetilde{D}_{C1}^{-1}\widetilde{N}_{C1}$ of C_1 satisfies

$$\frac{s^m}{\phi}\widetilde{D}_{C1} + \widetilde{N}_{C1}(N_{11} - \frac{s^m}{\phi}N_{12}Q_2\widetilde{N}_{21}) = I, \qquad (17)$$

equivalently, some RCF $N_{C1}D_{C1}^{-1}$ of C_1 satisfies

$$\frac{s^m}{\phi}D_{C1} + (N_{11} - \frac{s^m}{\phi}N_{12}Q_2\widetilde{N}_{21})N_{C1} = I. \qquad (18)$$

With $\widehat{C}_2 = \frac{\phi}{s^m}C_2$ as in (16), $\frac{\phi}{s^m}C_1$ is a controller such that $\mathcal{S}(P,\widehat{C})$ and $\mathcal{S}(P,\widehat{C}_2)$ are both $\mathcal{R}$-stable if and only if C_1 is given by

$$C_1 = \widetilde{D}_{C1}^{-1}\widetilde{N}_{C1} = (V_1 - Q_1\widehat{N}_{11})^{-1}(U_1 + \frac{s^m}{\phi}Q_1)$$

$$= N_{C1}D_{C1}^{-1} = (U_1 + \frac{s^m}{\phi}Q_1)(\widetilde{V}_1 - Q_1\widehat{N}_{11})^{-1}, \quad (19)$$

where $\widehat{N}_{11} := (N_{11} - \frac{s^m}{\phi}N_{12}Q_2\widetilde{N}_{21})$ and $U_1, V_1, \widetilde{V}_1 \in \mathcal{M}(\mathcal{R})$ satisfy

$$\begin{bmatrix} V_1 & U_1 \\ -\widehat{N}_{11} & \frac{s^m}{\phi}I \end{bmatrix} \begin{bmatrix} \frac{s^m}{\phi}I & -U_1 \\ \widehat{N}_{11} & \widetilde{V}_1 \end{bmatrix} = I. \quad (20)$$

c) The system $\mathcal{S}(P,\widehat{C})$, the system $\mathcal{S}(P,\widehat{C}_2)$, and the system $\mathcal{S}(P,\widehat{C}_1)$ are all $\mathcal{R}$-stable (equivalently, $(\widehat{C}_1,\widehat{C}_2) = (\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$ is a reliable decentralized controller pair with integral action) if and only if $C_2 = \widetilde{D}_{C2}^{-1}\widetilde{N}_{C2} = N_{C2}D_{C2}^{-1}$ is given by (16), $C_1 = \widetilde{D}_{C1}^{-1}\widetilde{N}_{C1} = N_{C1}D_{C1}^{-1}$ is given by (19), and

$$D_{22} + N_{C2}\widetilde{N}_{21}\widetilde{N}_{C1}N_{12} \text{ is } \mathcal{R}\text{-unimodular}, \quad (21)$$

equivalently,

$$\widetilde{D}_{22} + \widetilde{N}_{21}N_{C1}N_{12}\widetilde{N}_{C2} \text{ is } \mathcal{R}\text{-unimodular}. \quad (22)$$

□

3 Reliable decentralized stabilizability

By conditions (21)-(22) of Theorem 2.4, there exists a reliable decentralized controller pair with integral action $(\widehat{C}_1,\widehat{C}_2) = (\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$ if and only if there exist $\mathcal{R}$-stable matrices Q_1, Q_2 such that

$$D_{22} + (\widetilde{U}_2 + \frac{s^m}{\phi}D_{22}Q_2)\widetilde{N}_{21}(U_1 + \frac{s^m}{\phi}Q_1)N_{12}$$
$$\text{is } \mathcal{R}\text{-unimodular}, \quad (23)$$

equivalently,

$$\widetilde{D}_{22} + \widetilde{N}_{21}(U_1 + \frac{s^m}{\phi}Q_1)N_{12}(U_2 + \frac{s^m}{\phi}Q_2\widetilde{D}_{22})$$
$$\text{is } \mathcal{R}\text{-unimodular}. \quad (24)$$

Although all reliable decentralized controller pairs are characterized by (16), (19), and the equivalent conditions (21)-(22), this characterization does not explicitly describe how to choose $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ in order to satisfy the equivalent conditions (23)-(24).

Since strong $\mathcal{R}$-stabilizability of pseudosystems related to the original plant P play an important role in existence of reliable decentralized controller pairs, recall the following well-known definitions and facts [9]: An LTI system $\hat{P}$ is said to be strongly $\mathcal{R}$-stabilizable iff there is an $\mathcal{R}$-stable $\mathcal{R}$-stabilizing controller $\hat{C} \in \mathcal{M}(\mathcal{R})$ for $\hat{P}$ (in the standard full-feedback system). If $\mathcal{U} = \mathbb{C}_+$, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if it satisfies the *parity-interlacing-property*, i.e., $\hat{P}$ has an even number of poles between consecutive pairs of blocking zeros on the positive real-axis. For the general instability region $\mathcal{U}$, $\hat{P}$ is strongly $\mathcal{R}$-stabilizable if and only if $\hat{P}$ has an even number of $\mathcal{U}$-poles between consecutive pairs of real-axis blocking $\mathcal{U}$-zeros.

3.1 Theorem (*Stabilizability conditions*): Let $P \in \mathrm{R_p}^{n_o \times n_i}$. Let $n_o = n_i$ and $n_{oj} = n_{ij}$, $j = 1,2$.
a) *Necessary conditions:* If there exists a reliable decentralized controller pair $(\widehat{C}_1,\widehat{C}_2) = (\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$, then the following four necessary conditions on P hold:
i) P has an RCF ND^{-1} and an LCF $\widetilde{D}^{-1}\widetilde{N}$ of the form given by (13); P has no (transmission) zeros at $s = 0$, P_{22} has no (transmission) zeros at $s = 0$, P_{11} has no (transmission) zeros at $s = 0$ and $\mathrm{rank}N_{11}(0) = \mathrm{rank}(P_{11} - P_{12}\widetilde{U}_2\widetilde{D}_{22}P_{21})(0) = n_{o1}$.
ii) in (13), $N_{12}D_{22}^{-1}$ is an RCF of P_{12} and $\widetilde{D}_{22}^{-1}\widetilde{N}_{21}$ is an LCF of P_{21};
iii) P_{12} is strongly $\mathcal{R}$-stabilizable and P_{21} is strongly $\mathcal{R}$-stabilizable; furthermore, $\det D_{22}$ has the same sign the all real blocking $\mathcal{U}$-zeros of P_{12} and at all real blocking $\mathcal{U}$-zeros of P_{21}; this sign is the same as the sign of $\det D_{22}(\infty)$ when P_{12} or P_{21} is strictly-proper;
iv) the sign of

$$\det(\, D_{22}(0) + N_{22}^{-1}(0)\widetilde{N}_{21}(0)N_{11}^{-1}(0)N_{12}(0)\,)$$
$$= \det\begin{bmatrix} D_{22} & -\widetilde{U}_2\widetilde{N}_{21} \\ N_{12} & N_{11} \end{bmatrix}(0) \quad (25)$$

is the same as the sign of $\det D_{22}(\infty)$ when P_{12} or P_{21} is strictly-proper.
b) *Necessary and sufficient conditions:* Let P have an RCF ND^{-1} and an LCF $\widetilde{D}^{-1}\widetilde{N}$ of the form given by (13); let P, P_{22}, P_{11} have no (transmission) zeros at $s = 0$ and let $\mathrm{rank}N_{11}(0) = n_{o1}$. let $N_{12}D_{22}^{-1}$ be an RCF of P_{12} and $\widetilde{D}_{22}^{-1}\widetilde{N}_{21}$ be an LCF of P_{21}, where P_{12} or P_{21} is strictly-proper. With U_1 as in (20) and $N_{C2} = (\widetilde{U}_2 + D_{22}Q_2)$ by (16), define

$$\hat{P} := \frac{\phi}{s^m}N_{12}(D_{22} + N_{C2}\widetilde{N}_{21}U_1N_{12})^{-1}N_{C2}\widetilde{N}_{21}$$
$$= \frac{\phi}{s^m}P_{12}(I + N_{C2}\widetilde{N}_{21}U_1P_{12})^{-1}N_{C2}\widetilde{D}_{22}P_{21}. \quad (26)$$

There exist $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ satisfying the equivalent conditions (23)-(24) if and only if $\hat{P}$ is strongly $\mathcal{R}$-stabilizable for some $Q_2 \in \mathcal{M}(\mathcal{R})$. When P_{12} or P_{21} is strictly-proper, a reliable decentralized controller pair $(\widehat{C}_1,\widehat{C}_2) = (\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$ if and only if $\hat{P}$ is strongly $\mathcal{R}$-stabilizable for some $Q_2 \in \mathcal{M}(\mathcal{R})$. □

By Theorem 3.1, a reliable decentralized controller pair with integral action can be designed if and only if there exists an $\mathcal{R}$-stable Q_2 for which

$$\hat{P} = \frac{\phi}{s^m}P_{12}(I + (\widetilde{U}_2 + \frac{\phi}{s^m}D_{22}Q_2)\widetilde{N}_{21}U_1P_{12})^{-1}$$
$$\times\, (\widetilde{U}_2 + \frac{\phi}{s^m}D_{22}Q_2)\widetilde{D}_{22}P_{21}$$

is strongly $\mathcal{R}$-stabilizable. Clearly, there may not exist a reliable decentralized controller pair for some plants. We now study special plant cases where existence of reliable decentralized controller pairs is guaranteed.

3.2 Proposition (*Stable plants*):
Let $P \in \mathcal{R}^{n_o \times n_i}$ be $\mathcal{R}$-stable. Let $n_o = n_i$ and $n_{oj} = n_{ij}$, $j = 1, 2$. Let $\det P(0) \neq 0$.
a) There exists a reliable decentralized controller pair with integral action $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$ if and only if

$$\begin{array}{rrcl} \text{i)} & \det P_{11}(0) & \neq & 0\,, \\ \text{ii)} & \det P_{22}(0) & \neq & 0\,, \\ \text{and iii)} & \dfrac{\det P(0)}{\det P_{11}(0)\,\det P_{22}(0)} & > & 0\,. \end{array} \quad (27)$$

b) For $j = 1, 2$, let $P_{jj}(0)$ be nonsingular and (27) hold; then there exist $V_{jj}, U_{jj}, \widetilde{V}_{jj} \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} V_{jj} & U_{jj} \\ -P_{jj} & \frac{s^m}{\phi} I_{n_{oj}} \end{bmatrix} \begin{bmatrix} \frac{s^m}{\phi} I_{n_{oj}} & -U_{jj} \\ P_{jj} & \widetilde{V}_{jj} \end{bmatrix} = I. \quad (28)$$

All reliable decentralized controller pairs $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$ are parametrized by

$$\Big\{ (\widehat{C}_1, \widehat{C}_2) = (\tfrac{\phi}{s^m} C_1, \tfrac{\phi}{s^m} C_2) \;\mid\; \widehat{C}_j = \tfrac{\phi}{s^m} C_j,\; j = 1, 2,$$

$$C_j = N_{Cj} D_{Cj}^{-1} = (U_{jj} + \tfrac{\phi}{s^m} Q_j)(\widetilde{V}_{jj} - P_{jj} Q_j)^{-1} \Big\} \quad (29)$$

for some $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ such that

$$I - \frac{s^m}{\phi} P_{12} N_{C2} \frac{s^m}{\phi} P_{21} N_{C1} =$$

$$\frac{s^m}{\phi} P_{12}(U_{22} + \tfrac{\phi}{s^m} Q_2) P_{21}(U_{11} + \tfrac{s^m}{\phi} Q_1) \text{ is } \mathcal{R}\text{-unimodular} \quad (30)$$

and $(\widetilde{V}_{jj} - P_{jj} Q_j)$ is biproper. □

3.3 Remark (*Block-triangular plants*):
By Theorem 3.1, if there exists reliable decentralized controller pair $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$, then P has an RCF ND^{-1} and an LCF $\widetilde{D}^{-1}\widetilde{N}$ of the form given by (13), where $N_{22} D_{22}^{-1}$ is an RCF of P_{22}, $N_{12} D_{22}^{-1}$ is an RCF of P_{12} and $\widetilde{D}_{22}^{-1} \widetilde{N}_{21}$ is an LCF of P_{21}. Suppose that $P_{22} = 0$ or $P_{12} = 0$ (P is lower block-triangular) or $P_{21} = 0$ (P is upper block-triangular); equivalently, $N_{22} = 0$ or $N_{12} = 0$ or $\widetilde{N}_{21} = 0$. With one of these numerator matrices equal to zero, the corresponding pair $(0, D_{22})$ is right-coprime if and only if D_{22} is $\mathcal{R}$-unimodular, i.e., P is $\mathcal{R}$-stable. Therefore, whenever any of the sub-blocks P_{22}, P_{12} or P_{21} of the given plant is zero, then there exist reliable decentralized controller pairs if and only if P is $\mathcal{R}$-stable; hence, for any of these cases, the parametrization of all reliable decentralized controller pairs $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$ is given by (29) of Proposition 3.2, where (30) holds for all $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$ if P_{12} or P_{21} is zero. □

3.4 Theorem (*Conditions when P_{22} is SISO*):
Let $P \in \mathrm{R_p}^{n_o \times n_i}$, where $P_{11} \in \mathrm{R_p}^{n_{o1} \times n_{i1}}$, $P_{12} \in \mathrm{R_p}^{n_{o1} \times 1}$, $P_{21} \in \mathrm{R_p}^{1 \times n_{i1}}$, $P_{22} \in \mathrm{R_p}$ ($n_{o2} = n_{i2} = 1$). Let P_{12} or P_{21} be strictly-proper. Let $N_{22} D_{22}^{-1} = \widetilde{D}_{22}^{-1} \widetilde{N}_{22}$ be a coprime factorization of P_{22}. Let $\rho_1, \rho_2, \ldots, \rho_\gamma$ (arranged in ascending order) denote the distinct real $\mathcal{U}$-poles of P_{22} and let $\rho_{j_1}, \rho_{j_2}, \ldots, \rho_{j_\ell}$ (arranged in ascending order) denote those distinct real $\mathcal{U}$-poles of P_{22} for which the sign of $N_{22}(\rho_{j_k})$ is not equal to the sign of $N_{22}(\rho_{j_k+1})$, $1 \le k \le \ell$. There exists a reliable decentralized controller pair $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$ if and only if the four necessary conditions of Theorem 3.1(a) hold, and in addition, P_{22} has an even number of real $\mathcal{U}$-poles in each of the intervals $(\rho_{j_k}, \rho_{j_{k+1}+1})$, $1 \le k \le \ell - 1$, and (ρ_{j_ℓ}, ∞).
□

3.5 Corollary (*Sufficient conditions*):
Let $P \in \mathrm{R_p}^{n_o \times n_i}$, where $P_{11} \in \mathrm{R_p}^{n_{o1} \times n_{i1}}$, $P_{12} \in \mathrm{R_p}^{n_{o1} \times 1}$, $P_{21} \in \mathrm{R_p}^{1 \times n_{i1}}$, $P_{22} \in \mathrm{R_p}$ ($n_{o2} = n_{i2} = 1$). Let $N_{22} D_{22}^{-1}$ be an RCF of P_{22}. Let the four necessary conditions of Theorem 3.1(a) hold.
a) There exists a reliable decentralized controller pair $(\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$ if P_{22} has an even number of real $\mathcal{U}$-zeros between any pairs of its real $\mathcal{U}$-poles.
b) There exists a reliable decentralized controller pair $(\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$ if the sign of D_{22} is the same at all real $\mathcal{U}$-zeros of P_{22} as the sign of $D_{22}(\infty)$. □

Since the case of SISO channels (i.e., $P \in \mathrm{R_p}^{2 \times 2}$) is a special case, there exists a reliable decentralized controller pair $(\widehat{C}_1, \widehat{C}_2)$, with $C_j \in \mathrm{R_p}$ for $j = 1, 2$, if and only if the conditions of Theorem 3.4 hold.

Note that when $P_{22} \in \mathcal{M}(\mathrm{R_{sp}})$, the sufficient condition in Corollary 3.5(b) is equivalent to P_{22} being strongly $\mathcal{R}$-stabilizable; when P_{22} is not strictly-proper, this condition *implies* that P_{22} is strongly $\mathcal{R}$-stabilizable.

3.6 Theorem (*Conditions for MIMO channels*):
Let $P \in \mathrm{R_p}^{n_o \times n_i}$, $P_{11} \in \mathrm{R_p}^{n_{o1} \times n_{i1}}$, $P_{12} \in \mathrm{R_p}^{n_{o1} \times n_{i2}}$, $P_{21} \in \mathrm{R_p}^{n_{o2} \times n_{i1}}$, $P_{22} \in \mathrm{R_p}^{n_{o2} \times n_{i2}}$. Let the four necessary conditions of Theorem 3.1(a) hold. Let $(\widehat{C}_1, \widehat{C}_2) = (\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$.
a) Let either P_{12} or P_{21} be strictly-proper. Let $n_{o2} = n_{i2} > 1$. Let the sign of $\det D_{22}$ be the same as the sign of $\det D_{22}(\infty)$ at all real (transmission) $\mathcal{U}$-zeros of P_{12} coinciding with those of P_{21}. If $\operatorname{rank} P_{12} = n_{i2} \le n_{o1}$, $\operatorname{rank} P_{21} = n_{o2} \le n_{i1}$, then there exists a reliable decentralized controller pair $(\frac{\phi}{s^m} C_1, \frac{\phi}{s^m} C_2)$.
b) Let either P_{12} or P_{21} be strictly-proper. Let $P_{22} \in \mathrm{R_{sp}}^{n_{o2} \times n_{o2}}$. Let $\operatorname{rank} P_{22} = n_{o2} = n_{i2}$. Let $\operatorname{rank} P_{12} + \operatorname{rank} P_{21} > n_{o2} = n_{i2}$. Let the sign of $\det D_{22}$ be the same as the sign of $\det D_{22}(\infty)$ at all real (transmission) $\mathcal{U}$-zeros of P_{12} and of P_{21}. If the number of real (transmission) $\mathcal{U}$-zeros of $P_{22} = \widetilde{D}_{22}^{-1} \widetilde{N}_{22}$ be-

tween any pair of real blocking $\mathcal{U}$-zeros of $\widetilde{D}_{22}$ is even, then there exists a reliable decentralized controller pair $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$.

c) Let $P_{22} \in \mathrm{R_p}^{n_{o2}\times n_{i2}}$, where n_{o2} and n_{i2} are not both equal to 1. If $P_{12} \in \mathrm{R_p}^{n_{o1}\times n_{i2}}$ has an $\mathcal{R}$-stable left-inverse $P_{12}^I \in \mathcal{R}^{n_{i2}\times n_{o1}}$ and if $P_{21} \in \mathrm{R_p}^{n_{o2}\times n_{i1}}$ has an $\mathcal{R}$-stable right-inverse $P_{21}^I \in \mathcal{R}^{n_{i1}\times n_{o2}}$, then there exists a reliable decentralized controller pair $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$.

d) Let $P_{21} \in \mathrm{R_p}^{n_{o2}\times n_{i1}}$ have an $\mathcal{R}$-stable right-inverse $P_{21}^I \in \mathcal{R}^{n_{i1}\times n_{o2}}$. Let P_{11} and P_{12} be strictly-proper. Let P_{22} be strongly $\mathcal{R}$-stabilizable. If $M_{12}P_{12} = P_{22}$ for some $M_{12} \in \mathcal{R}^{n_{o2}\times n_{o1}}$, then there exists a reliable decentralized controller pair $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$.

e) Let $P_{12} \in \mathrm{R_p}^{n_{o1}\times n_{i2}}$ have an $\mathcal{R}$-stable left-inverse $P_{12}^I \in \mathcal{R}^{n_{i2}\times n_{o1}}$. Let P_{11} and P_{21} be strictly-proper. Let P_{22} be strongly $\mathcal{R}$-stabilizable. If $P_{21}M_{21} = P_{22}$ for some $M_{21} \in \mathcal{R}^{n_{i1}\times n_{i2}}$, then there exists a reliable decentralized controller pair $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$.

f) Let $P_{22} \in \mathrm{R_{sp}}^{n_{o2}\times n_{i2}}$. If $M_{12}P_{12} = P_{22}$ for some $M_{12} \in \mathcal{R}^{n_{o2}\times n_{o1}}$ and $P_{21}M_{21} = P_{22}$ for some $M_{21} \in \mathcal{R}^{n_{i1}\times n_{i2}}$, then there exists a reliable decentralized controller pair $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$. □

The conditions given in Theorem 3.6 are sufficient conditions for existence of reliable decentralized controllers in six cases. Several other cases can be derived from these six general cases. For example, under the assumptions of case (b), if either P_{21} has an $\mathcal{R}$-stable right-inverse $P_{21}^I \in \mathcal{R}^{n_{i1}\times n_{o2}}$ or P_{12} has an $\mathcal{R}$-stable left-inverse $P_{12}^I \in \mathcal{R}^{n_{i2}\times n_{o1}}$, then $\mathrm{rank}P_{12} + \mathrm{rank}P_{21} > n_{o2}$ holds since either $\mathrm{rank}P_{21} = n_{o2}$ or $\mathrm{rank}P_{12} = n_{i2} = n_{o2}$ so there exist reliable decentralized controllers. Note that some of the cases in Theorem 3.6 assume that the transfer-function P_{22} is not scalar; the case of scalar P_{22} is treated separately in Theorem 3.4, which provides necessary and sufficient conditions.

4 Conclusions

We considered the problem of designing reliable decentralized controllers which stabilize a given plant when both controllers act together and when either one of the controllers acts alone. The controllers are implemented with m integrators in each channel to achieve (at least) type-m integral action. We showed that there exist reliable decentralized controllers only if the sub-blocks P_{12} and P_{21} of the given plant P are strongly stabilizable. In Theorem 3.4, we established necessary and sufficient conditions for existence of reliable decentralized controller pairs $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$ when the sub-block P_{22} is SISO and in Theorem 3.6, we gave sufficient conditions when all sub-blocks of P are MIMO. We characterized all reliable decentralized controller pairs $(\frac{\phi}{s^m}C_1, \frac{\phi}{s^m}C_2)$. These results are crucial in deciding if a given plant is stabilizable using a decentralized configuration where each controller must maintain stability in case of failure of the other controller. Once the plant is found admissible for reliable decentralized stabilizability, the controllers are designed using the parametrizations (16), (19), where the controller parameter matrices Q_1 and Q_2 are selected to satisfy the unimodularity condition (23).

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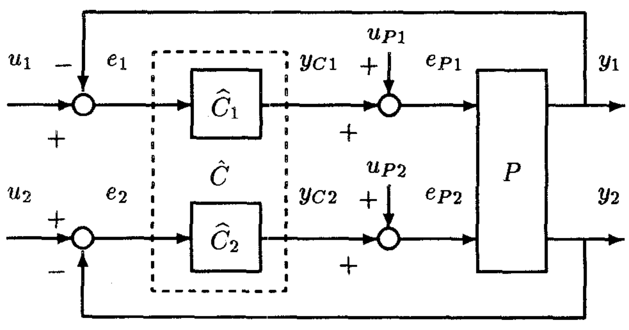

Figure 1: The decentralized system $\mathcal{S}(P,\hat{C})$