Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

A general sufficient condition for stability of interconnected nonlinear systems

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Abstract

A sufficient condition for stability of the feedback interconnection of two nonlinear stable systems is given as a generalization of the standard "small-gain theorem". Using the same input-output approach, the proposed extension reduces conservatism by relaxing the finite-gain stability assumption and by not requiring a boundedness result for all possible bounded inputs, and has simple geometric interpretations by utilizing graphs of upper-bounding functions.

1 Introduction

The "small-gain theorem" establishes a sufficient condition to ensure stability of the feedback interconnection of two stable nonlinear systems [1], [4]; it is central in many stability robustness results in the literature. The finite-gain setting of the theorem allows a natural extension of results on stable linear systems to the nonlinear case, and associates a "gain" with each of the nonlinear stable systems in the feedback interconnection. The strong result of the small-gain theorem requires no existence or uniqueness or continuity assumptions. Stability analysis is reduced to a simple scalar inequality condition: If the product of the gains is less than one, then the closed-loop feedback interconnection is finite-gain stable. A careful study shows that the result requires two ingredients: 1) bounded-input bounded-output stability of each subsystem, and 2) a crucial pair of inequality constraints resulting from the property of seminorms. In this note, we characterize bounded-input bounded-output stability of systems in a general form using non-decreasing upper-bounding functions. For a given bound on exogenous inputs, we state a sufficient condition to guarantee that all resulting signals in the feedback interconnection are bounded. The proposed condition is solely based on the crucial pair of inequality constraints and has a simple geometric interpretation using the graphs of two upper-bounding functions and their translations due to bounds on the exogenous signals. Hence, the level of conservatism in the standard finite-gain small-gain theorem is reduced due to adopting an upper-bounding function more general than an affine one, and due to incorporating the bound on exogenous inputs. This note was motivated by a generalization of the small-gain theorem in [3], which also allows general output-bounding functions. The conditions in [3] require additional assumptions to guarantee a bounded output for *any* bounded input.

2 Notation and Preliminaries

All nonlinear maps are causal, multi-input multi-output and defined over appropriate products of causal extension of the set $\mathcal{L}$ of bounded signals. The time-set $\mathcal{T}$ typically denotes nonnegative reals or integers. For $T \in \mathcal{T}$, let Π_T denote the usual truncation operator and $\|\cdot\|$ denote the associated norm used in describing the bounded signals in $\mathcal{L}$. The causal extension of $\mathcal{L}$ is denoted by $\mathcal{L}_e$. With a slight abuse of notation, $\|\cdot\|$ is also used in describing the product set of bounded signals $\mathcal{L}^n$. For a thorough treatment of general extended spaces within the input-output approach to nonlinear systems, see [1]. The extended space $\mathcal{L}_e$ is a means of incorporating unbounded signals in the analysis; however, although $\mathcal{L} \subset \mathcal{L}_e$, the component $(\mathcal{L})^C \backslash \mathcal{L}_e \neq \emptyset$, where $(\cdot)^C$ denotes the complement with respect to the set of all functions on $\mathcal{T}$. The nonempty intersection arises due to discontinuities which are not jump-discontinuities. Such signals which exhibit "finite escape time" are not covered within the scope of extended spaces; therefore domains restricted to a strictly proper subset of the input extended space might be necessary in describing the nonlinear maps. Hence, $\mathcal{L}$ describes the set of bounded signals and $\mathcal{L}_e \backslash \mathcal{L}$ denotes the set of unbounded signals (unbounded at infinity). An n_1-input n_2-output causal nonlinear map $\mathcal{P}$ is considered as $\mathcal{P} : \mathcal{U} \subset \mathcal{L}_e^{n_1} \to \mathcal{L}_e^{n_2}$, where $\mathcal{U}$ denotes the domain. With appropriate domain and range matchings, the map $\mathcal{FG}$ denotes the composition of two nonlinear causal maps $\mathcal{F}$ and $\mathcal{G}$. In an input-output approach to analysis and design of nonlinear systems, the notions of boundedness and stability are crucial for the subsequent results. Unlike the finite-dimensional linear time-invariant case, most of the properties depend on the particular framework used. The following definition sets up the framework used here [2].

Definition: A causal map $\mathcal{H} : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$ is said to be stable iff there exists a continuous nondecreasing $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|\mathcal{H}u\| \leq \varphi(\|u\|)\ \forall u \in \mathcal{L}^{n_i}$. $\diamond$

[1] Research supported by the National Science Foundation Grant ECS-9257932.

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The bounding φ in this definition need not be strictly increasing, or one-to-one, or onto, or subadditive.

3 Main Result

Theorem 3.1 states a sufficient boundedness condition applying the property of seminorms on the summing-junction equations of the unity-feedback interconnection:

3.1 Theorem: Let $\mathcal{H}_1 : \mathcal{L}_e^{n_1} \to \mathcal{L}_e^{n_2}$ and $\mathcal{H}_2 : \mathcal{L}_e^{n_2} \to \mathcal{L}_e^{n_1}$ be stable maps in the unity-feedback interconnection, where

$$e_1 = u_1 - \mathcal{H}_2 e_2 \quad ; \quad e_2 = u_2 + \mathcal{H}_1 e_1 \ . \qquad (1)$$

For a given $\alpha \in \mathbb{R}_+^2$, let $\|u_1\| \leq \alpha_1$ and $\|u_2\| \leq \alpha_2$. Let $\beta \in \mathbb{R}_+^2$ be such that

$$\left\{ x \in \mathbb{R}_+^2 \mid \begin{array}{lcl} x_1 & \leq & \alpha_1 + \varphi_2(x_2) \\ x_2 & \leq & \alpha_2 + \varphi_1(x_1) \end{array} \right\} \subset [0, \beta_1] \times [0, \beta_2]. \qquad (2)$$

Under these assumptions, if $(e_1, e_2) \in \mathcal{L}_e^{n_1} \times \mathcal{L}_e^{n_2}$ then $\|e_1\| \leq \beta_1$ and $\|e_2\| \leq \beta_2$. ◇

As in the standard finite-gain small-gain setting, no assumptions of existence, uniqueness, or continuity of solutions are made in Theorem 3.1. Although this note emphasizes a map setting, the theorem is still valid for relations. The proof is a simple exercise using causality, the truncation operator Π_T and the property of seminorms on equations (1): Let $(e_1, e_2) \in \mathcal{L}_e^{n_1} \times \mathcal{L}_e^{n_2}$. For any $T \in \mathcal{T}$,

$$\|\Pi_T e_1\| \leq \|\Pi_T u_1\| + \|\Pi_T \mathcal{H}_2 e_2\| \leq \alpha_1 + \varphi_2(\|\Pi_T e_2\|)$$
$$\|\Pi_T e_2\| \leq \|\Pi_T u_2\| + \|\Pi_T \mathcal{H}_1 e_1\| \leq \alpha_2 + \varphi_1(\|\Pi_T e_1\|).$$

Since (2) holds, $\|\Pi_T e_1\| \leq \beta_1$ and $\|\Pi_T e_2\| \leq \beta_2$, for all $T \in \mathcal{T}$, and the conclusion follows.

4 Application and Concluding Remarks

A simple two-dimensional graphics environment is all that is required in order to apply the result in Theorem 3.1. Optimization is not required unless the least-upper-bounding β is sought. Two graphs in $\mathbb{R}_+^2$, i.e., $\{(x_1, \alpha_2 + \varphi_1(x_1)),\ x_1 \in \mathbb{R}_+\}$ and $\{(\alpha_1 + \varphi_2(x_2), x_2) \mid x_2 \in \mathbb{R}_+\}$, are drawn. The desired intersection is the union of possibly disjoint sets formed by intersecting the region *below* the first graph and *above* the second graph. No conditions are imposed on the bounding functions or their compositions. In the case of affine upper-bounding functions with slopes k_1 and k_2 , as in the finite-gain small-gain theorem, a bounded intersection in the nonnegative quadrant is possible if and only if $k_1 k_2 \in [0, 1)$. Changing $\alpha \in \mathbb{R}_+^2$ corresponds to translating the two graphs appropriately. Since bounded intersections may not exist for all α in general, the user can easily see the effect of exogenous signal bounds and extract a tight bound before the sufficient condition fails.

Consider Figure 1, where the sufficient condition of Theorem 3.1 is satisfied. Although there is a finite β for a given α , the finite-gain stability setting would have been inapplicable since the bounding functions are not uniformly continuous.

Now consider the first plot of Figure 2, where the sufficient condition of Theorem 3.1 is satisfied for certain exogenous signal bounds, although the finite-gain approximation would have been inconclusive due to the maximum slopes of the upper-bounding functions. Changing the bound α_1 on the exogenous input u_1 from 0.25 to 0.75 would simply correspond to dragging the graph $\{(\varphi_2(x_2), x_2) \mid x_2 \in \mathbb{R}_+\}$ suitably to the right as seen in the second plot of Figure 2. Consequently, the sufficient condition no longer holds. Such a graphical interface allows the designer to visually extract tight bounds without any use of optimization.

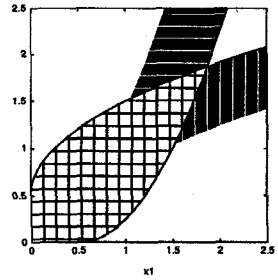

Figure 1: $\varphi_1(\cdot) = \varphi_2(\cdot) = \sqrt{(\cdot)}$ and $\alpha = [0.5\ 0.5]^T$: Below $(x_1, \alpha_2 + \varphi_1(x_1))$ and above $(\alpha_1 + \varphi_2(x_2), x_2)$.

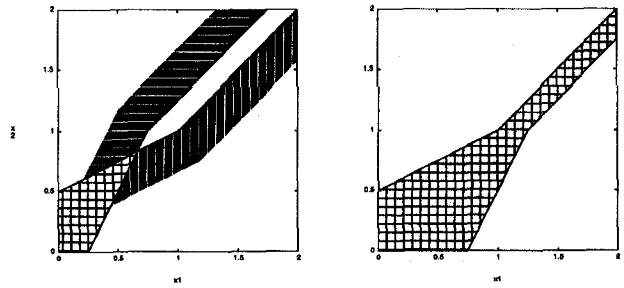

Figure 2: Feasible regions for $\alpha = [0.25\ 0.5]^T$ and $\alpha = [0.75\ 0.5]^T$ for two identical piecewise-linear upper-bounding functions.

Theorem 3.1 also yields a one-dimensional graphical interpretation using for example only $x_1 \leq \alpha_1 + \varphi_2(\alpha_2 + \varphi_1(x_1))$. However, this approach would not have the same simple interpretation in terms of the graphs of φ_1 and φ_2 since for each α_2, the graph of a new function $\varphi_2(\alpha_2 + \varphi_1(\cdot))$ would need to be computed. The two-dimensional approach above uses translations of the same pair of graphs for all α. The one-dimensional approach can be further simplified at the expense of additional assumptions, such as sub-additivity of φ_1 or φ_2. Regardless of which graphical interpretation of Theorem 3.1 is used, when at least one of the output-bounding functions φ_1 or φ_2 is uniformly bounded, there exists a bounded β for any bounded α.

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