[5] D. Czarkowski and M. K. Kazimierczuk, "Linear circuit models of PWM fly back and buck/boost converters," *IEEE Trans. Circuits Syst. I*, vol. 39, pp. 688–693, Aug. 1992.
[6] M. H. Rashid, *Power Electronics/Circuits, Devices and Application*. Englewood Cliffs, NJ: Prentice Hall, 1993.

Coprime Factorizations of Parallel, Cascade, Feedback Interconnections

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Abstract—**Doubly coprime factorizations of the transfer-functions of parallel, cascade, and feedback system interconnections are derived algebraically, extending the results of previous work on parallel interconnections. The factors of these doubly coprime representations derived here are important in characterizing all stabilizing controllers for these interconnections.**

I. Introduction and Preliminaries

This letter extends the results of [1] on right and left coprime factorizations of the transfer-function of parallel interconnections in terms of the constituent subsystems. Algebraic computation of the Bezout-identity terms in the doubly coprime factorization was left as an open problem in [1], which motivated this work. In this letter, these terms are derived without using a state-space representation. In addition to parallel interconnections, doubly coprime factorizations of cascade and feedback interconnections are also given here. These Bezout-identity terms are used in the characterization of all stabilizing controllers for P in the standard unity-feedback system (see for example [3], [2]); these terms play an important role also in reliable stability problems [1].

Let $\mathcal{U}$ contain the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). Let $\mathcal{R}$ denote proper rational functions which do not have any poles in the region of instability $\mathcal{U}$ and let R_p denote proper rational functions with real coefficients. The set of matrices whose entries are in $\mathcal{R}$ is denoted by $\mathcal{M}(\mathcal{R})$. A right-coprime-factorization (RCF) and a left-coprime-factorization (LCF) of $P \in R_p^{n_o \times n_i}$ are denoted by (N, D) and $(\tilde{D}, \tilde{N})$, where $D \in \mathcal{R}^{n_i \times n_i}$, $\tilde{D} \in \mathcal{R}^{n_o \times n_o}$, $N, \tilde{N} \in \mathcal{R}^{n_o \times n_i}$, D and $\tilde{D}$ are biproper and $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$.

Let (N, D) and $(\tilde{D}, \tilde{N})$ be an RCF and an LCF of P; then there exist $Y \in \mathcal{R}^{n_i \times n_i}$, $\tilde{Y} \in \mathcal{R}^{n_o \times n_o}$, $X, \tilde{X} \in \mathcal{R}^{n_i \times n_o}$ such that [3], [2]

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{1}$$

We refer to the description in (1) as a doubly coprime factorization and to the matrices $Y, \tilde{Y}, X, \tilde{X}$ as the Bezout-identity terms. The objective of this letter is to compute these Bezout-identity terms for parallel, cascade, and feedback interconnections of two subsystems P_1, P_2. Let (N_j, D_j) and $(\tilde{D}_j, \tilde{N}_j)$ be an RCF and an LCF of the given subsystems $P_j \in \mathcal{M}(R_p)$ for $j = 1, 2$; then for $j = 1, 2$, there exist $Y_j, \tilde{Y}_j, X_j, \tilde{X}_j \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} Y_j & X_j \\ -\tilde{N}_j & \tilde{D}_j \end{bmatrix} \begin{bmatrix} D_j & -\tilde{X}_j \\ N_j & \tilde{Y}_j \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2}$$

The terms in (2) will be used in deriving of the Bezout-identity terms $Y,\ \tilde{Y},\ X,\ \tilde{X}$ of the interconnection.

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II. Parallel Interconnections

Consider two linear, time-invariant (LTI) systems represented by their proper rational matrix transfer-functions $P_1 \in R_p^{n_o \times n_i}$, $P_2 \in R_p^{n_o \times n_i}$. Let (N_j, D_j) and $(\tilde{D}_j, \tilde{N}_j)$ be an RCF and an LCF of P_j for $j = 1, 2$. The transfer-function $P = P_1 + P_2 \in R_p^{n_o \times n_i}$ of the parallel interconnection of P_1 and P_2 is

$$\begin{aligned} P = P_1 + P_2 &= N_1 D_1^{-1} + N_2 D_2^{-1} = \tilde{D}_1^{-1}\tilde{N}_1 + \tilde{D}_2^{-1}\tilde{N}_2 \\ &= \tilde{D}_1^{-1}\big(\tilde{N}_1 + \tilde{D}_1\tilde{D}_2^{-1}\tilde{N}_2\big) = \big(N_1 D_1^{-1} D_2 + N_2\big) D_2^{-1}. \end{aligned}$$

It can be shown that $P = P_1 + P_2$ has no hidden-modes in the region of instability $\mathcal{U}$ if and only if the pair $(\tilde{D}_1, \tilde{D}_2)$ is right-coprime and the pair (D_1, D_2) is left-coprime. Therefore, we assume that these coprimeness conditions hold. If $(\tilde{D}_1, \tilde{D}_2)$ is a right-coprime pair, then there exist $V, U, \tilde{V}, \tilde{U}, T, S \in \mathcal{M}(\mathcal{R})$ of appropriate dimensions such that [2]

$$\begin{bmatrix} V & U \\ -T & S \end{bmatrix} \begin{bmatrix} \tilde{D}_2 & -\tilde{U} \\ \tilde{D}_1 & \tilde{V} \end{bmatrix} = \begin{bmatrix} I_{n_o} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{3}$$

Similarly, if (D_1, D_2) is a left-coprime pair, then there exist $Z, W, \tilde{Z}, \tilde{W}, \tilde{T}, \tilde{S} \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} Z & W \\ -D_2 & D_1 \end{bmatrix} \begin{bmatrix} \tilde{T} & -\tilde{W} \\ \tilde{S} & \tilde{Z} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_i} \end{bmatrix}. \tag{4}$$

By (2), (3), (4), the terms of (1) are given by

$$\begin{aligned} Y &= (Z + WY_1D_2 - WX_1N_2)(Y_2 + X_2V\tilde{N}_2 - X_2U\tilde{N}_1), \\ \tilde{Y} &= (\tilde{Y}_1 + N_1\tilde{Z}\tilde{X}_1 - N_2\tilde{W}\tilde{X}_1)(\tilde{V} + \tilde{D}_1\tilde{Y}_2\tilde{U} - \tilde{N}_1\tilde{X}_2\tilde{U}), \\ X &= (Z + WY_1D_2 - WX_1N_2)X_2U\tilde{D}_1 + WX_1, \\ \tilde{X} &= D_2\tilde{W}\tilde{X}_1(\tilde{V} + \tilde{D}_1\tilde{Y}_2\tilde{U} - \tilde{N}_1\tilde{X}_2\tilde{U}) + \tilde{X}_2\tilde{U} \\ N &= N_1\tilde{S} + N_2\tilde{T}, \qquad \tilde{N} = S\tilde{N}_1 + T\tilde{N}_2, \\ D &= D_2\tilde{T} = D_1\tilde{S}, \qquad \tilde{D} = S\tilde{D}_1 = T\tilde{D}_2. \end{aligned}$$

Therefore, as in [1], $(N_1\tilde{S} + N_2\tilde{T}, D_2\tilde{T})$ is an RCF and $(S\tilde{D}_1, S\tilde{N}_1 + T\tilde{N}_2)$ is an LCF of the parallel interconnection $P = P_1 + P_2 = (N_1\tilde{S} + N_2\tilde{T})(D_2\tilde{T})^{-1} = (S\tilde{D}_1)^{-1}(S\tilde{N}_1 + T\tilde{N}_2)$.

III. Cascade Interconnections

Consider two LTI systems represented by their transfer-functions $P_1 \in R_p^{n_o \times n_r}$, $P_2 \in R_p^{n_r \times n_i}$. Let (N_j, D_j) and $(\tilde{D}_j, \tilde{N}_j)$ be an RCF and an LCF of P_j for $j = 1, 2$. The transfer-function $P = P_1P_2 \in R_p^{n_o \times n_i}$ of the cascade interconnection of P_1 and P_2 is

$$\begin{aligned} P = P_1P_2 &= N_1D_1^{-1}N_2D_2^{-1} \\ &= \tilde{D}_1^{-1}\tilde{N}_1\tilde{D}_2^{-1}\tilde{N}_2 = N_1(\tilde{D}_2D_1)^{-1}N_2. \end{aligned}$$

It can be shown that $P = P_1P_2$ has no hidden-modes in the region of instability $\mathcal{U}$ if and only if the pair $(\tilde{N}_1, \tilde{D}_2)$ is right-coprime and the pair (D_1, N_2) is left-coprime. Therefore, we assume that these coprimeness conditions hold. If $(\tilde{N}_1, \tilde{D}_2)$ is a right-coprime pair, then there exist $J,\ F, \tilde{J},\ \tilde{F},\ M,\ G \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} J & F \\ -M & G \end{bmatrix}\begin{bmatrix} \tilde{D}_2 & -\tilde{F} \\ \tilde{N}_1 & \tilde{J} \end{bmatrix} = \begin{bmatrix} I_{n_r} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{5}$$

If (D_1, N_2) is a left-coprime pair, then there exist $R,\ H,\ \tilde{R},\ \tilde{H},\ \tilde{M},\ \tilde{G} \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} R & H \\ -N_2 & D_1 \end{bmatrix}\begin{bmatrix} \tilde{G} & -\tilde{H} \\ \tilde{M} & \tilde{R} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_r} \end{bmatrix}. \tag{6}$$

By (2), (5), (6),

$$\begin{aligned} &\begin{bmatrix} Y_1 J & X_1 + Y_1 F\tilde{D}_1 \\ -M & G\tilde{D}_1 \end{bmatrix}\begin{bmatrix} \tilde{D}_2 D_1 & -(\tilde{F} + \tilde{D}_2\tilde{X}_1\tilde{J}) \\ N_1 & \tilde{Y}_1\tilde{J} \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{n_r} & 0 \\ 0 & I_{n_o} \end{bmatrix}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\begin{bmatrix} RY_2 & H + RX_2D_1 \\ -\tilde{N}_2 & \tilde{D}_2 D_1 \end{bmatrix}\begin{bmatrix} D_2\tilde{G} & -(\tilde{X}_2 + D_2\tilde{H}\tilde{Y}_2) \\ \tilde{M} & \tilde{R}\tilde{Y}_2 \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_r} \end{bmatrix}. \end{aligned} \tag{8}$$

By (7), $(N_1, \tilde{D}_2 D_1)$ is a right-coprime pair and by (8), $(\tilde{D}_2 D_1, \tilde{N}_2)$ is a left-coprime pair; therefore, $(N_1, \tilde{D}_2 D_1, \tilde{N}_2)$ is a bicoprime factorization of the cascade interconnection $P = P_1 P_2 = N_1(\tilde{D}_2 D_1)^{-1} N_2$. We also obtain an RCF and an LCF of P and the terms of (1) from (2), (5), (6) as

$$\begin{aligned} Y &= RY_2 + (H + RX_2D_1)Y_1 J\tilde{N}_2 \\ \tilde{Y} &= \tilde{Y}_1\tilde{J} + N_1\tilde{R}\tilde{Y}_2(\tilde{F} + \tilde{D}_2\tilde{X}_1\tilde{J}), \\ X &= (H + RX_2D_1)(X_1 + Y_1F\tilde{D}_1), \\ \tilde{X} &= (\tilde{X}_2 + D_2\tilde{H}\tilde{Y}_2)(\tilde{F} + \tilde{D}_2\tilde{X}_1\tilde{J}), \\ N &= N_1\tilde{M}, \quad \tilde{N} = M\tilde{N}_2, \quad D = D_2\tilde{G}, \quad \tilde{D} = G\tilde{D}_1. \end{aligned}$$

Therefore, $(N_1\tilde{M}, D_2\tilde{G})$ is an RCF and $(G\tilde{D}_1, M\tilde{N}_2)$ is an LCF of the cascade interconnection $P = P_1 P_2 = N_1\tilde{M}(D_2\tilde{G})^{-1} = (G\tilde{D}_1)^{-1}M\tilde{N}_2$.

IV. Feedback Interconnections

Consider two LTI systems represented by their transfer-functions $P_1 \in R_p^{n_o \times n_i}$, $P_2 \in R_p^{n_i \times n_o}$. Let (N_j, D_j) and $(\tilde{D}_j, \tilde{N}_j)$ be an RCF and an LCF of P_j for $j = 1, 2$. The transfer-function of the feedback interconnection of P_1 and P_2 is given by $P = P_1(I + P_2P_1)^{-1}$. We assume that the interconnection is well-posed, equivalently, $(I + P_2P_1)^{-1}$ is proper. Using the coprime factorizations of the subsystems, the transfer-function P becomes

$$\begin{aligned} P &= P_1(I_{n_i} + P_2P_1)^{-1} = N_1D_1^{-1}\left(I_{n_i} + N_2D_2^{-1}N_1D_1^{-1}\right)^{-1} \\ &= \left(I_{n_o} + \tilde{D}_1^{-1}\tilde{N}_1\tilde{D}_2^{-1}\tilde{N}_2\right)^{-1}\tilde{D}_1^{-1}\tilde{N}_1 \\ &= N_1(\tilde{D}_2D_1 + \tilde{N}_2N_1)^{-1}\tilde{D}_2. \end{aligned}$$

It can be shown that $P = P_1(I + P_2P_1)^{-1}$ has no hidden-modes in the region of instability $\mathcal{U}$ if and only if the pair $(\tilde{N}_1, \tilde{D}_2)$ is right-coprime and the pair (D_2, N_1) is left-coprime. Therefore, we assume that these coprimeness conditions hold. If $(\tilde{N}_1, \tilde{D}_2)$ is a right-coprime pair, then there exist $\Delta,\ \Lambda,\ \tilde{\Delta}, \tilde{\Lambda},\ \Theta,\ \Sigma \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} \Delta & \Lambda \\ -\Theta & \Sigma \end{bmatrix}\begin{bmatrix} \tilde{D}_2 & -\tilde{\Lambda} \\ \tilde{N}_1 & \tilde{\Delta} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{9}$$

If (D_2, N_1) is a left-coprime pair, then there exist $\Psi,\ \Phi, \tilde{\Psi}, \tilde{\Phi}, \tilde{\Theta}, \tilde{\Sigma} \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} \Psi & \Phi \\ -N_1 & D_2 \end{bmatrix}\begin{bmatrix} \tilde{\Sigma} & -\tilde{\Phi} \\ \tilde{\Theta} & \tilde{\Psi} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{10}$$

By (2), (9), (10)

$$\begin{aligned} &\begin{bmatrix} Y_1\Delta & X_1 + Y_1\Lambda\tilde{D}_1 - Y_1\Delta\tilde{N}_2 \\ -\Theta & \Sigma\tilde{D}_1 + \Theta\tilde{N}_2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \tilde{D}_2D_1 + \tilde{N}_2N_1 & -(\tilde{\Lambda} + \tilde{D}_2\tilde{X}_1\tilde{\Delta} - \tilde{N}_2\tilde{Y}_1\tilde{\Delta}) \\ N_1 & \tilde{Y}_1\tilde{\Delta} \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\begin{bmatrix} \Phi X_2 & \Psi + \Phi Y_2N_1 - \Phi X_2D_1 \\ -\tilde{D}_2 & \tilde{D}_2D_1 + \tilde{N}_2N_1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} N_2\tilde{\Theta} + D_1\tilde{\Sigma} & -(\tilde{Y}_2 + N_2\tilde{\Psi}\tilde{X}_2 - D_1\tilde{\Phi}\tilde{X}_2) \\ \tilde{\Sigma} & \tilde{\Phi}\tilde{X}_2 \end{bmatrix} \\ &\quad = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_i} \end{bmatrix}. \end{aligned} \tag{12}$$

By (11), $(N_1, \tilde{D}_2D_1 + \tilde{N}_2N_1)$ is a right-coprime pair and by (12), $(\tilde{D}_2D_1 + \tilde{N}_2N_1, \tilde{D}_2)$ is a left-coprime pair; therefore, $(N_1, \tilde{D}_2D_1 + \tilde{N}_2N_1, \tilde{D}_2)$ is a bicoprime factorization of the feedback interconnection $P = P_1(I + P_2P_1)^{-1} = N_1(\tilde{D}_2D_1 + \tilde{N}_2N_1)^{-1}\tilde{D}_2$. We also obtain an RCF and an LCF of P and the terms of (1) from (2), (9), (10) as

$$\begin{aligned} Y &= \Phi X_2 + (\Psi + \Phi Y_2N_1 - \Phi X_2D_1)Y_1\Delta\tilde{D}_2, \\ \tilde{Y} &= \tilde{Y}_1\tilde{\Delta} + N_1\tilde{\Phi}\tilde{X}_2(\tilde{\Lambda} + \tilde{D}_2\tilde{X}_1\tilde{\Delta} - \tilde{N}_2\tilde{Y}_1\tilde{\Delta}), \\ X &= (\Psi + \Phi Y_2N_1 - \Phi X_2D_1)(X_1 + Y_1\Lambda\tilde{D}_1 - Y_1\Delta\tilde{N}_2), \\ \tilde{X} &= (\tilde{Y}_2 + N_2\tilde{\Psi}\tilde{X}_2 - D_1\tilde{\Phi}\tilde{X}_2)(\tilde{\Lambda} + \tilde{D}_2\tilde{X}_1\tilde{\Delta} - \tilde{N}_2\tilde{Y}_1\tilde{\Delta}), \\ N &= N_1\tilde{\Sigma}, \quad \tilde{N} = \Sigma\tilde{N}_1, \\ D &= D_1\tilde{\Sigma} + N_2\tilde{\Theta}, \quad \tilde{D} = \Sigma\tilde{D}_1 + \Theta\tilde{N}_2. \end{aligned}$$

Therefore, $(N_1\tilde{\Sigma}, D_1\tilde{\Sigma} + N_2\tilde{\Theta})$ is an RCF and $(\Sigma\tilde{D}_1 + \Theta\tilde{N}_2, \Sigma\tilde{N}_1)$ is an LCF of the feedback interconnection $P = P_1(I + P_2P_1)^{-1} = N_1\tilde{\Sigma}(D_1\tilde{\Sigma} + N_2\tilde{\Theta})^{-1} = (\Sigma\tilde{D}_1 + \Theta\tilde{N}_2)^{-1}\Sigma\tilde{N}_1$.

References

[1] K. D. Minto, "On coprime factorizations for parallel interconnections of systems," *IEEE Trans. Circuits Syst. I*, vol. 40, pp. 626–630, Sept. 1993.

[2] A. N. Gündeş and C. A. Desoer, *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators*, (Lecture Notes in Control and Information Sciences.) New York: Springer-Verlag, vol. 142, 1990.

[3] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: M.I.T. Press, 1985.