# Decoupling Linear Multiinput Multioutput Plants by Dynamic Output Feedback: An Algebraic Theory

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*Abstract*—**This paper presents an *algebraic* theory for the design of a decoupling compensator for linear time-invariant multiinput multioutput systems. The design method uses a two-input one-output compensator, which gives a convenient parametrization of *all diagonal* input–output (I/O) maps and *all* disturbance-to-output (D/O) maps achievable by a stabilizing compensator for a given plant. It is shown that this method has two degrees of freedom: *any* achievable diagonal I/O map and *any* achievable D/O map can be realized simultaneously by a choice of an appropriate compensator. The difference between *all* achievable *diagonal* and *nondiagonal* I/O maps and the "cost" of decoupling is discussed for some particular algebraic settings.**

## I. Introduction

In the design theory of linear time-invariant, multiinput multioutput (MIMO) systems, the characterization of all designs which can be achieved by a stabilizing controller for a given plant is a subject of great interest because it shows the limitations on achievable performance imposed by the plant model and the constraints of linearity and stability. Stabilizing compensators were first characterized by Youla *et al.* [31] for the lumped continuous and discrete-time cases. Later, an algebraic formulation was given by Desoer *et al.* [9] to include the lumped and distributed continuous-time and discrete-time cases. Using algebraic tools, Zames [32] considered stable plants, characterized all stabilizing compensators, and established bounds on closed-loop performance. His methods were used for design in [10]. Further results in parametrized form were given in [25], [5], [28], [24], and [29] until finally a general algebraic design procedure, which enables design with nonsquare plants and controllers and extends the parametrizations of [31] and [25] was obtained in [11].

This paper presents a general algebraic design method for all *diagonal* input–output (I/O) maps which can be achieved by a stabilizing two-input-one-output controller for a given plant. The design method is referred to as two-parameter compensation [30] or two-degrees-of-freedom design [17]. We consider the MIMO configuration $\Sigma\,(P, K)$ of Fig. 1, where the plant $P$ has an output $y_0$ and a measured output $y_m$ and the controller $K$ has two inputs: the exogenous input $v$ and the feedback signal $e_1 = u_1 - y_m$. Such two-parameter controllers were used, for example, in [1], [25], and [11]. This two-parameter compensation scheme enables us to design the I/O map independently of the D/O map and, therefore, requiring the compensator to diagonalize the I/O map leaves the stabilizing nature of the compensator intact. Furthermore, any plant, which satisfies the assumption to be given in Section II, can be stabilized and decoupled with a proper compensator, and unlike in one-parameter compensation schemes, decoupling brings no restrictions to those parameters of the compensator that are used in stabilization.

Some of the related work in this area can be summarized as follows. Decoupling of linear time-invariant multivariable systems over unique factorization domains is considered in [8]; necessary and sufficient conditions are established for the existence of a decoupling dynamic or static *state* feedback in the case that the system is internally *stable* and *reachable.* Furthermore, the stability preserving stable compensator is required to be invertible over the unique factorization domain. In the present paper, the plant is *not* assumed to be stable, dynamic *output* feedback is used, the compensator is not required to be stable, and if stable, it is not required to be invertible over the principal ring. Hammer and Khargonekar [16] give necessary and sufficient conditions for a plant $P$ to be decoupled using a one-parameter compensator *$C$ placed in the feedback loop,* and show that, in the lumped continuous-time case, there is no proper compensator which would decouple a plant whose inverse has off-diagonal polynomial terms: with strictly proper plant and proper compensator, the inverse of the resulting diagonal I/O map is $[P(I + CP)^{-1}]^{-1} = (I + CP)P^{-1}$, which approaches $P^{-1}$ as $|s| \to \infty$; hence the configuration proposed introduces the unnecessary constraint that the polynomial part of $P^{-1}$ must be diagonal. This problem does not arise with our two-parameter compensation scheme. Dion and Commault [14] study the row by row decoupling of a strictly proper system by dynamic *state* feedback defined by $u = F(s)x + Gu$ where $F(s)$ is a proper rational matrix and $G$ is a constant matrix; the equivalent compensator is a precompensator $B(s)G$, where $B(s)$ and its inverse are proper matrices. They give the conditions for decoupling by such a compensator and give the minimum McMillan degree achievable for the decoupled system (see [14] and the references therein). By restricting the plant $P(s)$ to approach diagonal dominance as $|s| \to \infty$, Zames and Bensoussan [33] include a study of decoupling with an arbitrarily small tolerance using a compensator in the feedback loop.

The system $\Sigma\,(P, K)$ shown in Fig. 1 represents a general configuration in which $y_o$, the output-of-interest, is not necessarily the same as the measured-output $y_m$, which is the feedback input to the compensator; furthermore, the disturbance $d$ is applied directly to the pseudostate of the plant rather than being an additive input as, for example, in [11]. The paper is organized as follows.

Section II defines the problem and states the stabilizability conditions. Section III builds the structures used for decoupling the I/O map, and presents the main results: the achievable diagonal I/O maps and the achievable D/O maps. Some examples and the conclusions are in Section IV.

The following is a list of the commonly used symbols.

a := b means a denotes b. $\theta_n$ is the n-vector of zeros. W.l.o.g. means without loss of generality. U.t.c. means under these conditions. If $\mathcal{H}$ is a ring, then $\mathcal{E}(\mathcal{H})$ denotes the set of matrices having all entries in $\mathcal{H}$. $\mathcal{R}_u$ denotes the proper rational functions

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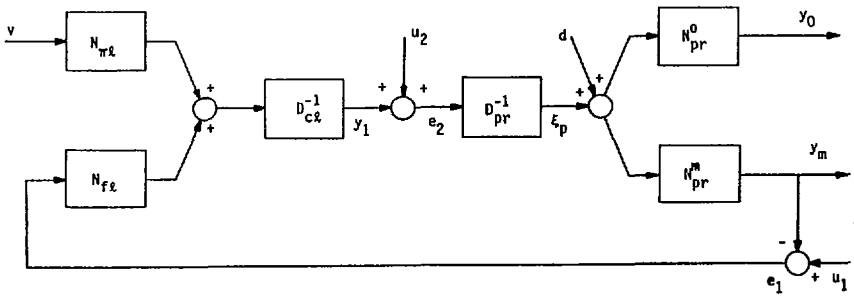


Fig. 1. The system $\Sigma$ $(P, K)$.

analytic in the region $\mathcal{U} \subset \mathbb{C}$, a symmetric subset of $\mathbb{C}$ which contains $\mathbb{C}_+$ and $\bar{\mathcal{U}} := \mathcal{U} \cup \{\infty\}$. $\mathbb{R}(s)$ denotes the scalar rational functions in $s$ with real coefficients, and $\mathbb{R}[s]$ denotes the scalar polynomials in s with real coefficients.

Throughout the paper, the properties of groups and of commutative rings are used; these and other standard algebraic terms can be found, for example, in [2], [7], [18], [21], [22], and [34]. The algebraic structure used here is similar to that of [11].

*Algebraic Structure [2, p. 55], [18, p. 393], [21, p. 69]:*

$\mathcal{H}$: A principal ring (principal ideal domain), i.e., an entire commutative ring in which every ideal is principal (e.g., $\mathcal{R}_{\mathcal{U}}$).

$\tilde{\mathcal{G}}$: The field of fractions over $\mathcal{H}$ [e.g., $\mathbb{R}(s)$].

$\mathcal{I}$: A multiplicative subset of $\mathcal{H}$, equivalently, $\mathcal{I} \subset \mathcal{H}$, $0 \notin \mathcal{I}$, and $x, y, \in \mathcal{I}$ implies that $xy \in \mathcal{I}$. W.l.o.g. $1 \in \mathcal{I}$ (e.g., $f \in \mathcal{I}$ if $f \in \mathcal{R}_{\mathcal{U}}$ and $f(\infty) = 1$).

$\mathcal{G} := \{n/d : n \in \mathcal{H}, d \in \mathcal{I}\}$, a subring of $\tilde{\mathcal{G}}$ (e.g., $\mathbb{R}_p(s)$, the ring of proper scalar rational functions).

$U(\mathcal{H}) := \{m \in \mathcal{H}: m^{-1} \in \mathcal{H}\}$, the group of units in $\mathcal{H}$ (e.g., $f \in U(\mathcal{H})$ if $f \in \mathcal{R}_{\mathcal{U}}$ and $f(s) \neq 0$ for all $s \in \bar{\mathcal{U}}$).

$\mathcal{G}_s := \{x \in \mathcal{G} : (1 + xy)^{-1} \in \mathcal{G}, \forall y \in \mathcal{G}\}$ (Jacobson radical of $\mathcal{G}$) (e.g., $\mathbb{R}_{p,o}(s)$, the set of strictly proper scalar rational functions).

Four examples of this algebraic structure are given in [11, Table I].

## II. Design Theory

### *A. Problem Description*

We consider the MIMO linear, time-invariant system $\Sigma$ $(P, K)$($^1\Sigma$ $(P, K)$) shown in Fig. 1 (Fig. 2). Given a plant $P$, we wish to design a controller $K$ with two inputs and one output such that the resulting feedback system is stable, $K$ has elements in $\mathcal{G}$, and the I/O map $v \mapsto y_o$ is *nonsingular and decoupled,* i.e., *diagonal.* We make the following assumptions on $\Sigma$ $(P, K)$.

*Assumptions on the System $\Sigma$ (P, K):*

(P) $P = [{}^{P^o}_{P^m}] \in \mathcal{G}^{2n \times n}$, and $\det P^o \neq 0$. Consequently, let

$$\begin{bmatrix} N^o_{pr} \\ \cdots \\ N^m_{pr} \end{bmatrix} D^{-1}_{pr},$$

with $D_{pr} \in \mathcal{H}^{n \times n}$, $N^o_{pr}$, $N^m_{pr} \in \mathcal{H}^{n \times n}$ and $\det D_{pr} \in \mathcal{I}$, $\det N^o_{pr} \neq 0$, be a right-coprime factorization (r.c.f.) of $P$.

(K) $K \in \mathcal{G}^{n \times 2n}$. Consequently, let $D^{-1}_{cl}[N_{\pi l} \vdots N_{fl}]$ with $D_{cl} \in \mathcal{H}^{n \times n}$, $N_{\pi l} \in \mathcal{H}^{n \times n}$, $N_{fl} \in \mathcal{H}^{n \times n}$, and $\det D_{cl} \in \mathcal{I}$ be a left-coprime factorization (l.c.f.) of $K$; we further assume that $\det (D_{cl}D_{pr} + N_{fl}N^m_{pr}) \in \mathcal{I}$.

It is understood that the subsystems $P$ and $K$, specified by their transfer functions, do not have any unstable hidden modes [3, sect. 4.2].

Under assumptions (P) and (K), the system $\Sigma$ $(P, K)$ in Fig. 1 is completely described by

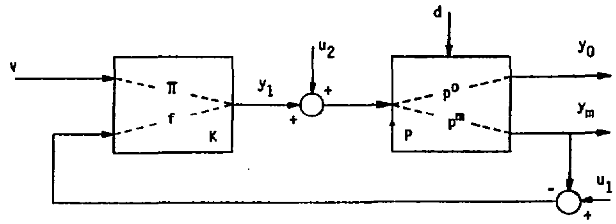


Fig. 2. The system $^1\Sigma$ $(P, K)$.

$$\begin{bmatrix} I_n & \vdots & -D_{pr} \\ \cdots & \vdots & \cdots \\ D_{cl} & \vdots & N_{fl}N^m_{pr} \end{bmatrix} \begin{bmatrix} y_1 \\ \\ \xi_p \end{bmatrix} = \begin{bmatrix} 0 & \vdots & 0 & \vdots & -I_n & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ N_{\pi l} & \vdots & N_{fl} & \vdots & 0 & \vdots & -N_{fl}N^m_{pr} \end{bmatrix} \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} I_n & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & N^o_{pr} \\ \cdots & \vdots & \cdots \\ 0 & \vdots & N^m_{pr} \end{bmatrix} \begin{bmatrix} y_1 \\ \xi_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_o \\ y_m \end{bmatrix} + \begin{bmatrix} 0 & & 0 & & 0 & & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & -N^o_{pr} \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & -N^m_{pr} \end{bmatrix} \begin{bmatrix} v \\ u_1 \\ u_2 \\ d \end{bmatrix}. \tag{2.2}$$

Let $u := (v^T, u_1^T, u_2^T, d^T)^T$, $\xi := (y_1^T, \xi_p^T)^T$, $y := (y_1^T, y_o^T, y_m^T)^T$. Then (2.1) and (2.2) are of the form

$$D\xi = N_l u \tag{2.3}$$

$$N_r\xi = y + Eu \tag{2.4}$$

where the matrices $D$, $N_l$, $N_r$, $E$, defined in an obvious manner from (2.1) and (2.2), have all their elements in $\mathcal{H}$.

For any $D_{cl} \in \mathcal{H}^{n \times n}$ and any $N_{fl} \in \mathcal{H}^{n \times n}$, define

$$D_h := D_{cl}D_{pr} + N_{fl}N^m_{pr}. \tag{2.5}$$

Note that $\det D = \det D_h$ and, by assumption (K), $\det D \in \mathcal{I}$.

Let assumptions (P) and (K) hold; then from (2.3) and (2.4) we obtain

$$H_{yu} = N_r D^{-1} N_l + E \in \mathcal{E}(\mathcal{G}). \tag{2.6}$$

Thus, $\det D \in \mathcal{I}$ is a sufficient condition for the well-posedness of $\Sigma$ $(P, K)$.

*Definition 2.1—($\mathcal{H}$-stability):* The system $\Sigma(P, K)$ is said to be $\mathcal{H}$-stable if and only if $H_{yu}: u \mapsto y$ satisfies $H_{yu} \in \mathcal{E}(\mathcal{H})$.

*Definition 2.2 (Stabilizing Controller):* Let the plant $P$ satisfy (P); the controller $K$ *is said to stabilize* $P$ iff $K$ satisfies assumption (K) and the resulting system $\Sigma(P, K)$ is $\mathcal{H}$-stable.

*Proposition 2.3 (Stabilizability of P):* Let $P$ satisfy (P), and in addition let $P^m \in \mathcal{G}_s^{n\times n}$. Then

i) $K$ stabilizes $P$ if and only if $\det D_h \in U(\mathcal{H})$;

ii) there is a compensator which stabilizes $P$ if and only if $(N_{pr}^m, D_{pr})$ is a right-coprime (r.c.) pair, i.e., there are matrices $U_{pr}^m$, $V_{pr}^m \in \mathcal{E}(\mathcal{H})$ such that

$$U_{pr}^m N_{pr}^m + V_{pr}^m D_{pr} = I_n. \tag{2.7}$$

*Remark (Normalization) [30]:* W.l.o.g. $K$ stabilizes $P$ if and only if

$$D_h = I. \tag{2.8}$$

*Proof of Proposition 2.3:* See [13].

## III. Achievable Performance of $\Sigma(P, K)$

In order to characterize all *diagonal* I/O maps which can be achieved by $\Sigma(P, K)$ for the given plant P, we introduce two diagonal matrices: $\Delta_L$ and $\Delta_R$.

*Construction of $\Delta_L$ and $\Delta_R$:* Let $P \in \mathcal{G}^{2n\times n}$; $K \in \mathcal{G}^{n\times 2n}$.

Let $n_{pk} \in \mathcal{H}^{1\times n}$ denote the $k$th row of $N_{pr}^o \in \mathcal{H}^{n\times n}$. For $k = 1, \cdots, n$, define $\Delta_{Lk}$ as a greatest common divisor (g.c.d.) over $\mathcal{H}$ of the elements of $n_{pk}$ [21, p. 71]: such $\Delta_{Lk}$ is well defined within a unimodular factor since $\mathcal{H}$ is a principal ring. Let the row-vector $\tilde{n}_{pk} \in \mathcal{H}^{1\times n}$ be defined by $n_{pk} = \Delta_{Lk}\tilde{n}_{pk}$. Let $\tilde{N}_{pr}^o \in \mathcal{H}^{n\times n}$ be defined as the matrix which has $\tilde{n}_{pk}$ as its $k$th row. Then

$$N_{pr}^o = \text{diag}\,(\Delta_{L1}, \cdots, \Delta_{Lk}, \cdots, \Delta_{Ln})\tilde{N}_{pr}^o =: \Delta_L \tilde{N}_{pr}^o \tag{3.1}$$

where $\Delta_L$ and $\tilde{N}_{pr}^o$ are not unique, since each $\Delta_{Lk}$ is only defined within a factor in $U(\mathcal{H})$. (In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, $\Delta_{Lk}$ "bookkeeps" the plant zeros in $\mathcal{U}$ that are common to all elements of the $k$th row of $N_{pr}^o$.) A similar factorization is used in [8].

The matrix $\tilde{N}_{pr}^o$ is not necessarily invertible over $\mathcal{H}^{n\times n}$; but by assumption (P), and from (3.1), $(\tilde{N}_{pr}^o)^{-1}$ has elements in the field of fractions $[\mathcal{H}][\mathcal{H}\setminus 0]^{-1}$ of the entire ring $\mathcal{H}$ since $\det \tilde{N}_{pr}^o \in \mathcal{H}$, and $\det N_{pr}^o = \det \Delta_L \det \tilde{N}_{pr}^o$, where $\Delta_L$ is nonsingular by construction [21, p. 69]. Let $m_{ij}/d_{ij}$ denote the $ij$th element of $(\tilde{N}_{pr}^o)^{-1}$, $i, j = 1, \cdots, n$, where $m_{ij}$, $d_{ij} \in \mathcal{H}$ are coprime; thus

$$(\tilde{N}_{pr}^o)^{-1} =: \left[\frac{m_{ij}}{d_{ij}}\right]. \tag{3.2}$$

For $j = 1, \cdots, n$, let $\Delta_{Rj}$ be a least common multiple (l.c.m.) of $d_{1j}, d_{2j}, \cdots, d_{nj}$ the elements of the $j$th column of $(\tilde{N}_{pr}^o)^{-1}$ [21, p. 72]. Each $\Delta_{Rj}$ is defined within a factor in $U(\mathcal{H})$. Define

$$\Delta_R := \text{diag}\,(\Delta_{R1}, \cdots, \Delta_{Rj}, \cdots, \Delta_{Rn}) \in \mathcal{H}^{n\times n}. \tag{3.3}$$

An extraction of a diagonal factor analogous to $\Delta_R$ is done in [10].

*Lemma 3.1:* Let $\tilde{N}_{pr}^o$ and $\Delta_R$ be defined by (3.1) and (3.3). Then $(\tilde{N}_{pr}^o)^{-1}\Delta_R \in \mathcal{H}^{n\times n}$.

*Proof:* Since $\Delta_{Rj}$ is an l.c.m. of $(d_{ij})_{i=1}^n$, for $i = 1, \cdots, n$, we have some $\bar{d}_{ij} \in \mathcal{H}$ such that

$$\Delta_{Rj} = d_{ij}\bar{d}_{ij}. \tag{3.4}$$

Then the $ij$th element of $(\tilde{N}_{pr}^o)^{-1}\Delta_R$ is $(m_{ij}/d_{ij})\Delta_{Rj} = m_{ij}\bar{d}_{ij} \in \mathcal{H}$ by (3.2) and (3.4). ■

### *The I/O Map $H_{y_ov}$ and the D/O Map $H_{y_od}$*

For any system $\Sigma(P, K)$ satisfying (P) and (K) (hence, for which $\det D_h \in \mathcal{J}$), (2.1) and (2.2) show that the I/O map $H_{y_ov}: v \mapsto y_o$ and the D/O map $H_{y_od}: d \mapsto y_o$ are given by

$$H_{y_ov} = N_{pr}^o D_h^{-1} N_{\pi l} \tag{3.5}$$

$$H_{y_od} = N_{pr}^o [I - D_h^{-1} N_{fl} N_{pr}^m] = N_{pr}^o D_h^{-1} D_{cl} D_{pr}. \tag{3.6}$$

Now if $K$ stabilizes $P$, by (2.8), (2.5), and (3.1) we obtain

$$H_{y_ov} = N_{pr}^o N_{\pi l} = \Delta_L \tilde{N}_{pr}^o N_{\pi l} \tag{3.7}$$

$$H_{y_od} = N_{pr}^o [I - N_{fl} N_{pr}^m] = N_{pr}^o D_{cl} D_{pr}. \tag{3.8}$$

We now use the relationships between the stabilizing controller $K$ and $\det D_h$ to give global parametrizations of a) the family of *all* diagonal I/O maps possible for a given plant with some *stabilizing controller, and* b) the family of *all* disturbance-to-output (D/O) maps possible for a given plant with some *stabilizing controller.*

*Definition 3.1 (Achievable Maps):* Let $P$ be a given plant that satisfies assumption (P); Roughly speaking, let $\mathcal{H}_{y_ov}(P)$ denote the set of all *achievable diagonal I/O* maps of $\Sigma(P, K)$ and let $\mathcal{H}_{y_od}(P)$ denote the set of all *achievable D/O* maps of $\Sigma(P, K)$; more precisely,

$$\mathcal{H}_{y_ov}(P) := \{H_{y_ov}: K \text{ stabilizes } P \text{ and the resulting I/O map } H_{y_ov} \text{ is diagonal and nonsingular}\} \tag{3.9}$$

$$\mathcal{H}_{y_od}(P) := \{H_{y_od}: K \text{ stabilizes } P \text{ and the resulting I/O map } H_{y_ov} \text{ is diagonal and nonsingular}\}. \tag{3.10}$$

The following theorem characterizes all the achievable diagonal nonsingular I/O maps and all the achievable D/O maps for $\Sigma(P, K)$.

*Theorem 3.2 (Achievable Diagonal I/O Maps and Achievable D/O Maps):* Consider the system $\Sigma(P, K)$ of Fig. 1. Let $P$ satisfy assumption (P) and let $(N_{pr}^m, D_{pr})$ be r.c. Let $D_{pl}^{-1}N_{pl}^m$ be an l.c.f. of $P^m$, where $D_{pl}, N_{pl}^m \in \mathcal{H}^{n\times n}$ and $\det D_{pl} \in \mathcal{J}$. Let $\Delta_L$, $\Delta_R$ be defined by (3.1) and (3.3) above. Then

i) any map $H_v \in \mathcal{H}^{n\times n}$ is an achievable diagonal, nonsingular I/O map of the $\mathcal{H}$-stable system $\Sigma(P, K)$ if and only if $H_v \in \mathcal{H}_{y_ov}(P)$, where

$$\mathcal{H}_{y_ov}(P) = \{\Delta_L \Delta_R Q_d : Q_d \in \mathcal{H}^{n\times n} \text{ is diagonal and nonsingular}\} \tag{3.11}$$

ii) any map $H_d \in \mathcal{H}^{n\times n}$ is an achievable D/O map of the $\mathcal{H}$-stable system $\Sigma(P, K)$ if and only if $H_d \in \mathcal{H}_{y_od}(P)$, where

$$\begin{aligned}\mathcal{H}_{y_od}(P) = \{&N_{pr}^o[I - (U_{pr}^m + RD_{pl})N_{pr}^m] \\ &= N_{pr}^o(V_{pr}^m - RN_{pl}^m)D_{pr} : R \in \mathcal{H}^{n\times n} \text{ s.t.} \\ &\cdot \det(V_{pr}^m - RN_{pl}^m) \in \mathcal{J} \\ &\text{where } V_{pr}^m,\ U_{pr}^m \text{ satisfy (2.7)}\}.\end{aligned} \tag{3.12}$$

*Comments:* 1) If decoupling were not required, the set of all achievable I/O maps of $\Sigma(P, K)$ would be given by

$$\mathcal{H}_{y_ov}(P) = \{N_{pr}^o Q = \Delta_L \tilde{N}_{pr}^o Q : Q \in \mathcal{H}^{n\times n}\} \tag{3.13}$$

and the set of all achievable D/O maps would still be given by (3.12) [11]. Requiring the I/O map to be diagonal adds a number of constraints to the set of maps in (3.13): i) $Q_d \in \mathcal{H}^{n\times n}$ must be *diagonal*; ii) we have $\Delta_L\Delta_R$ as a left factor of the I/O map $H_{y_ov}$ instead of just $\Delta_L$. In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, we can interpret the cost of decoupling as follows: the $\mathcal{U}$-zeros of $P^o: e_2 \mapsto y_o$ will always be the zeros of $H_{y_ov}$ whether the I/O map is decoupled or not. However, with decoupling, the multiplicity (as a zero of $\det H_{y_ov}$) of these $\mathcal{U}$-zeros may be greater than the multiplicity as a zero of $P^o$. This is due to $\Delta_R$: indeed, $\Delta_L$ is extracted directly from $N^o_{pr}$, and if $\tilde{N}^o_{pr}$ is invertible over $\mathcal{H}^{n\times n}$, the resulting I/O map will have the same $\mathcal{U}$-zeros as the original $P^o$ assuming that $Q_d$ brings no $\mathcal{U}$-zeros; but since $\Delta_R$ is constructed so that $N_{\pi l} = (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d \in \mathcal{E}(\mathcal{H})$, det $\Delta_R$ has a greater multiplicity of the same $\mathcal{U}$-zeros than $N^o_{pr}$ has. It is shown in the Appendix that if $n = 2$, det $\Delta_R$ = (det $\tilde{N}^o_{pr})^2$ within unit factors in $\mathcal{H}$. If $(\tilde{N}^o_{pr})^{-1} \in \mathcal{H}^{n\times n}$, the diagonal I/O maps are of the form $\Delta_L Q_d$.

2) The diagonalization of the I/O map is achieved by choosing $N_{\pi l}$; this choice is *independent* of the choice of $D_{cl}$ and $N_{fl}$, which appear in the D/O map. Similarly, $N_{\pi l}$ does not appear in the D/O map. Thus, the I/O map and the D/O map of the $\mathcal{H}$-stable $\Sigma$ $(P, K)$ can be specified independently: it is a two-degrees of freedom design [17]. The parameter $R$ appearing in the D/O map is related to the system stability, but the parameter $Q_d$ in (3.11) is only used in shaping the output.

3) It is important to note the constraints imposed on $H_{y_od}$ by the $\mathcal{U}$-zeros and the $\mathcal{U}$-poles of the plant when $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$. If $\Sigma$ $(P, K)$ is $\mathcal{H}$-stable and if $PF := PD_{cl}^{-1}N_{fl}$ is full normal rank in $\mathbb{R}_p(s)$, then:

a) If $z_o$ is a $\mathcal{U}$-zero of $N^o_{pr}$ (equivalently, $\exists\alpha \neq \theta_n$ such that $\alpha^* N^o_{pr}(z_o) = \theta_n$) then

$$\alpha^* N^o_{pr}(I - N_{fl}N^m_{pr})(z_o) = \alpha^* H_{y_od}(z_o) = \theta_n. \tag{3.14}$$

b) If $N^m_{pr}$ has full normal rank and if $z_m$ is a $\mathcal{U}$-zero of $N^m_{pr}$ (equivalently, $\exists\beta \neq \theta_n$ such that $N^m_{pr}(z_m)\beta = \theta_n$), then

$$N^o_{pr}(I - N_{fl}N^m_{pr})(z_m)\beta = N^o_{pr}(z_m)\beta = H_{y_od}(z_m)\beta. \tag{3.15}$$

c) If $p_o$ is a $\mathcal{U}$-pole of $P$ (equivalently, $\exists\gamma \neq \theta_n$ such that $D_{pr}(p_o)\gamma = \theta_n$), then

$$N^o_{pr}D_{cl}D_{pr}(p_o)\gamma = H_{y_od}(p_o)\gamma = \theta_n. \tag{3.16}$$

Thus, whenever either $N^o_{pr}$ or $N^m_{pr}$ has a $\mathcal{U}$-zero or when $P$ has a $\mathcal{U}$-pole, the D/O map is constrained by a vector-equality such as (3.14)–(3.16), respectively.

*Proof of Theorem 3.2:* (=>) We are given $P$ satisfying (P) and any diagonal nonsingular I/O map $H_v \in \mathcal{H}^{n\times n}$ and any D/O map $H_d \in \mathcal{H}^{n\times n}$ achieved by the $\mathcal{H}$-stable system $\Sigma$ $(P, K)$. Since $H_v$ is an achievable I/O map, $K$ satisfies assumption (K). We must show that $H_v$ is of the form $\Delta_L\Delta_R Q_d$ for some diagonal, nonsingular $Q_d \in \mathcal{H}^{n\times n}$ and $H_d$ is of the form $N^o_{pr}[I - (U^m_{pr} + RD_{pl})N^m_{pr}] = N^o_{pr}(V^m_{pr} - RN^m_{pl})D_{pr}$ for some $R \in \mathcal{H}^{n\times n}$ satisfying det $(V^m_{pr} - RN^m_{pl}) \in \mathcal{J}$.

Since $\Sigma$ $(P, K)$ is $\mathcal{H}$-stable, using (2.8), (3.5), (3.7), and (3.1), we see that the diagonal matrix $\Delta_L \in \mathcal{H}^{n\times n}$ is obviously a left-factor of $H_v$. It remains to show that $\Delta_R$ is also a factor. For a contradiction, suppose that for all diagonal $Q_d \in \mathcal{H}^{n\times n}$, $H_v$ is of the form

$$H_v = \Delta_L\tilde{\Delta}_R Q_d \tag{3.17}$$

where $\tilde{\Delta}_R$ is a *proper* factor of $\Delta_R$, and $Q_d \in \mathcal{H}^{n\times n}$ is nonsingular and diagonal. W.l.o.g. suppose, for example, that

$$\tilde{\Delta}_R = \text{diag}\ (\Delta_{R1}, \cdots, \Delta_{Rj-1}, \tilde{\Delta}_{Rj}, \Delta_{Rj+1}, \cdots, \Delta_{Rn}) \tag{3.18}$$

where, for a *nonunit prime* element $\delta_j \in \mathcal{H}$ [21, p. 72],

$$\Delta_{Rj} = \delta_j\tilde{\Delta}_{Rj}. \tag{3.19}$$

Then by (3.7) and (3.17)

$$\Delta_L\tilde{N}^o_{pr}N_{\pi l} = \Delta_L\tilde{\Delta}_R Q_d. \tag{3.20}$$

Since $\mathcal{H}$ is a principal ring, we may cancel the nonsingular left-factor $\Delta_L$ and invert $\tilde{N}^o_{pr}$ in (3.20) to obtain

$$N_{\pi l} = (\tilde{N}^o_{pr})^{-1}\tilde{\Delta}_R Q_d. \tag{3.21}$$

By (3.2) and (3.18)

$$N_{\pi l} = \left[\frac{m_{ij}}{d_{ij}}\right] \text{diag}\ (\Delta_{R1}, \cdots, \tilde{\Delta}_{Rj}, \cdots, \Delta_{Rn}) \cdot Q_d. \tag{3.22}$$

Recalling that $\Delta_{Rj}$ is by definition a l.c.m. of $(d_{ij})^n_{i=1}$ and by (3.19), for some $i$, we have

$$d_{ij} = \delta_j\tilde{d}_{ij} \tag{3.23}$$

where $\tilde{d}_{ij} \in \mathcal{H}$ is a factor of $\tilde{\Delta}_{Rj}$; i.e., there is a $\tilde{c}_{ij} \in \mathcal{H}$, possibly a unit, such that

$$\tilde{\Delta}_{Rj} = \tilde{d}_{ij}\tilde{c}_{ij}. \tag{3.24}$$

Hence, with $q_j \in \mathcal{H}$ denoting the $j$th (nonzero) diagonal entry of some general nonsingular diagonal $Q_d \in \mathcal{H}^{n\times n}$, we obtain the $ij$th element of $N_{\pi l}$ from (3.22)–(3.24) as

$$\frac{m_{ij}}{\delta_j}\tilde{c}_{ij}q_j. \tag{3.25}$$

Since $\delta_j \notin U(\mathcal{H})$ and in general $\delta_j$ is not a factor of $q_j$, (3.25) is not in $\mathcal{H}$. Therefore, except when the prime nonunit $\delta_j$ is a factor of $q_j$, $N_{\pi l} \notin \mathcal{H}^{n\times n}$, thus with $N_{\pi l}$ as in (3.21), there is a diagonal, nonsingular $Q_d \in \mathcal{H}^{n\times n}$ such that $K$ does not satisfy assumption (K). This contradicts the assumption that $K$ stabilizes $P$. Therefore, $H_v$ must be an element of the set in (3.11).

Now consider $H_d$. By (2.5) and (2.8),

$$N_{fl}N^m_{pr} + D_{cl}D_{pr} = I. \tag{3.26}$$

Viewing (3.26) as a *linear* matrix equation in $\mathcal{E}(\mathcal{H})$, we solve for $(D_{cl}, N_{fl})$ subject to det $D_{cl} \in \mathcal{J}$ so that $D_{cl}^{-1}N_{fl} \in \mathcal{G}^{n\times n}$: since $(N^m_{pr}, D_{pr})$ is an r.c. pair, from (2.7) we have

$$U^m_{pr}N^m_{pr} + V^m_{pr}D_{pr} = I \tag{3.27}$$

and since $N^m_{pr}D_{pr}^{-1} = D_{pl}^{-1}N^m_{pl} = P^m$, we have

$$D_{pl}N^m_{pr} - N^m_{pl}D_{pr} = 0. \tag{3.28}$$

The pair $(U^m_{pr}, V^m_{pr})$ in (3.27) is a particular solution to $(N_{fl}, D_{cl})$ in (3.26) and the pair $(D_{pl}, -N^m_{pl})$ is a particular solution to the *homogeneous* equation (3.28). Hence, any general solution of (3.26) is given by

$$N_{fl} = U^m_{pr} + RD_{pl} \tag{3.29a}$$

$$D_{cl} = V^m_{pr} - RN^m_{pl}. \tag{3.29b}$$

We now show that $R \in \mathcal{E}(\mathcal{H})$. Since $K$ satisfies (K), det $D_{cl} \in \mathcal{J}$; therefore, det $(V^m_{pr} - RN^m_{pl}) \in \mathcal{J}$. Since $(D_{pl}, N^m_{pl})$ are l.c., there exist $V_{pl}, U_{pl} \in \mathcal{E}(\mathcal{H})$ such that

$$D_{pl}V_{pl} + N^m_{pl}U_{pl} = I. \tag{3.30}$$

Thus, by (3.29a), (3.29b) and (3.30), we see that $R = R(D_{pl}V_{pl} + N^m_{pl}U_{pl}) = (N_{fl} - U^m_{pr})V_{pl} + (V^m_{pr} - D_{cl})U_{pl} = N_{fl}V_{pl} - D_{cl}U_{pl} \in \mathcal{E}(\mathcal{H})$ since $N_{fl}, D_{cl}, V_{pl}, U_{pl} \in \mathcal{E}(\mathcal{H})$.

From (3.9) and (3.29a), (3.29b) $H_d = N^o_{pr}[I - (U^m_{pr} + RD_{pl})N^m_{pr}] = N^o_{pr}(V^m_{pr} - RN^m_{pl})D_{pr}$. Therefore, the given $H_d$ is an element of the set (3.12).

(<=) For some diagonal nonsingular $Q_d \in \mathcal{H}^{n\times n}$, we are given $H_v = \Delta_L\Delta_R Q_d$, and for some $R \in \mathcal{H}^{n\times n}$, we are given $H_d = N^o_{pr}[I - (U^m_{pr} + RD_{pl})N^m_{pr}] = N^o_{pr}(V^m_{pr} - RN^m_{pl})D_{pr}$, where det $(V^m_{pr} - RN^m_{pl}) \in \mathcal{J}$. We must show that there exists a compensator $K$ which stabilizes $P$ and the $\mathcal{H}$-stable $\Sigma$ $(P, K)$ achieves the given $H_v$ and $H_d$.

Choose the controller $K := D_{cl}^{-1}[N_{\pi l} \vdots N_{fl}]$ with $N_{fl}$ and $D_{cl}$ as in (3.29a), (3.29b) and $N_{\pi l} = (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d$. By Lemma 3.1, $N_{\pi l} \in \mathcal{H}^{n\times n}$. Clearly, $D_{cl}, N_{fl} \in \mathcal{E}(\mathcal{H})$. Note that det $D_{cl} \in \mathcal{J}$ is guaranteed by the $R$ that was chosen. (Note that if $P^m \in \mathcal{G}_s^{n\times n}$, then $\det D_{cl} \in \mathcal{J}$ for all $R \in \mathcal{H}^{n\times n}$ since $N_{pl}^m, N_{pr}^m \in \mathcal{G}_s^{n\times n}$.) Now, by (2.5)

$$D_h = (V_{pr}^m - RN_{pl}^m)D_{pr} + (U_{pr}^m + RD_{pl})N_{pr}^m. \tag{3.31}$$

By (3.26) and (3.27), $D_h = I$. Rewriting (3.31) as

$$(V_{pr}^m - RN_{pl}^m)D_{pr} + [(\tilde{N}_{pr}^o)^{-1}\Delta_R Q_d \,\vdots\, (U_{pr}^m + RD_{pl})]\begin{bmatrix} \vartheta_{n\times n} \\ \cdots\cdots \\ N_{pr}^m \end{bmatrix} = I,$$

we see that $(D_{cl}, [N_{\pi l} \vdots N_{fl}])$ are l.c., and this $K$ satisfies (K). Since $\det D_h \in U(\mathcal{H})$, $\Sigma\ (P, K)$ is $\mathcal{H}$-stable by Proposition 2.3 i).

By (3.7), we calculate the I/O map: $H_{y_o v} = N_{pr}^o N_{\pi l} = \Delta_L \tilde{N}_{pr}^o (\tilde{N}_{pr}^o)^{-1}\Delta_R Q_d = H_v$. By (3.8), we calculate the D/O map: $H_{y_o d} = N_{pr}^o[I - N_{fl}N_{pr}^m] = N_{pr}^o[I - (U_{pr}^m + RD_{pl})N_{pr}^m] = N_{pr}^o D_{cl} D_{pr} = N_{pr}^0 (V_{pr}^m - RN_{pl}^m)D_{pr} = H_d$.

*Summary:* Given the setup of Theorem 3.2 and, in particular, the $Q_d$ and the $R$ of (3.11) and (3.12), the compensator $K$ that achieves the specified diagonal, nonsingular $H_v$ and the specified $H_d$ as in (3.11) and (3.12), and that stabilizes $P$ is given by the left-coprime factorization

$$D_{cl} = V_{pr}^m - RN_{pl}^m,\quad [N_{\pi l} \,\vdots\, N_{fl}] = [(\tilde{N}_{pr}^o)^{-1}\Delta_R Q_d \,\vdots\, U_{pr}^m + RD_{pl}].$$

## IV. Examples and Conclusions

In the following examples we concentrate on the diagonal I/O map $H_{y_o v}$, and show the design for the compensator parameter $N_{\pi l}$.

*Example 1:* In this example, $\mathcal{H} := \mathcal{R}(s, e^{-rs})$ is the principal ring where $\mathcal{R}(s, e^{-rs})$ denotes the *rational* functions which are *proper* in $s$, *analytic* in $\mathbb{C}_+$ and have coefficients in $\mathbb{R}[e^{-rs}]$. ($\mathbb{R}[e^{-rs}]$ is the ring of polynomials in $e^{-rs}$ with real coefficients.) Consider the $P^o$ given by (4.1) below: it is strictly proper but not $\mathcal{H}$-stable, and it has a simple zero at $s = 3$.

$$P^o(s, e^{-rs}) = \begin{bmatrix} \dfrac{e^{-s}}{s-1} & \vdots & \dfrac{1}{s-2} \\ \cdots & \vdots & \cdots \\ \dfrac{e^{-2s}}{s+1} & \vdots & \dfrac{e^{-s}}{s-1} \end{bmatrix} \notin \mathcal{H}^{2\times 2}. \tag{4.1}$$

A r.c.f. of $P^o$ is given by

$$P^o = N_{pr}^o D_{pr}^{-1} = \begin{bmatrix} \dfrac{e^{-s}}{s+2} & \vdots & \dfrac{s-1}{(s+1)^2} \\ \cdots & \vdots & \cdots \\ \dfrac{(s-1)e^{-2s}}{(s+1)(s+2)} & \vdots & \dfrac{(s-2)e^{-s}}{(s+1)^2} \end{bmatrix} \cdot \operatorname{diag}\left[\frac{s-1}{s+2}, \frac{(s-1)(s-2)}{(s+1)^2}\right]^{-1}.$$

Then

$$N_{pr}^o = \Delta_L \tilde{N}_{pr}^o = \operatorname{diag}\left[\frac{1}{s+2}, \frac{e^{-s}}{s+1}\right] \cdot \begin{bmatrix} e^{-s} & \vdots & \dfrac{(s-1)(s+2)}{(s+1)^2} \\ \cdots & \vdots & \cdots \\ \dfrac{(s-1)e^{-2s}}{s+2} & \vdots & \dfrac{s-1}{s+1} \end{bmatrix}.$$

Here, $\Delta_L$ and $\tilde{N}_{pr}^o$ are not unique; $\Delta_L$ extracts a zero at $\infty$ from the rational part of each row of $N_{pr}^o$. From

$$(\tilde{N}_{pr}^o)^{-1} = \begin{bmatrix} \dfrac{(s-2)(s+1)}{(s-3)e^{-s}} & \vdots & \dfrac{-(s-1)(s+2)}{(s-3)e^{-s}} \\ \cdots & \vdots & \cdots \\ \dfrac{-(s-1)(s+1)^2}{(s-3)(s+2)} & \vdots & \dfrac{(s+1)^2}{(s-3)} \end{bmatrix} \notin \mathcal{H}^{2\times 2},$$

we obtain

$$\Delta_R = \operatorname{diag}\left[\frac{(s-3)e^{-s}}{(s+1)^2}, \frac{(s-3)e^{-s}}{(s+1)^2}\right],$$

and

$$N_{\pi l} = (\tilde{N}_{pr}^o)^{-1}\Delta_R Q_d = \begin{bmatrix} \dfrac{s-2}{s+1} & \vdots & \dfrac{-(s-1)(s+2)}{(s+1)^2} \\ \cdots & \vdots & \cdots \\ \dfrac{-(s-1)e^{-s}}{s+2} & \vdots & e^{-s} \end{bmatrix} Q_d.$$

Note that each diagonal entry of $\Delta_R$ is equal to $\det \tilde{N}_{pr}^o$. Consequently, $\det \Delta_R = (\det \tilde{N}_{pr}^o)^2$, and the number of the $\mathbb{C}_+$-zeros of the diagonal I/O map is increased. Here,

$$H_{y_o v} = \Delta_L \Delta_R Q_d = \operatorname{diag}\left[\frac{(s-3)e^{-s}}{(s+2)(s+1)^2}, \frac{(s-3)e^{-2s}}{(s+1)^3}\right] Q_d$$

has a zero of multiplicity *two* at $s = 3$ and it may have other $\mathbb{C}_+$-zeros due to $Q_d \in \mathcal{H}^{2\times 2}$. Comparing this to the $\mathbb{C}_+$-zeros of det $N_{pr}^o$, we see that the cost of decoupling is the increased number of $\mathbb{C}_+$-zeros (due to $\Delta_R$) and the restriction that $Q_d$ be diagonal.

*Example 2:* Let $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$, where $\mathcal{U} = \mathbb{C}_+$. $P^o$ is given by (4.2): it is proper but not $\mathcal{H}$-stable; $P^o$ has a zero of multiplicity two at $s = 1$, a zero at $s = 2$ and two zeros at infinity.

$$P^o(s) = \begin{bmatrix} \dfrac{s-1}{(s-3)(s+2)} & \vdots & \dfrac{1}{s+2} & \vdots & \dfrac{(s-1)(s-2)}{(s+1)(s+2)} \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ \dfrac{s+1}{s-3} & \vdots & 1 & \vdots & \dfrac{s-2}{s+2} \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & \dfrac{1}{(s-1)(s+1)} & \vdots & \dfrac{s-2}{(s+1)(s+2)} \end{bmatrix} \notin \mathcal{H}^{3\times 3}. \tag{4.2}$$

An r.c.f. of $P^o$ is given by

$$N_{pr}^o D_{pr}^{-1} = \begin{bmatrix} \dfrac{s-1}{(s+1)(s+2)} & \vdots & \dfrac{s-1}{(s+1)(s+2)} & \vdots & \dfrac{(s-1)(s-2)}{(s+1)(s+2)} \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 1 & \vdots & \dfrac{s-1}{s+1} & \vdots & \dfrac{s-2}{s+2} \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & \dfrac{1}{(s+1)^2} & \vdots & \dfrac{s-2}{(s+1)(s+2)} \end{bmatrix} \operatorname{diag}\left[\frac{s-3}{s+1}, \frac{s-1}{s+1}, 1\right]^{-1}.$$

Then,

$$\tilde{N}^o_{pr} = \Delta_L \tilde{N}^o_{pr}$$

$$= \operatorname{diag}\left[\frac{s-1}{s+2},\ 1,\ \frac{1}{s+1}\right] \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} & \frac{s-2}{s+1} \\ 1 & \frac{s-1}{s+1} & \frac{s-2}{s+2} \\ 0 & \frac{1}{s+1} & \frac{s-2}{s+2} \end{bmatrix}.$$

$\Delta_L$ and $\tilde{N}^o_{pr}$ are not unique and $\Delta_L$ extracts a zero at $s = 1$ from the first row of $N^o_{pr}$, and a zero at $\infty$ from the third row of $N^o_{pr}$. Now

$$(\tilde{N}^o_{pr})^{-1} = \begin{bmatrix} \frac{(s-2)(s+1)}{s-1} & \frac{1}{s-1} & \frac{-(s^2-3)}{s-1} \\ \frac{-(s+1)^2}{s-1} & \frac{s+1}{s-1} & \frac{(s+1)^2}{s-1} \\ \frac{(s+1)(s+2)}{(s-1)(s-2)} & \frac{-(s+2)}{(s-1)(s-2)} & \frac{-2(s+2)}{(s-1)(s-2)} \end{bmatrix} \notin \mathcal{H}^{3\times 3}$$

and

$$\Delta_R = \operatorname{diag}\left[\frac{(s-1)(s-2)}{(s+1)^2(s+2)},\ \frac{(s-1)(s-2)}{(s+1)(s+2)},\ \frac{(s-1)(s-2)}{(s+1)^2(s+2)}\right].$$

(The first and the third diagonal entries of $\Delta_R$ are equal to det $\tilde{N}^o_{pr}$.) Then,

$$N_{\pi l} = \begin{bmatrix} \frac{(s-2)^2}{(s+1)(s+2)} & \frac{s-2}{(s+1)(s+2)} & \frac{-(s^2-3)(s-2)}{(s+1)^2(s+2)} \\ \frac{-(s-2)}{s+2} & \frac{s-2}{s+2} & \frac{(s-2)}{s+2} \\ \frac{1}{s+1} & \frac{-1}{s+1} & \frac{-2}{(s+1)^2} \end{bmatrix} Q_d,$$

and

$$H_{y_o v} = \Delta_L \Delta_R Q_d$$

$$= \operatorname{diag}\left[\frac{(s-1)^2(s-2)}{(s+1)^2(s+2)^2},\ \frac{(s-1)(s-2)}{(s+1)(s+2)},\ \frac{(s-1)(s-2)}{(s+1)^3(s+2)}\right] Q_d,$$

where $Q_d \in \mathcal{H}^{3\times 3}$ is diagonal and nonsingular. The closed-loop diagonal I/O map $H_{y_o v}$ has a zero of multiplicity three at $s = 2$ and three zeros at $\infty$. $H_{y_o v}$ may have other $\mathbb{C}_+$-zeros due to $Q_d$. The cost of decoupling is the increased number of $\mathbb{C}_+$-zeros (due to $\Delta_R$) and the restriction that $Q_d$ be diagonal.

*Example 3:* In this example we design a decoupling compensator for the $P^o$ given in (4.3), which is the model of a "boiler subsystem" in [19]. Johansson and Koivo apply the inverse Nyquist array method of Rosenbrock in the design of a multivariable controller for this system. Let $\mathcal{H} := \mathcal{R}(s, e^{-rs})$.

$$P^o(s, e^{-rs}) = \begin{bmatrix} \frac{-e^{-2s}}{10s+1} & \frac{-1}{10s+1} \\ 0 & \frac{e^{-10s}}{60s+1} \end{bmatrix} \in \mathcal{H}^{2\times 2}. \tag{4.3}$$

An r.c.f. of $P^o$ is given by $D_{pr} = I$, $N^o_{pr} = P^o$. Then

$$\Delta_L = \operatorname{diag}\left[\frac{1}{7s+1},\ \frac{1}{40s+1}\right] \text{ and } (\tilde{N}^o_{pr})^{-1}$$

$$= \begin{bmatrix} \frac{-(10s+1)e^{2s}}{7s+1} & \frac{-(60s+1)e^{12s}}{(40s+1)} \\ 0 & \frac{(60s+1)e^{10s}}{(40s+1)} \end{bmatrix}.$$

From this, we obtain $\Delta_R = \operatorname{diag}[e^{-2s}, e^{-12s}]$, and

$$N_{\pi l} = (\tilde{N}^o_{pr})^{-1}\Delta_R Q_d$$

$$= \begin{bmatrix} \frac{-(10s+1)}{(7s+1)} & \frac{(60s+1)}{(40s+1)} \\ 0 & \frac{(60s+1)e^{-2s}}{(40s+1)} \end{bmatrix} Q_d,$$

where $Q_d \in \mathcal{H}^{2\times 2}$ is *diagonal* and *nonsingular*. Finally,

$$H_{y_o v} = \Delta_L \Delta_R Q_d = \operatorname{diag}\left[\frac{e^{-2s}}{7s+1},\ \frac{e^{-12s}}{40s+1}\right] Q_d.$$

The closed-loop I/O map is diagonal and the time-constants are reduced from 10 s and 60 s to 7 s and 40 s, respectively.

## Conclusions

Without decoupling, the set of all achievable I/O maps of $\Sigma$ $(P, K)$ is given by (3.13). The compensator parameter $N_{\pi l}$, which is used in designing the I/O map, is made $\mathcal{H}$-stable by an appropriate choice of a diagonal $\mathcal{H}$-stable matrix $\Delta_R$ defined by (3.3). Finally, the set of all achievable diagonal nonsingular I/O maps is given by (3.11), where $\Delta_L$ appears as a left factor of both diagonal and nondiagonal achievable I/O maps.

The examples of this section clearly illustrate the cost involved in decoupling the I/O map while requiring that it be $\mathcal{H}$-stable; this cost is reflected by $\Delta_R$ and $Q_d$: $\Delta_R$ must be chosen so that $N_{\pi l}$ is $\mathcal{H}$-stable; $Q_d \in \mathcal{H}^{n\times n}$ must be diagonal. In the case that $\mathcal{H} = \mathcal{R}_{\mathcal{U}}$ (or $\mathcal{H} = \mathcal{R}(s, e^{-rs})$ as in Example 1) the presence of $\Delta_R$ in the diagonal I/O map results in increasing the number of $\mathcal{U}$-zeros. If $N^o_{pr} \in \mathcal{H}^{2\times 2}$, det $\Delta_R$ has exactly twice as many $\mathcal{U}$-zeros as det $\tilde{N}^o_{pr}$ (for a proof, see the Appendix.) This design method has two degrees of freedom: decoupling the I/O map has no effect on the D/O map. The D/O map is designed using the parameters $D_{cl}$ and $N_{fl}$ of the compensator. The only compensator parameter used in the I/O map is $N_{\pi l}$.

Four classes of systems for which the results of this paper are valid can be found in [11, Table I].

## Appendix

Let $n = 2$. Let $\tilde{N}^o_{pr}$, $\Delta_L$, $\Delta_R$ be defined as in Section III. U.t.c., det $\Delta_R$ = (det $\tilde{N}^o_{pr})^2 u$, where $u \in U(\mathcal{H})$.

*Proof:* Let

$$\tilde{N}^o_{pr} = \begin{bmatrix} n_{11} & \vdots & n_{12} \\ \cdots & \vdots & \cdots \\ n_{21} & \vdots & n_{22} \end{bmatrix} \in \mathcal{H}^{2\times 2}$$

where, by construction of $\Delta_L$, $(n_{11}, n_{12})$ is a coprime pair. With $\delta := \det \tilde{N}^o_{pr}$, the first and the second columns of $(\tilde{N}^o_{pr})^{-1}$ are $(n_{22}/\delta, -n_{21}/\delta)$ and $(-n_{12}/\delta, n_{11}/\delta)$, respectively. Now, any *irreducible* common factor that cancels in $n_{22}/\delta$ will not cancel in $-n_{21}/\delta$ since $(n_{22}, -n_{21})$ are coprime. Thus, a least *common* denominator for the first column is $\delta$. The same holds for the second column and hence, $\Delta_R = \operatorname{diag}(\delta, \delta)$. Then $\det \Delta_R = (\det \tilde{N}^o_{pr})^2$, times a factor in $U(\mathcal{H})$.

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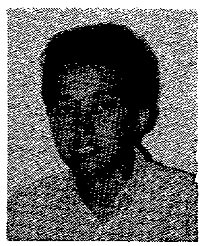

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