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Simultaneous stabilizability of P and KP

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Abstract

It is shown that there exists a common controller which simultaneously stabilizes any given nominal plant P and perturbed plant KP for any given positive real constant K in the standard linear, time-invariant, multi-input multi-output unity-feedback system. The class of plants such that P and KP can be simultaneously stabilized for negative K is also determined.

Keywords: Simultaneous stabilization; Stabilizing controller; Parity-interlacing-property; Multiplicative perturbation

1. Introduction

It is well-known that any given proper plant P can be stabilized by a proper controller C in the standard linear, time-invariant (LTI), multi-input multi-output (MIMO) unity-feedback system configuration. The set of all stabilizing controllers can be obtained using coprime factorizations of the plant's transfer-function P. Now consider the problem of stabilizing the given P simultaneously with KP, where K is a known real constant. If P is stable, it follows from the small-gain theorem (see for example [5]) that there exist common stabilizing controllers for P and KP for any $K \neq 0$; however, if P is not stable, it is not possible to conclude existence of simultaneously stabilizing controllers using this result unless K is assumed to be "sufficiently small".

In this paper, it is shown that P and KP can be simultaneously stabilized for any given P and any given positive real constant K. Equivalently, there exists a stabilizing controller C for any given plant P, such that the controller KC also stabilizes P for any positive real constant K. Simultaneous stabilizability of P and $2P$ was proven in [5]; however, the proof given there cannot be extended to $K \neq 2$. The proof given here for the case of general K uses the Smith–McMillan form to show that the pseudo-plant associated with P and KP satisfies the parity-interlacing-property. The main result of this note, Theorem 3.1, states *existence* of simultaneously stabilizing controllers for P and KP for any positive K and characterizes the class of plants such that P and KP can be simultaneously stabilized for negative K. The simultaneously stabilizing controllers can be constructed as stable stabilizing controllers of an associated pseudo-plant.

Notation: Let $\mathscr{U}$ contain the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The set of real numbers, the ring of proper rational functions which do not have any poles in the region of instability $\mathscr{U}$, the sets of proper and strictly proper rational functions with real coefficients are denoted by $\mathbb{R}$, $\mathscr{R}$, $\mathrm{R_p}$, $\mathrm{R_{sp}}$, respectively. The set of matrices whose entries are in $\mathscr{R}$ is denoted by $\mathscr{M}(\mathscr{R})$; M is called $\mathscr{R}$-stable iff $M \in \mathscr{M}(\mathscr{R})$; an $\mathscr{R}$-stable M is called $\mathscr{R}$-unimodular iff $M^{-1} \in \mathscr{M}(\mathscr{R})$. For $M \in \mathscr{M}(\mathscr{R})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s \in \partial\mathscr{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ and $\partial\mathscr{U}$ denote the maximum singular value and the boundary of $\mathscr{U}$. A right-coprime-factorization (RCF) and a left-coprime-factorization (LCF) of

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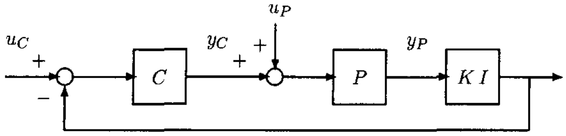

Fig. 1. The system $S(KI,P,C)$.

$P\in \mathrm{R}_{\mathrm{p}}^{n_o\times n_i}$ are denoted by (N,D) and $(\tilde{D},\tilde{N})$, where $N,D,\tilde{N},\tilde{D}\in\mathscr{M}(\mathscr{R})$, D and $\tilde{D}$ are biproper and $P=ND^{-1}=\tilde{D}^{-1}\tilde{N}$. Let rank $P=r$. A $z_0\in\mathscr{U}$ is called a (transmission) $\mathscr{U}$-zero of P iff $\operatorname{rank}P(z_0)<r$, equivalently, $\operatorname{rank}N(z_0)=\operatorname{rank}\tilde{N}(z_0)<r$; $z_0\in\mathscr{U}$ is called a blocking $\mathscr{U}$-zero of P iff $P(z_0)=0$, equivalently, $N(z_0)=0=\tilde{N}(z_0)$; $s_0\in\mathscr{U}$ is called a $\mathscr{U}$-pole of P iff it is a pole of some entry of P, equivalently, $\det D(s_0)=0=\det\tilde{D}(s_0)$. The identity map is denoted by I. $a:=b$ means a is defined as b.

2. Preliminaries

Consider the LTI, MIMO system $S(KI,P,C)$ (Fig. 1), where $P\in\mathrm{R}_{\mathrm{p}}^{n_o\times n_i}$ and $C\in\mathrm{R}_{\mathrm{p}}^{n_i\times n_o}$ represent the plant and the controller. The real constant $K\in\mathbb{R}$ represents a known multiplicative perturbation. The nominal plant P is not necessarily $\mathscr{R}$-stable. If $K=1$, then $S(KI,P,C)$ becomes the nominal unity-feedback system $\mathscr{S}(P,C)$. It is assumed that P and C do not have any hidden modes associated with eigenvalues in $\mathscr{U}$ and that the system $S(KI,P,C)$ is well-posed.

Definitions 2.1 (*$\mathscr{R}$-stability, $\mathscr{R}$-stabilizing controller*). The system $S(KI,P,C)$ is said to be $\mathscr{R}$-stable iff the closed-loop transfer-function from $u:=[u_P^{\mathrm{T}},u_C^{\mathrm{T}}]^{\mathrm{T}}$ to $y:=[y_P^{\mathrm{T}},y_C^{\mathrm{T}}]^{\mathrm{T}}$ is $\mathscr{R}$-stable. The controller C is called an $\mathscr{R}$-stabilizing controller for $P\in\mathrm{R}_{\mathrm{p}}^{n_o\times n_i}$ iff $C\in\mathrm{R}_{\mathrm{p}}^{n_i\times n_o}$ and $\mathscr{S}(P,C)$ is $\mathscr{R}$-stable. The controller C is called a simultaneously $\mathscr{R}$-stabilizing controller for P and KP iff C is an $\mathscr{R}$-stabilizing controller for $P\in\mathrm{R}_{\mathrm{p}}^{n_o\times n_i}$ and $S(KI,P,C)$ is $\mathscr{R}$-stable; P and KP are said to be simultaneously $\mathscr{R}$-stabilizable iff there exists a simultaneously $\mathscr{R}$-stabilizing controller C for P and KP .

Facts 2.2 ($\mathscr{R}$-stability of $S(KI,P,C)$, all $\mathscr{R}$-stabilizing controllers [5, 3, 4]). *Let (N,D) be any RCF and $(\tilde{D},\tilde{N})$ be any LCF of $P\in\mathrm{R}_{\mathrm{p}}^{n_o\times n_i}$; let $(\tilde{D}_C,\tilde{N}_C)$ be any LCF of C.*

(i) *The system* S(KI,P,C) *is $\mathscr{R}$-stable if and only if $(\tilde{D}_CD+\tilde{N}_CKN)$ is $\mathscr{R}$-unimodular.*

(ii) *The controller $C\in\mathscr{M}(\mathrm{R}_{\mathrm{p}})$ is an $\mathscr{R}$-stabilizing controller for P if and only if C is given by*

$$\begin{aligned}C&=(V-Q\tilde{N})^{-1}(U+Q\tilde{D})\\&=(\tilde{U}+DQ)(\tilde{V}-NQ)^{-1}\end{aligned}\tag{1}$$

for some $\mathscr{R}$-stable $Q\in\mathscr{R}^{n_i\times n_o}$ such that $(V-Q\tilde{N})$ is biproper (which holds for all $Q\in\mathscr{M}(\mathscr{R})$ when P is strictly proper), where $U,V,\tilde{U},\tilde{V}$ are $\mathscr{R}$-stable matrices such that

$$\begin{bmatrix}V & U\\ -\tilde{N} & \tilde{D}\end{bmatrix}\begin{bmatrix}D & -\tilde{U}\\ N & \tilde{V}\end{bmatrix}=I.\tag{2}$$

(iii) *Let $U,V\in\mathscr{M}(\mathscr{R})$ be as in (2). The controller C is a simultaneously $\mathscr{R}$-stabilizing controller for P and KP if and only if C is given by (1), where $Q\in\mathscr{M}(\mathscr{R})$ is such that $(V-Q\tilde{N})$ is biproper and*

$$\begin{aligned}&(V-Q\tilde{N})D+(U+Q\tilde{D})KN\\&\quad=I_{n_i}+(K-1)(U+Q\tilde{D})N\\&\qquad\text{is }\mathscr{R}\text{-unimodular.}\end{aligned}\tag{3}$$

Equivalently, P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if there exists $\mathscr{R}$-stable Q such that (3) *holds.*

Remarks 2.3 (i) If the constant $K\in\mathbb{R}$ is equal to zero, then the system $S(KI,P,C)$ becomes an open-loop system. For internal $\mathscr{R}$-stability, each of the subsystems P and C must then be $\mathscr{R}$-stable. It is therefore obvious that for $K=0$, P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if P is $\mathscr{R}$-stable. This also follows from Fact 2.2(i) because when $K=0$, $(\tilde{D}_CD+\tilde{N}_CKN)=\tilde{D}_CD$ is $\mathscr{R}$-unimodular if and only if $\tilde{D}_C$ and D are $\mathscr{R}$ unimodular, equivalently, C and P are both $\mathscr{R}$-stable. Note that all $\mathscr{R}$-stable controllers that $\mathscr{R}$-stabilize $P\in\mathscr{M}(\mathscr{R})$ are given by $C=(I-QP)^{-1}Q$, where $Q\in\mathscr{M}(\mathscr{R})$ is such that $(I-QP)$ is $\mathscr{R}$-unimodular.

(ii) If P is $\mathscr{R}$-stable, then P and KP are simultaneously $\mathscr{R}$-stabilizable for *any* $K\in\mathbb{R}$. One choice for a simultaneously $\mathscr{R}$-stabilizing controller is obviously $C=0$; nonzero controllers can be found using the *small-gain* condition (see for example [5]) as follows. If $P\in\mathscr{M}(\mathscr{R})$, then $(N,D)=(P,I_{n_i})$ is an RCF of P and a solution for (2) is given by $U=0$, $V=I_{n_i}$. If an $\mathscr{R}$-stable $Q\in\mathscr{R}^{n_i\times n_o}$ is chosen so that $\|Q\|<\|(K-1)P\|^{-1}$, then (3) holds since $(I_{n_i}+(K-1)QP)$ is $\mathscr{R}$-unimodular and, hence, P and KP are simultaneously $\mathscr{R}$-stabilizable.

If P is not $\mathscr{R}$-stable, there may not exist $\mathscr{R}$-stable Q such that $\|(K-1)(U+Q\tilde{D})N\| < 1$, and, hence, the existence of $Q \in \mathscr{M}(\mathscr{R})$ satisfying (3) cannot be concluded using this (sufficient) small-gain condition without restricting K.

(iii) Using a well-known result (see for example [5, 6, 1, 2]), by Fact 2.2 (iii), P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if the "pseudo-plant" $P_S := (I_{n_i}+(K-1)UN)^{-1}(K-1)\tilde{D}N$ can be strongly $\mathscr{R}$-stabilized. The simultaneously $\mathscr{R}$-stabilizing controller for P and KP is given by (1), where $Q \in \mathscr{M}(\mathscr{R})$ is any strongly $\mathscr{R}$-stabilizing controller for the pseudo-plant P_S.

Fact 2.4 (Smith–McMillan form [5]). *Let $P \in \mathrm{R}_p^{n_o \times n_i}$. Let* rank $P =: r$, *where $r \leqslant \min\{n_o, n_i\}$. There exist $\mathscr{R}$-unimodular matrices $L \in \mathscr{R}^{n_o \times n_o}$, $R \in \mathscr{R}^{n_i \times n_i}$ such that*

$$P = L \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o-r)\times(n_i-r)} \end{bmatrix} \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I_{(n_i-r)} \end{bmatrix} R$$

$$= L \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I_{(n_o-r)} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o-r)\times(n_i-r)} \end{bmatrix} R, \quad (4)$$

$$\Lambda := \operatorname{diag}[\lambda_1 \ \cdots \ \lambda_r], \quad \Psi := \operatorname{diag}[\psi_1 \ \cdots \ \psi_r],$$

where, for $j = 1,\ldots,r$, the (numerator and denominator) invariant-factors λ_j and ψ_j satisfy the following: $\lambda_j \in \mathscr{R}$, $\psi_j \in \mathscr{R}$, ψ_j is biproper; for $j = 1,\ldots,r-1$, λ_j divides λ_{j+1}, and ψ_{j+1} divides ψ_j; for $j = 1,\ldots,r$, the pair (λ_j, ψ_j) is coprime, equivalently, there exist $u_j \in \mathscr{R}$, $v_j \in \mathscr{R}$ such that

$$v_j\psi_j + u_j\lambda_j = 1. \quad (5)$$

3. Main results

We now show that P and KP are simultaneously $\mathscr{R}$-stabilizable for any real positive constant K; we also show that only certain classes of plants can be simultaneously $\mathscr{R}$-stabilized with KP when K is negative.

Theorem 3.1 (Simultaneous $\mathscr{R}$-stabilizability of P and KP). *Let $P \in \mathrm{R}_p^{n_o \times n_i}$ be any given plant and let $K \in \mathbb{R}$ be a given constant.*

(a) *Let $K > 0$; then P and KP are simultaneously $\mathscr{R}$-stabilizable.*

(b) *Let $K < 0$. Consider the Smith–McMillan form (4) of P. Let $s_i \in \mathbb{R} \cap \mathscr{U}$, $i \in \{1,\ldots,m\}$, be such that*

$\psi_{\ell_i}(s_i) = 0$ for some $\ell_i \in \{1,\ldots,r\}$, and

$\psi_j(s_i) \neq 0$ for all $j \in \{1,\ldots,r\}$ such that $j \geqslant (\ell_i+1)$, and

$\lambda_j(s_i) = 0$ for all $j \in \{1,\ldots,r\}$ such that $j \geqslant (\ell_i+1)$.

Let $\{\ell_1,\ldots,\ell_m\}$ denote the set of indices corresponding to the set $\{s_1,\ldots,s_m\}$.

(i) *Suppose that P has no real blocking $\mathscr{U}$-zeros (including infinity); then P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if the indices in the set $\{\ell_1,\ldots,\ell_m\}$ are either all even or all odd.*

(ii) *Suppose that P has at least one real blocking $\mathscr{U}$-zero (including infinity); then P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if all of the indices in the set $\{\ell_1,\ldots,\ell_m\}$ are even.*

(c) *Let $K = 0$; then P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if $P \in \mathscr{M}(\mathscr{R})$.*

Corollary 3.2 (Conditions for simultaneous $\mathscr{R}$-stabilizability of scalar P and KP). *Let $P \in \mathrm{R}_p$ be any given (scalar) plant and let $K \in \mathbb{R}$, $K < 0$ be a given negative constant. Then P and KP are simultaneously $\mathscr{R}$-stabilizable if and only if P either has no real $\mathscr{U}$-zeros (including infinity) or has no real $\mathscr{U}$-poles.*

Proof of Theorem 3.1 . The case for $K = 0$ is obvious as explained in Remark 2.3(i). We prove cases (a) and (b) in detail.

Let (N,D) be any RCF and $(\tilde{D},\tilde{N})$ be any LCF of P; let $U, V, \tilde{U}, \tilde{V} \in \mathscr{M}(\mathscr{R})$ satisfy (2). By Fact 2.2(iii), there exist simultaneously $\mathscr{R}$-stabilizing controllers for P and KP if and only if there exists $Q \in \mathscr{M}(\mathscr{R})$ such that (3) holds. By (2), $K \neq 0$ implies

$$(I_{n_i} - K^{-1}(K-1)UN)(I_{n_i} + (K-1)UN) + (K^{-1}(K-1)U\tilde{V})((K-1)\tilde{D}N) = I_{n_i}, \quad (6)$$

and, hence, the pair $((I_{n_i}+(K-1)UN), (K-1)\tilde{D}N)$ is right-coprime. Therefore, there exists $Q \in \mathscr{M}(\mathscr{R})$ such that (3) holds if and only if the pair $(I_{n_i} + (K-1)UN, (K-1)\tilde{D}N)$ satisfies the parity-interlacing-property, equivalently, $\det(I_{n_i}+(K-1)UN)$ has the same sign at all real blocking $\mathscr{U}$-zeros of $\tilde{D}N$. If $K = 1$, as expected since this corresponds to the nominal system $\mathscr{S}(P,C)$, the parity-interlacing-property is satisfied because $\det(I+(K-1)UN) = 1$. We now investigate the parity-interlacing-property when $K \neq 1$: Consider the Smith–McMillan form (4) of P. Any RCF (N,D) and any LCF $(\tilde{D},\tilde{N})$ of

P is given in terms of this Smith–McMillan form as

$$(N,D) = \left(L \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o-r)\times(n_i-r)} \end{bmatrix} M, \right.$$

$$\left. R^{-1} \begin{bmatrix} \Psi & 0 \\ 0 & I_{(n_i-r)} \end{bmatrix} M \right), \tag{7}$$

$$(\tilde D,\tilde N) = \left(\tilde M \begin{bmatrix} \Psi & 0 \\ 0 & I_{(n_o-r)} \end{bmatrix} L^{-1}, \right.$$

$$\left. \tilde M \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o-r)\times(n_i-r)} \end{bmatrix} R \right) \tag{8}$$

for some $\mathscr{R}$-unimodular $M \in \mathscr{M}(\mathscr{R})$ and for some $\mathscr{R}$-unimodular $\tilde M \in \mathscr{M}(\mathscr{R})$. Let $U_D := \operatorname{diag}[u_1 \cdots u_r]$, $V_D := \operatorname{diag}[v_1 \cdots v_r]$; then by (5), $(V_D\Psi + U_D\Lambda) = I_r$. A solution for $U, V, \tilde U, \tilde V$ satisfying (2) is

$$U := M^{-1} \begin{bmatrix} U_D & 0 \\ 0 & 0_{(n_i-r)\times(n_o-r)} \end{bmatrix} L^{-1},$$

$$V := M^{-1} \begin{bmatrix} V_D & 0 \\ 0 & I_{(n_i-r)} \end{bmatrix} R,$$

$$\tilde U := R^{-1} \begin{bmatrix} U_D & 0 \\ 0 & 0_{(n_i-r)\times(n_o-r)} \end{bmatrix} \tilde M^{-1},$$

$$\tilde V := L \begin{bmatrix} V_D & 0 \\ 0 & I_{(n_o-r)} \end{bmatrix} \tilde M^{-1}. \tag{9}$$

By (9),

$$\det(I_{n_i} + (K-1)UN)$$

$$= \det\left(I_{n_i} + (K-1) \begin{bmatrix} U_D\Lambda & 0 \\ 0 & 0_{(n_i-r)\times(n_i-r)} \end{bmatrix} \right)$$

$$= \prod_{j=1}^{r} (1 + (K-1)u_j\lambda_j). \tag{10}$$

By (7) and (8), since $M, \tilde M \in \mathscr{M}(\mathscr{R})$ are $\mathscr{R}$-unimodular,

$$(\tilde D N)(s_0)$$

$$= \left(\tilde M \begin{bmatrix} \Psi & 0 \\ 0 & I_{(n_o-r)} \end{bmatrix} L^{-1} L \begin{bmatrix} \Lambda & 0 \\ 0 & 0_{(n_o-r)\times(n_i-r)} \end{bmatrix} M \right)(s_0)$$

$$= 0$$

if and only if $(\Psi\Lambda)(s_0) = 0$, equivalently, $(\psi_j\lambda_j)(s_0) = 0$ for $j = 1,\ldots,r$. Since (λ_j, ψ_j) is coprime, $(\psi_j\lambda_j)(s_0) = 0$ means that either $\psi_j(s_0) = 0$ or $\lambda_j(s_0) = 0$. For any $s_0 \in \mathbb{R} \cap \mathscr{U}$ such that $(\tilde D N)(s_0) = 0$, there are only two possibilities (s_0 is either a blocking zero of P, or s_0 is a pole of P which appears as a zero of the smallest invariant factor ψ_r or which coincides with a zero):

Case 1: Suppose that $\psi_j(s_0) \neq 0$ for all $j \in \{1,\ldots,r\}$, equivalently, $\det \tilde D(s_0) \neq 0$; then $(\tilde D N)(s_0) = 0$ implies that $N(s_0) = 0$, i.e., s_0 is a blocking $\mathscr{U}$-zero of P. Therefore $\det(I_{n_i} + (K-1)UN)(s_0) = 1$.

Case 2: Suppose that $\psi_\ell(s_0) = 0$ for some $\ell \in \{1,\ldots,r\}$, but $\psi_j(s_0) \neq 0$ for $j > \ell$; then $\psi_j(s_0) = 0$ for all $j \leqslant \ell$ because ψ_{j+1} divides ψ_j. Since (λ_j, ψ_j) is coprime, $\lambda_\ell(s_0) \neq 0$. But $(\tilde D N)(s_0) = 0$ implies $(\psi_j\lambda_j)(s_0) = 0$ for $j = 1,\ldots,r$; therefore $\lambda_j(s_0) = 0$ for all $j \geqslant (\ell+1)$. By (5), $(u_j\lambda_j)(s_0) = 1$ for all $j \leqslant \ell$ and $(u_j\lambda_j)(s_0) = 0$ for all $j \geqslant (\ell+1)$. By (10),

$$\det(I_{n_i} + (K-1)UN)(s_0)$$

$$= \prod_{j=1}^{r} (1 + (K-1)(u_j\lambda_j)(s_0)) = K^\ell. \tag{11}$$

Note that if $\ell = r$, then the smallest invariant factor $\psi_r(s_0) = 0$ and, hence, $\lambda_j(s_0) \neq 0$ for $j = 1,\ldots,r$. In this case, $\det(I_{n_i} + (K-1)UN)(s_0) = K^r$.

(a) Let $K > 0$. By (11), for all real blocking $\mathscr{U}$-zeros of $(\tilde D N)(s_0) = 0$ as in Case 2 above, $\det(I_{n_i} + (K-1)UN)(s_0) = K^\ell > 0$; since this sign agrees with the positive sign at all other blocking $\mathscr{U}$-zeros described in Case 1, the pair $((I_{n_i} + (K-1)UN), (K-1)\tilde D N)$ satisfies the parity-interlacing-property and, hence, there exist simultaneously $\mathscr{R}$-stabilizing controllers for P and KP for any given $K > 0$.

(b) Let $K < 0$. Suppose that $\tilde D N$ has m real blocking $\mathscr{U}$-zeros $s_1,\ldots,s_m$ as described in Case 2 above and the corresponding indices are $\ell_1,\ldots,\ell_m$; i.e., $\psi_{\ell_i}(s_i) = 0$ for $i \in \{1,\ldots,m\}$. By (11), the parity-interlacing-property is satisfied for these zeros if and only if the sign of $\det(I_{n_i} + (K-1)UN)(s_i) = K^{\ell_i}$ remains the same for all s_i; but when $K < 0$, the sign of K^{ℓ_i} is the same for all $i \in \{1,\ldots,m\}$ if and only if the indices $\ell_1,\ldots,\ell_m$ are either all odd or all even numbers. Now if $\tilde D N$ has any real blocking $\mathscr{U}$-zero s_0 as described in Case 1, then $N(s_0) = 0$, i.e., s_0 is a real blocking $\mathscr{U}$-zero of P. Since $\det(I_{n_i} + (K-1)UN)(s_0) = 1$ is positive at these zeros, K^{ℓ_i} has to be positive so that the sign does not change; but when $K < 0$, K^{ℓ_i} is positive if and only if all of the indices $\ell_1,\ldots,\ell_m$ are even numbers.

This proves cases (a) and (b). It remains to show that the simultaneously $\mathscr{R}$-stabilizing controllers, whenever they exist, can be chosen *proper*. If the plant is strictly proper, then $N \in \mathcal{M}(\mathrm{R_{sp}})$, and, hence, $(V - Q\tilde{N})$ is biproper for all $Q \in \mathcal{M}(\mathcal{R})$; therefore the controllers are proper for any choice of Q. For proper plants which are not *strictly* proper, the choice of $Q \in \mathcal{M}(\mathcal{R})$ satisfying (3) should be modified to ensure that the simultaneously $\mathcal{R}$-stabilizing controllers are proper by choosing $Q \in \mathcal{M}(\mathcal{R})$ to satisfy the additional condition that $(V - Q\tilde{N})$ is biproper. One way to do this is explained here for $K > 0$; the case of negative K is similar whenever the conditions on the indices are satisfied and is omitted since ensuring the properness of the controllers is a technicality: For any $P \in \mathcal{M}(\mathrm{R_p})$, there exists a solution of (2) with $U \in \mathcal{M}(\mathrm{R_{sp}})$; let $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{M}(\mathcal{R})$ be any solution satisfying (2). Let $W \in \mathcal{R}^{n_i \times n_o}$ be any $\mathcal{R}$-stable matrix such that $W(\infty) = -X(\infty)\tilde{D}^{-1}(\infty)$. Define $U := (X + W\tilde{D})$, $V := (Y - W\tilde{N})$, $\tilde{U} := (\tilde{X} + DW)$, $\tilde{V} := (\tilde{Y} - NW)$; then by construction, $U(\infty) = 0$. Let $a \in \mathbb{R}$, $-a \in \mathbb{C}\backslash\mathcal{U}$; define $Q =: (1/(s+a))\hat{Q}$, where $\hat{Q} \in \mathrm{R}_{\mathrm{sp}}^{n_i \times n_o}$ is such that (3) holds, i.e., $[I_{n_i} + (K-1)UN + \hat{Q}(K-1)(1/(s+a))\tilde{D}N]$ is $\mathcal{R}$-unimodular. (This last claim follows by showing that, when $K > 0$, $\det(I_{n_i} + (K-1)UN)(s_0) > 0$ for any $s_0 \in \mathbb{R} \cap \mathcal{U}$ such that $((1/(s+a))\tilde{D}N)(s_0) = 0$; from (6), $\det(I_{n_i} + (K-1)UN) > 0$ at any real blocking $\mathcal{U}$-zero of $(\tilde{D}N)$; now the only additional blocking $\mathcal{U}$-zero of $((1/(s+a))\tilde{D}N)$ is at infinity, where $\det(I_{n_i} + (K-1)UN)(\infty) = 1$ since $U(\infty) = 0$). Since U and Q are strictly proper, $(V - Q\tilde{N})$ is biproper because $(V - Q\tilde{N})(\infty)D(\infty) = I - (U + (1/(s+a))\hat{Q}\tilde{D})(\infty)N(\infty) = I$; the corresponding controller $C = (V - (1/(s+a))\hat{Q}\tilde{N})^{-1}(U + (1/(s+a))\hat{Q}\tilde{D})$ is in fact strictly proper. □

Proof of Corollary 3.2. As in the proof of Theorem 3.1 , let (N, D) be any coprime factorization of P; let $U, V \in \mathcal{M}(\mathcal{R})$ satisfy $VD + UN = 1$. There exist simultaneously $\mathcal{R}$-stabilizing controllers for P and KP if and only if the pair $(1 + (K-1)UN, (K-1)DN)$ satisfies the parity-interlacing-property, equivalently, $(1 + (K-1)UN)$ has the same sign at all real $\mathcal{U}$-zeros of DN. But when $K < 0$, this sign cannot be the same if P has both real $\mathcal{U}$-zeros and real $\mathcal{U}$-poles: At the real $\mathcal{U}$-zeros of N, $(1 + (K-1)UN) = 1 > 0$ and at the real $\mathcal{U}$-zeros of D, $(1 + (K-1)UN) = K < 0$. Therefore, the parity-interlacing-property is satisfied if and only if P either has no real $\mathcal{U}$-zeros or it has no real $\mathcal{U}$-poles.

An alternate proof follows from Theorem 3.1 (b): If P has any real $\mathcal{U}$-poles, then the only denominator invariant factor index is $\ell = 1$ because P is scalar; since the index is an odd number, the parity-interlacing-property is satisfied if and only if P does not have real $\mathcal{U}$-zeros when it has real $\mathcal{U}$-poles. □

Comments 3.3 (*Special cases for simultaneous $\mathcal{R}$-stabilizability of P and KP*). Theorem 3.1 shows that P and KP are simultaneously $\mathcal{R}$-stabilizable for any given plant $P \in \mathcal{M}(\mathrm{R_p})$ and any given *positive* real constant K. If K is negative, then P and KP are simultaneously $\mathcal{R}$-stabilizable only for certain classes of plants. In the single-input single-output case, the characterization of this class of plants is simple; as stated in Corollary 3.2, plants in this class are allowed to have either real $\mathcal{U}$-poles or real $\mathcal{U}$-zeros, but not both. Note that $\infty \in \mathcal{U}$ so when $K < 0$, if P is strictly proper, then it can be simultaneously $\mathcal{R}$-stabilized with KP if and only if it has no real $\mathcal{U}$-poles.

For MIMO plants, when $K < 0$, the characterization of the class of P which can be simultaneously $\mathcal{R}$-stabilized with KP is not as simple since it requires a careful account of all possible real blocking $\mathcal{U}$-zeros of the product $(\tilde{D}N)$. The blocking $\mathcal{U}$-zeros of $\tilde{D}$ and of N are obviously among the blocking $\mathcal{U}$-zeros of $(\tilde{D}N)$. However, there may also be additional blocking $\mathcal{U}$-zeros: If P is strictly proper or it has at least one real blocking $\mathcal{U}$-zero, then $N(z) = 0$ for some $z \in \mathbb{R} \cap \mathcal{U}$, which is clearly a blocking $\mathcal{U}$-zero of $(\tilde{D}N)$. Considering the Smith–McMillan form in (4), all other real blocking $\mathcal{U}$-zeros of $(\tilde{D}N)$ are either $\mathcal{U}$-zeros of the smallest invariant factor ψ_r of the denominator or they are $\mathcal{U}$-zeros of some denominator invariant factor ψ_ℓ, which also appear immediately as a zero of the next numerator invariant factor $\lambda_{(\ell+1)}$. It is clear that the conditions in Theorem 3.1 (b) are satisfied for the following special cases:

(i) If P has no real $\mathcal{U}$-poles, then there are no s_i as described in Theorem 3.1 (b); therefore P and KP are simultaneously $\mathcal{R}$-stabilizable for any $K < 0$.

(ii) If P has no real $\mathcal{U}$-zeros (including infinity), then none of the numerator invariant factors λ_i have real $\mathcal{U}$-zeros and, hence, the only blocking $\mathcal{U}$-zeros of $(\tilde{D}N)$ are due to the smallest denominator invariant factor ψ_r, i.e., the only index $\ell_i = r$; therefore P and KP are simultaneously $\mathcal{R}$-stabilizable for any $K < 0$.

(iii) If P has no real $\mathcal{U}$-poles coinciding with $\mathcal{U}$-zeros, then the only s_i are due to $\psi_r(s_i) = 0$, i.e., the only index $\ell_i = r$. In this case, if P has real blocking $\mathcal{U}$-zeros (including infinity), then P and KP are simultaneously $\mathcal{R}$-stabilizable for any $K < 0$ if and only if the rank r is even. If P is full (row or column) rank, i.e., if $r = \min\{n_o, n_i\}$, then obviously these comments on r apply to the smaller of the number of inputs (n_i) and outputs (n_o).

4. Conclusions

It is shown that P and KP are simultaneously $\mathcal{R}$-stabilizable for any given P and any given positive real constant $K > 0$. This result is equivalent to existence of an $\mathcal{R}$-stabilizing controller C such that KC is also an $\mathcal{R}$-stabilizing controller for P. It is also shown that P and KP are not always simultaneously stabilizable for $K < 0$ and a complete characterization of the class of plants that can be simultaneously $\mathcal{R}$-stabilized with KP is given in Theorem 3.1 .

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