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Reliable stabilization with integral action in decentralized control systems

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Abstract

Reliable stabilization with integral action is studied in the linear, time-invariant, multi-input, multi-output, two-channel decentralized control system, where the plant is stable. The objective is to achieve closed-loop stability when both controllers act together and when each controller acts alone. Necessary and sufficient conditions are obtained for existence of block-diagonal decentralized controllers that ensure reliable stabilization and integral action. All decentralized controllers with integral action that provide reliable stabilization are characterized.

1. Introduction

We consider reliable stabilization with integral action in the linear, time-invariant (LTI), multi-input, multi-output (MIMO), two-channel decentralized system $\mathcal{S}(P, \frac{1}{s}C_D)$, shown in Figure 1. The goal of reliable stabilization is to find a pair of controllers C_1, C_2 such that the closed-loop system $\mathcal{S}(P, \frac{1}{s}C_D)$ is stable when both controllers act together and when each controller acts alone. Furthermore, the closed-loop system has integral action at all outputs due to the integrators at each of the two channels. In this system, the plant P is stable. The integrators in the controller $\frac{1}{s}C_D = \frac{1}{s}\operatorname{diag}[\,C_1 \quad C_2\,]$ provide Type-I closed-loop response. The block-diagonal controller C_D is not assumed to be stable. The model of controller failure used here is that when a controller fails, it is replaced by zero. Since the system is required to remain stable when either one of the controllers fails, it is assumed that the failure is recognized and the corresponding controller is taken out of service (i.e., the states in the controller implementation are all set to zero, the initial conditions and the outputs of the channel that failed are set to zero for all inputs). Clearly, stability is maintained when both controllers are set to zero as well since the open-loop plant is stable. The integrators in the control channels guarantee that the closed-loop system achieves asymptotic tracking of step inputs. If one controller fails, this integral action is still present in the outputs of the channel with the active controller.

After the introduction of multi-controller systems in [6], [7], the problem of reliable stabilization was studied using full-feedback controllers ([4], [3]) and decentralized controllers [8]. Integral action in the decentralized configuration was considered with scalar channels assuming that C_1 and C_2 are stable [5], [1]. Conditions on the steady-state gain of the plant were developed in [1] for the case of scalar channels with stable C_D assuming that the fully decentralized channels have gain uncertainty between 0 and 1.

The objective of this paper is to present *necessary and sufficient* conditions on P for existence of block-diagonal decentralized controllers that ensure reliable stabilization with integral action. The main result (Theorem 3.5) states that decentralized controllers with integral action exist if and only if a simple positivity condition holds on the determinant of P and it's sub-blocks evaluated at zero. All decentralized controllers with integral action that provide reliable stabilization are characterized. When C_1 and C_2 are square, it is shown that one of the two controllers can be stable. Fixing one of the controllers, the other controller is designed by finding a strongly stabilizing controller for a pseudo-plant. Strongly stabilizing controllers can be designed for this pseudo-plant using the interpolation methods of [9] for the scalar case. We also give an alternate method of designing the second controller explicitly without interpolation for a restricted class of plants.

The results apply to discrete-time systems as well as continuous-time systems; for the case of discrete-time systems, all evaluations and poles at $s = 0$ would be interpreted at 1.

Notation: Let $\mathcal{U}$ contain the extended closed right-half-plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). Let $\mathbb{R}$ denote the set of real numbers. Let $\mathrm{R_p}$ ($\mathrm{R_{sp}}$) denote proper (strictly-proper) rational functions with real coefficients; $\mathcal{R} \subset \mathrm{R_p}$ denotes proper

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rational functions which have no poles in the region of instability $\mathcal{U}$; $\mathcal{M}(\mathcal{R})$ denotes the set of matrices whose entries are in $\mathcal{R}$. A matrix M is called $\mathcal{R}$-stable iff $M \in \mathcal{M}(\mathcal{R})$; $M \in \mathcal{M}(\mathcal{R})$ is called $\mathcal{R}$-unimodular iff M^{-1} is also $\mathcal{R}$-stable. For $M \in \mathcal{M}(\mathcal{R})$, the norm $\|\cdot\|$ is defined as $\| M \| = \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$. A right-coprime-factorization (RCF) of $P_{jj} \in \mathrm{R_p}^{n_j \times m_j}$ is denoted by (N_{jj}, D_{jj}), where $N_{jj}, D_{jj} \in \mathcal{M}(\mathcal{R})$, D_{jj} is biproper and $P_{jj} = N_{jj} D_{jj}^{-1}$. Similarly, an RCF of $C_j \in \mathrm{R_p}^{m \times n}$ is denoted by (N_{Cj}, D_{Cj}), where $N_{Cj}, D_{Cj} \in \mathcal{M}(\mathcal{R})$, D_{Cj} is biproper and $C_j = N_{Cj} D_{Cj}^{-1}$. The identity map (of dimension n) is denoted by I_n. $a := b$ means a is defined as b.

2. Analysis

We consider the LTI, MIMO, two-channel decentralized control system $\mathcal{S}(P, \frac{1}{s} C_D)$ shown in Figure 1; P and $\frac{1}{s} C_D$ represent the transfer-functions of the plant and the controller, respectively, where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{R}^{n \times m}$, $P_{jj} \in \mathcal{R}^{n_j \times m_j}$, $j = 1, 2$, $n = n_1 + n_2$, $m = m_1 + m_2$, $\frac{1}{s} C_D = \frac{1}{s} \operatorname{diag}[\, C_1 \quad C_2 \,] \in \mathrm{R_p}^{m \times n}$, $C_j \in \mathrm{R_p}^{m_j \times n_j}$. It is assumed that $\mathcal{S}(P, \frac{1}{s} C_D)$ is a well-posed system and that the plant and the controller do not have any hidden-modes associated with eigenvalues in the region of instability $\mathcal{U}$. The plant is $\mathcal{R}$-stable. The controller $\frac{1}{s} C_D$ is not $\mathcal{R}$-stable; the block-diagonal decentralized controller C_D is not necessarily stable. Let (N_{Cj}, D_{Cj}) be an RCF of C_j, $j = 1, 2$; $N_C := \operatorname{diag}[\, N_{C1} \quad N_{C2} \,]$, $D_C := \operatorname{diag}[\, D_{C1} \quad D_{C2} \,]$; then (N_C, D_C) is an RCF of C_D. For integral action at each output, it is necessary that C_D has no (transmission) zeros at $s = 0$, i.e.,

$$\operatorname{rank} N_{Cj}(0) = n_j \ , \quad j = 1, 2 \ ;$$

this necessary condition implies that the number of outputs can not exceed the number of inputs $(n_j \le m_j)$ for each channel. Let $-a \in \mathbb{R} \setminus \mathcal{U}$; then $\frac{1}{s}$ can be factored as $\frac{1}{s} = (\frac{s}{s+a})^{-1} \frac{1}{s+a}$. For $j = 1, 2$, let $\frac{s}{s+a} D_{Cj} \xi_{Cj} = \frac{1}{s+a} e_{Cj}$, $N_{Cj} \xi_{Cj} = y_{Cj}$. Let $u_P := [u_{P1}^T, u_{P2}^T]^T$, $u_C := [u_{C1}^T, u_{C2}^T]^T$, $y_P := [y_{P1}^T, y_{P2}^T]^T$, $y_C := [y_{C1}^T, y_{C2}^T]^T$. The system $\mathcal{S}(P, \frac{1}{s} C_D)$ is described as:

$$\left[\frac{s}{s+a} \begin{bmatrix} D_{C1} & 0 \\ 0 & D_{C2} \end{bmatrix} + \frac{1}{s+a} P \begin{bmatrix} N_{C1} & 0 \\ 0 & N_{C2} \end{bmatrix} \right] \begin{bmatrix} \xi_{C1} \\ \xi_{C2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+a} I_n & -\frac{1}{s+a} P \end{bmatrix} \begin{bmatrix} u_C \\ u_P \end{bmatrix} , \quad (1)$$

$$\begin{bmatrix} N_C \\ P N_C \end{bmatrix} \begin{bmatrix} \xi_{C1} \\ \xi_{C2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} u_C \\ u_P \end{bmatrix} = \begin{bmatrix} y_C \\ y_P \end{bmatrix} . \quad (2)$$

Equations (1)-(2) are of the form $D_H \xi = u$, $y = N \xi + G u$. Since $\frac{1}{s} C_D$ is strictly-proper for any proper C_1, C_2, the system $\mathcal{S}(P, \frac{1}{s} C_D)$ is well-posed, i.e., the transfer-function $H : (u_P, u_C) \mapsto (y_P, y_C)$ is proper.

Now suppose that one of the controllers fails, i.e., is set equal to zero; it is assumed that the failure is recognized and the corresponding controller is taken out of service. When C_2 is set equal to zero, the system is called $\mathcal{S}(P, \frac{1}{s} C_1)$; similarly, when C_1 is set equal to zero, the system is called $\mathcal{S}(P, \frac{1}{s} C_2)$. Consider the system $\mathcal{S}(P, \frac{1}{s} C_1)$, where C_2 is set equal to zero and the outputs y_{C2} of the second control channel are not observed. A description of $\mathcal{S}(P, \frac{1}{s} C_1)$ similar to that of $\mathcal{S}(P, \frac{1}{s} C_D)$ is given by:

$$\left[\frac{s}{s+a} D_{C1} + \frac{1}{s+a} P_{11} N_{C1} \right] \xi_{C1} = \left[\frac{1}{s+a} I_{n_1} \quad -\frac{1}{s+a} [\, I_{n_1} \quad 0 \,] P \right] \begin{bmatrix} u_{C1} \\ u_P \end{bmatrix} , \quad (3)$$

$$\begin{bmatrix} N_{C1} \\ P_{11} N_{C1} \\ 0 \end{bmatrix} \xi_{C1} + \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} u_{C1} \\ u_P \end{bmatrix} = \begin{bmatrix} y_{C1} \\ y_P \end{bmatrix} . \quad (4)$$

A similar description can be obtained for $\mathcal{S}(P, \frac{1}{s} C_2)$, where C_1 is set equal to zero and the outputs y_{C1} of the first control channel are not observed.

3. Stability

We now investigate the $\mathcal{R}$-stability of the system $\mathcal{S}(P, \frac{1}{s} C_D)$ under normal operation and under the complete failure of either the first or the second controller. It is assumed that if one of the controllers fails, the failure is recognized and the controller that failed is taken out of service.

3.1. Definitions (*Reliable controller pair*):

1) The system $\mathcal{S}(P, \frac{1}{s} C_D)$ is said to be $\mathcal{R}$-stable iff the closed-loop transfer-function H from (u_C, u_P) to (y_C, y_P) is $\mathcal{R}$-stable. Similarly, for $j = 1, 2$, the system $\mathcal{S}(P, \frac{1}{s} C_j)$ is $\mathcal{R}$-stable iff the transfer-function H_j from (u_{Cj}, u_P) to (y_{Cj}, y_P) is $\mathcal{R}$-stable.

2) The pair $(\frac{1}{s} C_1, \frac{1}{s} C_2)$ is called a *decentralized reliable controller pair with integral action* iff $C_1, C_2 \in \mathcal{M}(\mathrm{R_p})$, and the systems $\mathcal{S}(P, \frac{1}{s} C_D)$, $\mathcal{S}(P, \frac{1}{s} C_1)$, $\mathcal{S}(P, \frac{1}{s} C_2)$ are all $\mathcal{R}$-stable. □

3.2. Lemma (*Closed-loop stability*):

Let $P \in \mathcal{R}^{n \times m}$. For $j = 1, 2$, let (N_{Cj}, D_{Cj}) be an RCF of C_j, and let $\operatorname{rank} N_{Cj}(0) = n_j$.

i) The system $\mathcal{S}(P, \frac{1}{s} C_D)$ is $\mathcal{R}$-stable if and only if $D_H := \frac{s}{s+a} D_C + \frac{1}{s+a} P N_C = \begin{bmatrix} \frac{s}{s+a} D_{C1} + \frac{1}{s+a} P_{11} N_{C1} & \frac{1}{s+a} P_{12} N_{C2} \\ \frac{1}{s+a} P_{21} N_{C1} & \frac{s}{s+a} D_{C2} + \frac{1}{s+a} P_{22} N_{C2} \end{bmatrix}$

is $\mathcal{R}$-unimodular.

ii) The system $\mathcal{S}(P, \frac{1}{s}C_1)$ is $\mathcal{R}$-stable if and only if $D_{H1} := \left[\frac{s}{s+a} D_{C1} + \frac{1}{s+a} P_{11} N_{C1} \right]$ is $\mathcal{R}$-unimodular.

iii) The system $\mathcal{S}(P, \frac{1}{s}C_2)$ is $\mathcal{R}$-stable if and only if $D_{H2} := \left[\frac{s}{s+a} D_{C2} + \frac{1}{s+a} P_{22} N_{C2} \right]$ is $\mathcal{R}$-unimodular. $\square$

Proof: We prove **(i)** using the system description (1)-(2); (ii) and (iii) follow similarly from (3)-(4). The closed-loop transfer-function of $\mathcal{S}(P, \frac{1}{s}C_D)$ is given by $H = N_R D_H{}^{-1} N_L + G$, where N_R, D_H, N_L, $G \in \mathcal{M}(\mathcal{R})$. We show that (N_R, D_H, N_L) is a bicoprime triple: Since (N_C, D_C) is right-coprime, rank $\begin{bmatrix} D_C \\ N_C \end{bmatrix} = n$ for all $s \in \mathcal{U}$ [2]; hence, rank $\begin{bmatrix} \frac{s}{s+a} D_C \\ N_C \end{bmatrix} = n$ for all $s \in \mathcal{U} \setminus \{0\}$ since $-a$ is not in $\mathcal{U}$. Since rank $N_C(0) = n$ by assumption, rank $\begin{bmatrix} \frac{s}{s+a} D_C \\ N_C \end{bmatrix}(0) = n$. Therefore,

$$\text{rank} \begin{bmatrix} D_H \\ N_R \end{bmatrix} = \text{rank} \begin{bmatrix} I & \frac{1}{s+a}P & 0 \\ 0 & I & 0 \\ 0 & P & I \end{bmatrix} \begin{bmatrix} \frac{s}{s+a} D_C \\ N_C \\ 0 \end{bmatrix}$$

$= \text{rank} \begin{bmatrix} \frac{s}{s+a} D_C \\ N_C \end{bmatrix} = n$ for all $s \in \mathcal{U}$, i.e., (N_R, D_H) is right-coprime. Similarly, rank$[D_H \;\; N_C] =$

$$\text{rank} \begin{bmatrix} \frac{s}{s+a} D_C & \frac{1}{s+a} I_n & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ P N_C & I & -P \\ 0 & 0 & I \end{bmatrix} =$$

rank $\left[\frac{s}{s+a} D_C \quad \frac{1}{s+a} I_n \right]$ for all $s \in \mathcal{U}$. The matrix in the last equality has full row-rank for all $s \in \mathcal{U}$ because $\frac{1}{s+a} I_n$ drops rank only at infinity, but rank$(\frac{s}{s+a} D_C) = n$ at infinity since D_C is biproper. Therefore, rank$[D_H \quad N_L] = n$ for all $s \in \mathcal{U}$, equivalently, (D_H, N_L) is left-coprime. Since it is shown that (N_R, D_H, N_L) is a bicoprime triple, it follows using standard arguments that $H \in \mathcal{M}(\mathcal{R})$ if and only if $D_H{}^{-1} \in \mathcal{M}(\mathcal{R})$ ([9], [2]). $\square$

From the three conditions of Lemma 3.2, it is clear that the following conditions on P are necessary for $\mathcal{R}$-stability when the two controllers act together and when each controller acts alone.

3.3. Corollary (*Necessary conditions on P*):

i) If the system $\mathcal{S}(P, \frac{1}{s}C_D)$ is $\mathcal{R}$-stable, then rank$P(0) = n$, i.e., $(\frac{s}{s+a}I_n, \frac{1}{s+a}P)$ is left-coprime.

ii) If the system $\mathcal{S}(P, \frac{1}{s}C_1)$ is $\mathcal{R}$-stable, then rank $P_{11}(0) = n_1$, i.e., the pair $(\frac{s}{s+a}I_{n_1}, \frac{1}{s+a} P_{11})$ is left-coprime.

iii) If the system $\mathcal{S}(P, \frac{1}{s}C_2)$ is $\mathcal{R}$-stable, then rank $P_{22}(0) = n_2$, i.e., the pair $(\frac{s}{s+a}I_{n_2}, \frac{1}{s+a} P_{22})$ is left-coprime. $\square$

Proof: By Lemma 3.2, the system $\mathcal{S}(P, \frac{1}{s}C_D)$ is $\mathcal{R}$-stable if and only if D_H is $\mathcal{R}$-unimodular, equivalently, rank$D_H = n$ for all $s \in \mathcal{U}$. Therefore, rank$D_H(0) = $ rank$(\frac{1}{a}P(0) N_C(0)) = n$, implying that rank$P(0) = n$ whenever $\mathcal{S}(P, \frac{1}{s}C_D)$ is stable. The pair $(\frac{s}{s+a}I_n, \frac{1}{s+a}P)$ is left-coprime if and only if rank $\left[\frac{s}{s+a}I_n \quad \frac{1}{s+a}P \right] = n$, which is full row-rank if and only if rank$P(0) = n$ since $\frac{s}{s+a}I_n$ only drops rank at $s = 0$. This establishes (i); (ii) and (iii) follow similarly from D_{H1} and D_{H2}. $\square$

The three conditions of Corollary 3.3 are necessary for the existence of decentralized reliable controllers with integral action; therefore we assume from now on that rank$P(0) = n$ and for $j = 1, 2$, rank$P_{jj}(0) = n_j$. If rank$P_{jj}(0) = n_j$, equivalently, $(\frac{s}{s+a}I_{n_j}, \frac{1}{s+a}P_{jj})$ is left-coprime, then there exist V_j, U_j, $\tilde{V}_j$, $\tilde{U}_j$, D_{jj}, $N_{jj} \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} V_j & U_j \\ -\frac{1}{s+a}P_{jj} & \frac{s}{s+a}I_{n_j} \end{bmatrix} \begin{bmatrix} D_{jj} & -\tilde{U}_j \\ N_{jj} & \tilde{V}_j \end{bmatrix} = \begin{bmatrix} I_{m_j} & 0 \\ 0 & I_{n_j} \end{bmatrix}, \quad \text{for } j = 1, 2. \tag{5}$$

When P_{11}, P_{22} are square, i.e., $n_1 = m_1$ and $n_2 = m_2$, then P is also square; P_{12} and P_{21} are not necessarily square since it is not assumed that $n_1 = n_2$. If each P_{jj} is square, then the necessary conditions of Corollary 3.3 become $P(0)$ is nonsingular and $P_{jj}(0)$ is nonsingular for $j = 1, 2$. When $n_j = m_j$, if $P_{jj}(0)$ is nonsingular, then there exist $V_j, U_j, \tilde{V}_j \in \mathcal{M}(\mathcal{R})$ such that

$$\begin{bmatrix} V_j & U_j \\ -\frac{1}{s+a}P_{jj} & \frac{s}{s+a}I_{n_j} \end{bmatrix} \begin{bmatrix} \frac{s}{s+a}I_{n_j} & -U_j \\ \frac{1}{s+a}P_{jj} & \tilde{V}_j \end{bmatrix} = I_{2n_j}, \quad \text{for } j = 1, 2. \tag{6}$$

In (5), (N_{jj}, D_{jj}) is a right-coprime-factorization of $\frac{1}{s}P_{jj}$. When P_{jj} is square, $(\frac{1}{s+a}P_{jj}, \frac{s}{s+a}I_{n_j})$ is an RCF of $\frac{1}{s}P_{jj}$ and $U_j = \tilde{U}_j$. Since the only $\mathcal{U}$-poles of $\frac{1}{s}P_{jj}$ are at $s = 0$, det $D_{jj}(0) = 0$. However, $D_{jj}(0) \neq 0$ when $m_j > n_j$.

3.4. Lemma (*Decentralized reliable controllers*): Let $P \in \mathcal{R}^{n \times m}$, $P_{jj} \in \mathcal{R}^{n_j \times m_j}$. Let rank$P(0) = n$ and rank$P_{jj}(0) = n_j$ for $j = 1, 2$.

i) For $j = 1, 2$, the system $\mathcal{S}(P, \frac{1}{s}C_j)$ is $\mathcal{R}$-stable if and only if, for some $Q_j \in \mathcal{R}^{n_j \times n_j}$, $C_j = N_{Cj} D_{Cj}^{-1}$, where

$$N_{Cj} = (\tilde{U}_j + D_{jj} Q_j), \quad D_{Cj} = (\tilde{V}_j - N_{jj} Q_j). \tag{7}$$

ii) For $j = 1, 2$, let $\mathcal{S}(P, \frac{1}{s}C_j)$ be $\mathcal{R}$-stable, i.e., let C_j be as in (7). The system $\mathcal{S}(P, \frac{1}{s}C_D)$ is $\mathcal{R}$-stable if and only if

$$I_{n_1} - \frac{1}{s+a} P_{12} N_{C2} \frac{1}{s+a} P_{21} N_{C1}$$
$$= I_{n_1} - \frac{1}{(s+a)^2} P_{12} (\tilde{U}_2 + D_{22} Q_2) P_{21} (\tilde{U}_1 + D_{11} Q_1)$$
$$\text{is } \mathcal{R}\text{-unimodular.} \qquad (8)$$

Proof: By Lemma 3.2, $\mathcal{S}(P, \frac{1}{s} C_j)$ is $\mathcal{R}$-stable if and only if D_{Hj} is $\mathcal{R}$-unimodular, equivalently, for some RCF (N_{Cj}, D_{Cj}) of C_j, $\left[\frac{s}{s+a} D_{Cj} + \frac{1}{s+a} P_{jj} N_{Cj}\right] = I_{n_j}$. The expression $C_j = N_{Cj} D_{Cj}^{-1} = (U_j + D_{jj} Q_j)(\tilde{V}_j - N_{jj} Q_j)^{-1}$ is obtained by finding all solutions of this matrix equality using the Bezout identity (5). For $j = 1, 2$, substituting this solution for N_{Cj}, D_{Cj}, D_H becomes $D_H = \begin{bmatrix} I_{n_1} & \frac{1}{s+a} P_{12} N_{C2} \\ \frac{1}{s+a} P_{21} N_{C1} & I_{n_2} \end{bmatrix}$, which is $\mathcal{R}$-unimodular if and only if (8) holds. □

Suppose that $m_j = n_j$, i.e., $P \in \mathcal{R}^{n \times n}$, $P_{jj} \in \mathcal{R}^{n_j \times n_j}$. In this case, by setting $D_{jj} = \frac{s}{s+a} I_{n_j}$, $N_{jj} = \frac{1}{s+a} P_{jj}$ and $U_j = \tilde{U}_j$ as in (6), Lemma 3.4 is stated as follows: Let $P(0)$ be nonsingular and $P_{jj}(0)$ be nonsingular for $j = 1, 2$. For $j = 1, 2$, the system $\mathcal{S}(P, \frac{1}{s} C_j)$ is $\mathcal{R}$-stable if and only if, for some $Q_j \in \mathcal{R}^{n_j \times n_j}$, $C_j = N_{Cj} D_{Cj}^{-1}$, where

$$N_{Cj} = (U_j + \tfrac{s}{s+a} Q_j), \; D_{Cj} = (\tilde{V}_j - \tfrac{1}{s+a} P_{jj} Q_j). \quad (9)$$

For $j = 1, 2$, let $\mathcal{S}(P, \frac{1}{s} C_j)$ be $\mathcal{R}$-stable, i.e., let C_j be as in (9). The system $\mathcal{S}(P, \frac{1}{s} C_D)$ is $\mathcal{R}$-stable if and only if

$$I_{n_1} - \frac{1}{s+a} P_{12} N_{C2} \frac{1}{s+a} P_{21} N_{C1}$$
$$= I_{n_1} - \frac{1}{(s+a)^2} P_{12} (U_2 + \tfrac{s}{s+a} Q_2) P_{21} (U_1 + \tfrac{s}{s+a} Q_1)$$
$$\text{is } \mathcal{R}\text{-unimodular.} \qquad (10)$$

By Lemma 3.4, there exist reliable decentralized integral controllers if and only if $\mathcal{R}$-stable matrices Q_1, Q_2 can be found such that (8) is $\mathcal{R}$-unimodular (or (10) is $\mathcal{R}$-unimodular if the channels are square). If P is block-triangular ($P_{12} = 0$ or $P_{21} = 0$) then (8) obviously holds for all $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$. However (8) does not explicitly state whether such Q_1, Q_2 exist. Theorem 3.5 establishes necessary and sufficient conditions for existence of Q_1, Q_2 by checking ranks of real matrices associated with $P(0)$. Corollary 3.6 gives an equivalent condition for square channels.

3.5. Theorem (*Existence of decentralized reliable controllers with integral action*):
Let $P \in \mathcal{R}^{n \times m}$ and $P_{jj} \in \mathcal{R}^{n_j \times m_j}$. There exists a decentralized reliable controller pair with integral action for P if and only if there exist real matrices $\mathbf{R_1} \in \mathbb{R}^{m_1 \times n_1}$ and $\mathbf{R_2} \in \mathbb{R}^{m_2 \times n_2}$ such that

i) $\quad P_{11}(0)\, \mathbf{R_1} = I_{n_1}$,

ii) $\quad P_{22}(0)\, \mathbf{R_2} = I_{n_2}$, and

iii) $\quad \det(I_{n_1} - P_{12}(0)\, \mathbf{R_2}\, P_{21}(0)\, \mathbf{R_1}) > 0$. (11)

Proof: *Necessity:* Let $(\frac{1}{s} C_1, \frac{1}{s} C_2)$ be a decentralized reliable controller pair with integral action; by Definition 3.1, the systems $\mathcal{S}(P, \frac{1}{s} C_D)$, $\mathcal{S}(P, \frac{1}{s} C_1)$, $\mathcal{S}(P, \frac{1}{s} C_2)$ are $\mathcal{R}$-stable. By Corollary 3.3, $\mathcal{S}(P, \frac{1}{s} C_j)$ $\mathcal{R}$-stable implies that rank $P_{jj}(0) = n_j$. By Lemma 3.4, C_j is given by (9), where (10) is $\mathcal{R}$-unimodular. Defining $X := -\frac{1}{(s+a)^2} P_{12} N_{C2} P_{21}$, by (10), $(I_{n_1} - \frac{1}{(s+a)^2} P_{12} N_{C2} P_{21} N_{C1}) = (I_{n_1} + X N_{C1})$ is $\mathcal{R}$-unimodular. Since $\det(I_{n_1} + X N_{C1})$ is a unit of $\mathcal{R}$, it must have the same sign at all $s \in \mathcal{U}$. Since $X(\infty) = 0$, clearly $\det(I_{n_1} + X N_{C1})(\infty) = 1$. So $\det(I_{n_1} + X N_{C1})(s)$ must be positive for all $s \in \mathcal{U}$ since it is a unit of $\mathcal{R}$. From (5), $\frac{s}{s+a} D_{Cj} + \frac{1}{s+a} P_{jj} N_{Cj} = I_{n_j}$ implies that $a^{-1} N_{Cj}(0) = (\tilde{U}_j + D_{11} Q_j)(0)$ is a right-inverse of $P_{jj}(0)$, i.e., $N_{Cj}(0) = a\, \mathbf{R_j}$. Therefore, $X(0) N_{C1}(0) = -P_{12}(0)\, \mathbf{R_2}\, P_{21}(0)\, \mathbf{R_1}$ and hence, $\det(I_{n_1} + X N_{C1})(0) = \det(I_{n_1} - P_{12}(0)\, \mathbf{R_2}\, P_{21}(0)\, \mathbf{R_1})$ must be positive for some $\mathbf{R_j}$ such that $P_{jj} \mathbf{R_j} = I_{n_j}$.

Sufficiency: If rank $P_{jj}(0) = n_j$, then for $j = 1, 2$, $P_{jj}(0)$ has a right-inverse. By assumption, (11) holds for some $\mathbf{R_1}$, $\mathbf{R_2}$. Let C_j be given by by (9). By (6), $\frac{s}{s+a} D_{Cj} + \frac{1}{s+a} P_{jj} N_{Cj} = I_{n_j}$. Choose $Q_2 \in \mathcal{M}(\mathcal{R})$ for C_2 such that $N_{C2}(0) = (\tilde{U}_2 + D_{22} Q_2)(0) = a\, \mathbf{R_2}$ for the particular right-inverse $\mathbf{R_2}$ of $P_{22}(0)$ which satisfies (11). Similarly, there is an $A_1 \in \mathcal{M}(\mathcal{R})$ such that $(\tilde{U}_1 + D_{11} A_1)(0) = a\, \mathbf{R_1}$ for the right-inverse $\mathbf{R_1}$ of $P_{11}(0)$ which satisfies (11). With N_{C2} fixed, define $X := -\frac{1}{(s+a)^2} P_{12} N_{C2} P_{21}$ and define $U := (\tilde{U}_1 + D_{11} A_1)$. The goal is to show that the pair $((I_{n_1} + XU), X D_{11})$ satisfies the parity-interlacing-property. Since $X(\infty) = 0$, $(I_{n_1} + XU)(\infty) = I_{n_1}$ implies that $(I_{n_1} + XU)$ is nonsingular. The first step is to show that the pair $((I_{n_1} + XU), X D_{11})$ is left-coprime. For $s = 0$, $X(0) = -a^{-1} P_{12}(0)\, \mathbf{R_2}\, P_{21}(0)$ and $U(0) = (\tilde{U}_1 + D_{11} A_1)(0) = a\, \mathbf{R_1}$; therefore, by (11), $I_{n_1} + X(0) U(0) = I_{n_1} - P_{12}(0)\, \mathbf{R_2}\, P_{21}(0)\, \mathbf{R_1}$ is nonsingular. Therefore, rank $[(I_{n_1} + XU) \quad X D_{11}] = n_1$ for $s = 0$. Since the only $\mathcal{U}$-poles of $\frac{1}{s} P_{jj}$ are at $s = 0$, $D_{11}{}^{-1}(s)$ is $\mathcal{R}$-stable for all $s \in \mathcal{U}$ except for $s = 0$. Therefore, rank $[(I_{n_1} + XU) \quad X D_{11}] =$ rank $[(I_{n_1} + XU) \quad X D_{11}] \begin{bmatrix} I_{n_1} & 0 \\ -D_{11}{}^{-1} U & I_{m_1} \end{bmatrix} =$ rank $[I_{n_1} \quad X D_{11}] = n_1$ for all $s \in \mathcal{U}$, $s \neq 0$. Since this shows that rank $[(I_{n_1}+XU) \quad XD_{11}] = n_1$ for all $s \in \mathcal{U}$, the pair $((I_{n_1}+XU), XD_{11})$ is left-coprime. Now the second step is to show that $\det(I_{n_1}+XU)$ has the same sign at all real blocking $\mathcal{U}$-zeros of XD_{11}. At the blocking $\mathcal{U}$-zeros of X, $\det(I_{n_1}+XU)(s_o)$ is positive for all $s_o \in \mathcal{U}$ such that $X(s_o) = 0$. Now the only other possible blocking $\mathcal{U}$-zero of XD_{11} is at $s = 0$ because $\det D_{11}(s) = 0$ only if $s = 0$. But by (11), $\det(I_{n_1}+X(0)U(0))$ is positive.

Since $((I_{n_1}+X(\widetilde{U}_1+D_{11}A_1)), D_{11}X)$ satisfies the parity-interlacing-property, there exists $\hat{Q}_1 \in \mathcal{M}(\mathcal{R})$ such that $(I_{n_1}+X(\widetilde{U}_1+D_{11}A_1)+XD_{11}\hat{Q}_1)$ is $\mathcal{R}$-unimodular. Choose Q_1 in (7) as $Q_1 = A_1+\hat{Q}_1$. Then by Lemma 3.4, condition (8) implies that the pair C_1, C_2 is a decentralized reliable controller pair with integral action. □

3.6. Corollary (*Existence of decentralized reliable controllers with integral action; square channels*):
Let $P \in \mathcal{R}^{n\times n}$ and $P_{jj} \in \mathcal{R}^{n_j\times n_j}$. There exists a decentralized reliable controller pair with integral action for P if and only if,

$$\text{i)} \qquad \det P_{11}(0) \neq 0 \,,$$

$$\text{ii)} \qquad \det P_{22}(0) \neq 0 \,, \text{ and}$$

$$\text{iii)} \qquad \frac{\det P(0)}{\det P_{11}(0)\,\det P_{22}(0)} > 0. \qquad (12)$$

Proof: Replacing $\mathbf{R_1}$ and $\mathbf{R_2}$ by the unique inverses $P_{11}(0)^{-1}$ and $P_{22}(0)^{-1}$, (12) follows by expanding $\det P(0)$ as $\det(I_{n_1} - P_{12}(0)P_{22}(0)^{-1}P_{21}(0)P_{11}(0)^{-1})$ $\det P_{11}(0)\det P_{22}(0)$. The necessity proof follows the same lines as in the proof of Theorem 3.5. In the proof of sufficiency, $Q_2 \in \mathcal{M}(\mathcal{R})$ can be chosen arbitrarily since $N_{C2}(0) = (\widetilde{U}_2 + \frac{s}{s+a}Q_2)(0) = a\,P_{22}(0)^{-1}$. Similarly, A_1 can be taken as zero since $\widetilde{U}_1(0) = U_1(0) = a\,P_{11}(0)^{-1}$. Using similar arguments, it can be shown that the pair $((I_{n_1}+XU_1), \frac{s}{s+a}X)$ satisfies the parity-interlacing-property. In the case of square channels, $\frac{s}{s+a}X$ always has a blocking $\mathcal{U}$-zero at $s = 0$. □

3.7. Remarks

1) In Theorem 3.5, for $j = 1,2$, the real matrices $\mathbf{R_j}$ correspond to right-inverses of $P_{jj}(0)$. If $m_j > n_j$ then $\mathbf{R_j}$ is not unique. When $m_j = n_j$, using the (unique) inverse $P_{jj}^{-1}(0)$, (11) becomes $\det(I_{n_1} - P_{12}(0)P_{22}(0)^{-1}P_{21}(0)P_{11}(0)^{-1}) > 0$, which is equivalent to (12) of Corollary 3.6.

2) If the system has single-input single-output channels ($n_1 = n_2 = 1$), then $P_{jj} \in \mathcal{R}$ are scalar and the conditions of Corollary 3.6 are simplified as follows: There exists a decentralized reliable controller pair with integral action if and only if $P_{11}(0) \neq 0$, $P_{22}(0) \neq 0$, $P_{11}^{-1}(0)P_{22}^{-1}(0)P_{21}(0)P_{12}(0) < 1$.

3) The decentralized controller C_D is proper for all choices of $Q_1, Q_2 \in \mathcal{M}(\mathcal{R})$. With C_j as in (7) (or (9) for square channels), (5) implies $\frac{s}{s+a}D_{Cj} + \frac{1}{s+a}P_{jj}N_{Cj} = I_{n_j}$. Therefore, D_{Cj} is biproper since $D_{Cj}(\infty) = I_{n_j}$ and hence, C_j is proper.

4) By Lemma 3.4, the systems $\mathcal{S}(P, \frac{1}{s}C_D)$, $\mathcal{S}(P, \frac{1}{s}C_1)$, $\mathcal{S}(P, \frac{1}{s}C_2)$ are all $\mathcal{R}$-stable if and only if C_j is given by (7) and (8) is $\mathcal{R}$-unimodular. Defining $X := -\frac{1}{(s+a)^2}P_{12}N_{C2}P_{21}$, (8) is equivalent to $(I_{n_1} - \frac{1}{(s+a)^2}P_{12}N_{C2}P_{21}\widetilde{U}_1) - \frac{1}{(s+a)^2}P_{12}N_{C2}P_{21}D_{11}Q_1 = (I_{n_1}+X\widetilde{U}_1)+XD_{11}Q_1$ is $\mathcal{R}$-unimodular. There exists $Q_1 \in \mathcal{M}(\mathcal{R})$ such that $(I_{n_1}+X\widetilde{U}_1)+XD_{11}Q_1$ is $\mathcal{R}$-unimodular if and only if the "pseudo-plant" $(I_{n_1}+X\widetilde{U}_1)^{-1}XD_{11}$ is strongly stabilizable, equivalently, the pair $((I_{n_1}+X\widetilde{U}_1), XD_{11})$ satisfies the parity-interlacing-property [9]. Therefore, the statement of Theorem 3.5 can be interpreted as, there is a stable stabilizing controller for this "pseudo-plant" if and only if (11) holds.

For square channels ($n_1 = m_1$, $n_2 = m_2$), (10) is equivalent to $(I_{n_1} - \frac{1}{(s+a)^2}P_{12}N_{C2}P_{21}U_1) - \frac{1}{(s+a)^2}P_{12}N_{C2}P_{21}\frac{s}{s+a}Q_1 = (I_{n_1}+XU_1)+\frac{s}{s+a}XQ_1$ is $\mathcal{R}$-unimodular. There exists $Q_1 \in \mathcal{M}(\mathcal{R})$ such that $(I_{n_1}+XU_1)+\frac{s}{s+a}XQ_1$ is $\mathcal{R}$-unimodular if and only if the "pseudo-plant" $(I_{n_1}+XU_1)^{-1}\frac{s}{s+a}X$ is strongly stabilizable, equivalently, the pair $((I_{n_1}+XU_1), \frac{s}{s+a}X)$ satisfies the parity-interlacing-property. Therefore, Corollary 3.6 can be interpreted as, there is a stable stabilizing controller for this "pseudo-plant" if and only if (12) holds.

5) For the case of square channels ($n_1 = m_1$, $n_2 = m_2$), $\frac{s}{s+a}X$ has a blocking $\mathcal{U}$-zero at $s = 0$. But if $m_1 > n_1$, then $D_{11}(0) \neq 0$ although $\det D_{11}(0) = 0$. Therefore, XD_{11} may or may not have a blocking $\mathcal{U}$-zero at $s = 0$.

6) Consider the case of square channels ($n_1 = m_1$, $n_2 = m_2$). If $P(0)$ satisfies the conditions of Corollary 3.6, then a decentralized reliable controller pair with integral action can be designed as follows: Choose any $Q_2 \in \mathcal{M}(\mathcal{R})$; C_2 is given by (9), where the numerator is now fixed as $N_{C2} = (U_2 + \frac{s}{s+a}Q_2)$. The $\mathcal{R}$-stable matrix Q_1 should be constructed as any stable stabilizing controller for the "pseudo-plant" $(I_{n_1}+XU_1)^{-1}\frac{s}{s+a}X$. For the scalar case, the interpolation method in [9, section 3.3] can be used to obtain all stable stabilizing controllers.

The description of Q_1 as any stable stabilizing controller for $(I_{n_1}+XU_1)^{-1}\frac{s}{s+a}X$ is not explicit. Although interpolation gives the parametrization of the entire set for the scalar case, the method does not extend to the MIMO case. We give an alternate method of explicitly finding a sub-class of stable stabilizing controllers for the MIMO "pseudo-plant" $(I_{n_1} + XU_1)^{-1} \frac{s}{s+a} X$ using the binomial expansion. This method applies only to those P which have lower or upper block-triangular structure at $s = 0$. We explain the case $P_{12}(0) = 0$; the case $P_{21}(0) = 0$ is entirely similar: For $j = 1, 2$, let rank $P_{jj}(0) = n_j$ (recall that this condition is necessary). Suppose that $P_{12} = \frac{s}{s+a} R_{12}$ for some $R_{12} \in \mathcal{M}(\mathcal{R})$. Defining $X =: \frac{s}{s+a} \hat{X}$, $(I_{n_1} - \frac{1}{(s+a)^2} P_{12} N_{C2} P_{21} U_1) - \frac{1}{(s+a)^2} P_{12} N_{C2} P_{21} \frac{s}{s+a} Q_1 = (I_{n_1} + XU_1) + \frac{s}{s+a} X Q_1$ becomes $I_{n_1} - \frac{1}{(s+a)^2} \frac{s}{s+a} R_{12} N_{C2} P_{21} U_1 - \frac{1}{(s+a)^2} P_{12} N_{C2} \frac{s}{s+a} R_{12} \frac{s}{s+a} Q_1 = I_{n_1} + \frac{s}{s+a} \hat{X}(U_1 + \frac{s}{s+a} Q_1)$ is $\mathcal{R}$-unimodular. Choose Q_1 as

$$Q_1 := U_1 \hat{X} \sum_{m=2}^{k} r_m \, k^{-m} \, (\tfrac{s}{s+a} \, U_1 \, \hat{X} \,)^{m-2} U_1, \quad (13)$$

where k is any integer such that $k > \| \frac{s}{s+a} \hat{X} U_1 \|$ and r_m are the binomial coefficients. Then $I_{n_1} + \frac{s}{s+a} \hat{X}(U_1 + \frac{s}{s+a} Q_1) = (I + \frac{1}{k} \frac{s}{s+a} \hat{X} U_1)^k$ is $\mathcal{R}$-unimodular. Therefore, using any arbitrary $Q_2 \in \mathcal{M}(\mathcal{R})$ and then Q_1 of (13) in the expression (9) for C_j, we constructed a decentralized reliable controller pair $(\frac{1}{s} C_1, \frac{1}{s} C_2)$.

7) Suppose that there exists a decentralized reliable controller pair with integral action, equivalently, the conditions of Corollary 3.6 hold. One of the controllers, say C_2, in $(\frac{1}{s} C_1, \frac{1}{s} C_2)$ can always be designed to be $\mathcal{R}$-stable. Since the only $\mathcal{U}$-pole of $\frac{1}{s} P_{22}$ is at $s = 0$, the pair $((\frac{s}{s+a} I_{n_2}), \frac{1}{s+a} P_{22})$ satisfies the parity-interlacing-property and hence, $\frac{1}{s+a} P_{22} (\frac{s}{s+a} I_{n_2})^{-1} = \frac{1}{s} P_{22}$ is strongly stabilizable. Therefore there exists Q_2 such that the denominator D_{C2} in (9) is $\mathcal{R}$-unimodular and hence, C_2 is $\mathcal{R}$-stable for such Q_2. Now the other controller C_1 may or may not be $\mathcal{R}$-stable; there exists an $\mathcal{R}$-stable C_1 if and only if there exists a Q_1 such that (10) is $\mathcal{R}$-unimodular and $(\widetilde{V}_1 - \frac{s}{s+a} P_{11} Q_1)$ is also $\mathcal{R}$-unimodular.

8) Consider the case of square channels ($n_1 = m_1$, $n_2 = m_2$). The characterization of C_j given by (9) relies on the solution of the Bezout identity (6). For the special case of scalar channels, one solution for $V_j, U_j, \widetilde{V}_j \in \mathcal{R}$ of (6) can be obtained explicitly as follows: Consider the numerator and the denominator polynomials of $P_{jj} \in \mathcal{R}$, i.e., $P_{jj} = \frac{x(s)}{y(s)}$; the degree of $x(s)$ does not exceed the degree of $y(s)$. Let $U_j = a \, P_{jj}{}^{-1}(0) = a \frac{y(0)}{x(0)}$, $V_j = \widetilde{V}_j = \frac{\gamma(s)}{y(s)}$, where $\gamma(s)$ is the polynomial (of the same degree as $y(s)$) given by $s\,\gamma(s) = (s + a)\, y(s) - a \, P_{jj}{}^{-1}(0)\, x(s)$. It follows by simple calculation that this solution for the scalar $U_j, V_j = \widetilde{V}_j \in \mathcal{R}$ satisfies (6). This simple method of finding a solution for the Bezout identity (6) can be extended to the case of MIMO square channels by applying the same method to the largest invariant-factor of P_{jj} using the Smith-form [9]: Consider the Smith-form $P_{jj} = L_j \Lambda_j R_j$ of $P_{jj} \in \mathcal{R}^{n_j \times n_j}$, where L_j and R_j are $\mathcal{R}$-unimodular and the diagonal matrix $\Lambda_j(0)$ is nonsingular since we assume that $P_{jj}(0)$ is nonsingular. Let λ_{n_j} denote the largest invariant-factor of Λ_j and consider the polynomial factorization of $\lambda_{n_j} =: \frac{x(s)}{y(s)}$. Let $u := a \lambda_{n_j}^{-1}(0)$ and $v = \frac{\gamma(s)}{y(s)}$, where the polynomial $\gamma(s)$ of the same order as $y(s)$ is given by $s\,\gamma(s) = (s + a)\, y(s) - a\, \lambda_{n_j}^{-1}(0)\, x(s)$. Since $\lambda_{n_j} \Lambda_j^{-1}$ is $\mathcal{R}$-stable, a solution for (6) is $U_j = u \, \lambda_{n_j} R_j^{-1} \Lambda_j^{-1} L_j^{-1} = u \, \lambda_{n_j} P_{jj}{}^{-1}$, $V_j = \widetilde{V}_j = v \, I_{n_j}$.

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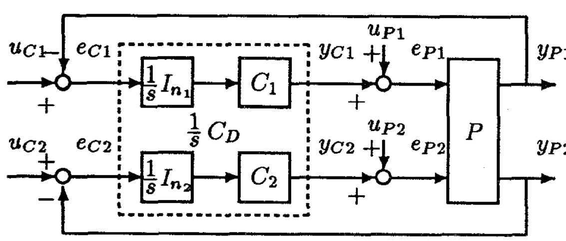

Figure 1: The decentralized system $\mathcal{S}(P, \frac{1}{s} C_D)$