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Reliable stabilization of linear plants using a two-controller configuration

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Abstract: A reliable controller design method is developed for linear, time-invariant, multi-input multi-output control systems; two controllers are designed to stabilize the closed-loop system when acting together and acting independently if one fails. All reliable controllers which achieve closed-loop stability are characterized for strongly stabilizable plants using a factorization approach.

Keywords: Reliable control systems; reliable stabilization; controller design; strong stabilization.

1. Introduction

Reliable stabilization using the linear, time-invariant (LTI), multi-input, multi-output (MIMO) two-controller system configuration $\mathscr{S}(P, C_1, C_2)$ (Figure 1) is considered in this paper, where P is the given plant and C_1, C_2 are the two controllers. The reliable stabilization problem aims to find (if it exists) a reliable controller pair (C_1, C_2) such that the system $\mathscr{S}(P, C_1, C_2)$ is stable when both controllers are acting together (normal mode) and when each controller is acting alone (failure mode). The failure of a controller is modeled by setting its transfer function equal to zero.

A multi-controller system configuration that achieves reliable stabilization was introduced in [4, 5]. Factorization methods were used to study the reliable stability of this configuration in [8, 10, 3]. Reliable stabilization using a two-channel decentralized control system was considered in [6], eliminating the sharing of input and output channels. A methodology for the design of reliable control systems guaranteeing stability and H_∞ disturbance attenuation was developed recently in [7]. Sufficient conditions for reliable stability, with the two-controller configuration used here, were given in [10]. In [3], it was stated that a given plant can be reliably stabilized using this two-controller configuration if and only if it is strongly stabilizable (i.e., it can be stabilized using a stable controller) in the standard unity-feedback system. In this paper, all reliable controller pairs (C_1, C_2) are characterized and a design method to achieve reliable controller pairs is developed for strongly stabilizable plants. It is shown that one of the two controllers can be an arbitrary strongly stabilizing controller. A simple example is given to illustrate the method and to show that neither one of the controllers has to be stable in order to achieve reliable stabilization.

The results apply to continuous-time as well as discrete-time systems.

Notation. Let $\mathscr{U}$ be a subset of the field $\mathbb{C}$ of complex numbers, $\mathscr{U}$ is closed and symmetric about the real axis, $\pm\infty \in \mathscr{U}$, $\mathbb{C} \backslash \mathscr{U}$ is nonempty. Let $\mathscr{R}_{\mathscr{U}}$, $\mathbb{R}_{\mathrm{p}}(s)$ and $\mathbb{R}_{\mathrm{sp}}(s)$ be the ring of proper rational functions which have no

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poles in $\mathscr{U}$, the ring of proper rational functions and the set of strictly proper rational functions of s (with real coefficients). The group of units of $\mathscr{R}_{\mathscr{U}}$ is $\mathscr{J}$ and the set of non-strictly proper elements of $\mathscr{R}_{\mathscr{U}}$ is $\mathscr{I} = \mathscr{R}_{\mathscr{U}} \backslash \mathbb{R}_{sp}(s)$. The set of matrices with entries in $\mathscr{R}_{\mathscr{U}}$ is denoted $\mathscr{M}(\mathscr{R}_{\mathscr{U}})$. A matrix M is called $\mathscr{R}_{\mathscr{U}}$-stable iff $M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$; $M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is $\mathscr{R}_{\mathscr{U}}$-unimodular iff det $M \in \mathscr{J}$, where det denotes the determinant. The identity matrix of size n is denoted I_n. Let the norm of an $\mathscr{R}_{\mathscr{U}}$-stable matrix $M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ be defined as $\| M \| = \sup_{s \in \partial\mathscr{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ denotes the maximum singular value and $\partial\mathscr{U}$ denotes the boundary of $\mathscr{U}$. Let (N_P, D_P) denote a right-coprime-factorization (RCF) and $(\tilde{D}_P, \tilde{N}_P)$ denote a left-coprime-factorization (LCF) of $P \in \mathbb{R}_p(s)^{no \times ni}$, where N_P, $\tilde{N}_P \in \mathscr{R}_{\mathscr{U}}{}^{no \times ni}$, $D_P \in \mathscr{R}_{\mathscr{U}}{}^{ni \times ni}$, $\tilde{D}_P \in \mathscr{R}_{\mathscr{U}}{}^{no \times no}$, $P = N_P D_P^{-1} = \tilde{D}_P^{-1} \tilde{N}_P$; det $D_P \in \mathscr{I}$ (equivalently, det $\tilde{D}_P \in \mathscr{I}$) if and only if $P \in \mathscr{M}(\mathbb{R}_p(s))$. Similarly, (N_C, D_C) denotes an RCF and $(\tilde{D}_C, \tilde{N}_C)$ denotes an LCF of $C \in \mathbb{R}_p(s)^{ni \times no}$. := is used for 'defined as'; i.e., $a := b$ (or $b =: a$) means a is defined as b.

2. Problem formulation and preliminaries

The LTI, MIMO two-controller feedback configuration $\mathscr{S}(P, C_1, C_2)$, which is shown in Figure 1, is used for the reliable stabilization problem considered here. In this system, $P: e_P \mapsto y_P$ represents the transfer function of the plant, $C_1: e_{C_1} \mapsto y_{C_1}$ and $C_2: e_{C_2} \mapsto y_{C_2}$ represent the transfer functions of the two controllers; the input vector is $u := [u_P^{\mathrm{T}}\ u_{C_1}^{\mathrm{T}}\ u_{C_2}^{\mathrm{T}}]^{\mathrm{T}}$, the output vector is $y := [y_P^{\mathrm{T}}\ y_{C_1}^{\mathrm{T}}\ y_{C_2}^{\mathrm{T}}]^{\mathrm{T}}$ and the closed-loop transfer function is $H_{yu}(P, C_1, C_2): u \mapsto y$.

If one of the two controllers, say $C_2 = 0$, the system $\mathscr{S}(P, C_1, C_2)$ becomes the standard unity-feedback system called $\mathscr{S}(P, C)$, in which case the input vector, the output vector and the closed-loop transfer function would be $u := [u_P^{\mathrm{T}}\ u_C^{\mathrm{T}}]^{\mathrm{T}}$, $y := [y_P^{\mathrm{T}}\ y_C^{\mathrm{T}}]^{\mathrm{T}}$, $H_{yu}(P, C): u \mapsto y$.

Assumption 2.1. (i) The plant $P \in \mathbb{R}_p(s)^{no \times ni}$.

(ii) The controllers $C_1, C_2 \in \mathbb{R}_p(s)^{ni \times no}$.

(iii) The system $\mathscr{S}(P, C_1, C_2)$ is well-posed; equivalently, the closed-loop transfer function $H_{yu}(P, C_1, C_2) \in \mathscr{M}(\mathbb{R}_p(s))$.

(iv) P, C_1, C_2 have no hidden-$\mathscr{U}$-modes.

The system $\mathscr{S}(P, C_1, C_2)$ in Figure 1 can be described using coprime factorizations as follows: let (N_P, D_P) be any RCF of P. For $j = 1, 2$, let $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ be any LCF of C_j. Using $D_P \xi_P = e_P$, $N_P \xi_P = y_P$, $\tilde{D}_{C_j} y_{C_j} = \tilde{N}_{C_j} e_{C_j}$, the following description of $\mathscr{S}(P, C_1, C_2)$ is obtained:

$$\begin{bmatrix} \tilde{D}_{C_1} D_P + \tilde{N}_{C_1} N_P & -\tilde{D}_{C_1} \\ \tilde{N}_{C_2} N_P & \tilde{D}_{C_2} \end{bmatrix} \begin{bmatrix} \xi_P \\ y_{C_2} \end{bmatrix} = \begin{bmatrix} \tilde{D}_{C_1} & \tilde{N}_{C_1} & 0 \\ 0 & 0 & \tilde{N}_{C_2} \end{bmatrix} \begin{bmatrix} u_P \\ u_{C_1} \\ u_{C_2} \end{bmatrix}, \tag{2.1}$$

$$\begin{bmatrix} N_P & 0 \\ D_P & -I_{ni} \\ 0 & I_{ni} \end{bmatrix} \begin{bmatrix} \xi_P \\ y_{C_2} \end{bmatrix} = \begin{bmatrix} y_P \\ y_{C_1} \\ y_{C_2} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -I_{ni} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_P \\ u_{C_1} \\ u_{C_2} \end{bmatrix}. \tag{2.2}$$

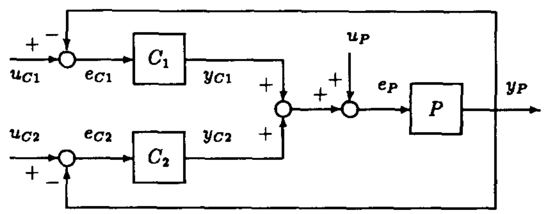

Fig. 1. The system $\mathscr{S}(P, C_1, C_2)$.

Similarly, let $(\tilde{D}_P, \tilde{N}_P)$ be any LCF of P. For $j = 1, 2$, let (N_{C_j}, D_{C_j}) be any RCF of C_j. Using $D_{C_j}\xi_{C_j} = e_{C_j}$, $N_{C_j}\xi_{C_j} = y_{C_j}$, $\tilde{D}_P y_P = \tilde{N}_P e_P$, a dual description of $\mathscr{S}(P, C_1, C_2)$ is obtained

$$\begin{bmatrix} \tilde{D}_P D_{C_1} + \tilde{N}_P N_{C_1} & \tilde{N}_P N_{C_2} \\ -D_{C_1} & D_{C_2} \end{bmatrix} \begin{bmatrix} \xi_{C_1} \\ \xi_{C_2} \end{bmatrix} = \begin{bmatrix} -\tilde{N}_P & \tilde{D}_P & 0 \\ 0 & -I_{no} & I_{no} \end{bmatrix} \begin{bmatrix} u_P \\ u_{C_1} \\ u_{C_2} \end{bmatrix}, \tag{2.3}$$

$$\begin{bmatrix} -D_{C_1} & 0 \\ N_{C_1} & 0 \\ 0 & N_{C_2} \end{bmatrix} \begin{bmatrix} \xi_{C_1} \\ \xi_{C_2} \end{bmatrix} = \begin{bmatrix} y_P \\ y_{C_1} \\ y_{C_2} \end{bmatrix} - \begin{bmatrix} 0 & I_{no} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_P \\ u_{C_1} \\ u_{C_2} \end{bmatrix}. \tag{2.4}$$

Definition 2.2. (a) The system $\mathscr{S}(P, C_1, C_2)$ is said to be *$\mathscr{R}_{\mathscr{U}}$-stable* iff the closed-loop transfer function $H_{yu}(P, C_1, C_2) \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. Similarly, $\mathscr{S}(P, C)$ is $\mathscr{R}_{\mathscr{U}}$-stable iff $H_{yu}(P, C) \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$.

(b) The controller C is an *$\mathscr{R}_{\mathscr{U}}$-stabilizing controller* for P in the system $\mathscr{S}(P, C)$ iff $C \in \mathbb{R}_p(s)^{ni \times no}$ and $\mathscr{S}(P, C)$ is $\mathscr{R}_{\mathscr{U}}$-stable. The set $S(P) := \{C \mid C \in \mathbb{R}_p(s)^{ni \times no}$ and $\mathscr{S}(P, C)$ is $\mathscr{R}_{\mathscr{U}}$-stable$\}$ is called the *set of all $\mathscr{R}_{\mathscr{U}}$-stabilizing controllers* for P in the system $\mathscr{S}(P, C)$.

(c) The pair (C_1, C_2) is called a *reliable controller pair* for P in the system $\mathscr{S}(P, C_1, C_2)$ iff

(i) C_1 is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P (in the system $\mathscr{S}(P, C)$),

(ii) C_2 is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P (in the system $\mathscr{S}(P, C)$) and

(iii) $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable.

The set $S_2(P) := \{(C_1, C_2) \mid C_j$ is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for $P, j = 1, 2$, and $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable$\}$ is called the *set of all reliable controller pairs* for P in the system $\mathscr{S}(P, C_1, C_2)$. □

3. Conditions for reliable stability

The *reliable stabilization problem* considered here deals with finding two proper controllers such that the system $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable when the two controllers are acting together and also when one of the controllers becomes zero. By Definition 2.2, this reliable stabilization problem can be solved if and only if there exists a reliable controller pair (C_1, C_2) for the given plant P.

Lemma 3.1 ($\mathscr{R}_{\mathscr{U}}$-stability of the system $\mathscr{S}(P, C_1, C_2)$). *Let (N_P, D_P) be any* RCF *and $(\tilde{D}_P, \tilde{N}_P)$ be any* LCF *of P. For $j = 1, 2$, let (N_{C_j}, D_{C_j}) be any* RCF *and $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ be any* LCF *of C_j. Then $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if*

$$D_H := \begin{bmatrix} \tilde{D}_{C_1} D_P + \tilde{N}_{C_1} N_P & -\tilde{D}_{C_1} \\ \tilde{N}_{C_2} N_P & \tilde{D}_{C_2} \end{bmatrix} \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular;} \tag{3.5}$$

equivalently,

$$\tilde{D}_H := \begin{bmatrix} \tilde{D}_P D_{C_1} + \tilde{N}_P N_{C_1} & \tilde{N}_P N_{C_2} \\ -D_{C_1} & D_{C_2} \end{bmatrix} \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular.} \quad \square \tag{3.6}$$

Proof. The description of $\mathscr{S}(P, C_1, C_2)$ given by (2.1), (2.2) is of the form $D_H \xi = N_L u$, $N_R \xi = y + G_H u$, where D_H, N_L, N_R, G_H are $\mathscr{R}_{\mathscr{U}}$-stable. Since (D_H, N_L) is a left-coprime and (N_R, D_H) is a right-coprime, $H_{yu}(P, C_1, C_2) = N_R D_H^{-1} N_L + G_H \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ if and only if $D_H^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$; therefore, $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if D_H is $\mathscr{R}_{\mathscr{U}}$-unimodular. The $\mathscr{R}_{\mathscr{U}}$-unimodularity condition on $\tilde{D}_H$ follows similarly from the alternate description of $\mathscr{S}(P, C_1, C_2)$ in (2.3), (2.4). □

Corollary 3.2. *Under the assumptions of Lemma* 3.1, (C_1, C_2) *is a reliable controller pair if and only if for* $j = 1, 2$,

$$\tilde{D}_{C_j} D_P + \tilde{N}_{C_j} N_P \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular} \tag{3.7}$$

and the $\mathscr{R}_{\mathscr{U}}$*-unimodularity condition* (3.5) *holds for the matrix* D_H; *equivalently,*

$$\tilde{D}_P D_{C_j} + \tilde{N}_P N_{C_j} \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular} \tag{3.8}$$

and the $\mathscr{R}_{\mathscr{U}}$*-unimodularity condition* (3.6) *holds for the matrix* $\tilde{D}_H$.

Proof. C_j is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P if and only if (3.7) holds for any LCF $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ of C_j [8, 2]. By Lemma 3.1, $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if D_H in (3.5) is $\mathscr{R}_{\mathscr{U}}$-unimodular for any LCF $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ of C_j. The equivalent conditions on the RCF of C_j follow similarly. □

Lemma 3.3 (Conditions for reliable controller pairs). *Let* (N_P, D_P) *by any* RCF *and* $(\tilde{D}_P, \tilde{N}_P)$ *be any* LCF *of* P. *Then* (C_1, C_2) *is a reliable controller pair if and only if for some* LCF $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ *of* C_j, $j = 1, 2$,

$$\tilde{D}_{C_j} D_P + \tilde{N}_{C_j} N_P = I_{ni} \tag{3.9}$$

and

$$\tilde{D}_{C_2} + \tilde{N}_{C_2} N_P \tilde{D}_{C_1} = \tilde{D}_{C_2}(I_{ni} - D_P \tilde{D}_{C_1}) + \tilde{D}_{C_1} \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular.} \tag{3.10}$$

Equivalently, (C_1, C_2) *is a reliable controller pair if and only if for some* RCF (N_{C_j}, D_{C_j}) *of* C_j,

$$\tilde{D}_P D_{C_j} + \tilde{N}_P N_{C_j} = I_{no} \tag{3.11}$$

and

$$D_{C_2} + D_{C_1} \tilde{N}_P N_{C_2} = (I_{ni} - D_{C_1} \tilde{D}_P) D_{C_2} + D_{C_1} \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular.} \tag{3.12}$$

Proof. C_j is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P if and only if (3.9) holds for some LCF $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ of C_j [8, 2]. By Lemma 3.1, $\mathscr{S}(P, C_1, C_2)$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if D_H in (3.5) is $\mathscr{R}_{\mathscr{U}}$-unimodular for any LCF $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ of C_j. Hence, for the LCF satisfying (3.9), by elementary-row operations (over $\mathscr{R}_{\mathscr{U}}$), D_H is $\mathscr{R}_{\mathscr{U}}$-unimodular if and only if (3.10) holds, where the second equality follows from (3.9) since $\tilde{N}_{C_j} N_P = I_{ni} - \tilde{D}_{C_j} D_P$. Conditions (3.11)–(3.12) on the RCF of C_j are obtained similarly. □

4. Reliable controller pair design

The plant P is said to be strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable if there is an $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller $C \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ for P (in the system $\mathscr{S}(P, C)$). If $\mathscr{U} = \mathbb{C}_+$, P is strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable if and only if it satisfies the *parity interlacing property*, i.e., P has an even number of poles between pairs of blocking zeros on the positive real axis [8, 9]. From a coprime factorizations viewpoint, P is strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable if and only if, for any RCF (N_P, D_P) of P, there exists an $\mathscr{R}_{\mathscr{U}}$-unimodular $\tilde{D}_C$ such that $\tilde{D}_C D_P + \tilde{N}_C N_P$ is $\mathscr{R}_{\mathscr{U}}$-unimodular for some $\mathscr{R}_{\mathscr{U}}$-stable $\tilde{N}_C$. Therefore, C is an $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P if and only if $\tilde{D}_C D_P + \tilde{N}_C N_P$ is $\mathscr{R}_{\mathscr{U}}$-unimodular for any RCF (N_P, D_P) of P and any LCF $(\tilde{D}_C, \tilde{N}_C)$ of C, where $\tilde{D}_C$ is $\mathscr{R}_{\mathscr{U}}$-unimodular [8].

For P strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable, $C_S \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is an $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller if and only if there exists an RCF (N_P, D_P) of P such that

$$D_P + C_S N_P = I_{ni}; \tag{4.13}$$

equivalently,

$$\begin{bmatrix} I_{ni} & C_S \\ -N_P & I_{no} - N_P C_S \end{bmatrix} \begin{bmatrix} D_P & -C_S \\ N_P & I_{no} \end{bmatrix} = I_{ni+no}. \tag{4.14}$$

Therefore, if P is strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable, then it has an RCF (N_P, D_P) and an LCF $(\tilde{D}_P, \tilde{N}_P)$ with $N_P = \tilde{N}_P$; from (4.14), $(\tilde{D}_P, \tilde{N}_P) = ((I_{no} - N_P C_S), N_P)$, where $C_S \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P. Now the set $S(P)$ of all $\mathscr{R}_{\mathscr{U}}$-stabilizing controllers for a strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable plant P can be described in terms of any $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller C_S as follows: Let (N_P, D_P) be the RCF of P which satisfies (4.13). Then

$$S(P) = \{C = C_S + (I_{ni} - QN_P)^{-1}Q \mid Q \in \mathscr{R}_{\mathscr{U}}^{ni \times no}, \det(I_{ni} - QN_P) \in \mathscr{I}\}. \tag{4.15}$$

The requirement that $Q \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ satisfies $\det(I_{ni} - QN_P) \in \mathscr{I}$ is equivalent to C being proper. If P is strictly proper, then $\det(I_{ni} - QN_P) \in \mathscr{I}$ for *all* $Q \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$.

The following condition for the existence of a reliable controller pair was stated in [3]. The assumption that the plant is strictly proper is a technicality which is used here only in the sufficiency proof to ensure that the second controller is proper; to illustrate that this condition is not necessary for the existence of reliable controller pairs, the plant in Example 4.7 is chosen non-strictly proper.

Theorem 4.1 (Existence of reliable controller pairs (*Minto and Ravi* [3])). *Let $P \in \mathbb{R}_{sp}(s)^{ni \times no}$. There exists a reliable controller pair (C_1, C_2) for P if and only if P is strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable.*

Proof. *Necessity:* Let (C_1, C_2) be a reliable controller pair, then by Lemma 3.3, $\tilde{D}_{C_2} + \tilde{N}_{C_2} N_P \tilde{D}_{C_1} =: M$ is $\mathscr{R}_{\mathscr{U}}$-unimodular, where $(\tilde{D}_{C_j}, \tilde{N}_{C_j})$ is a LCF of C_j satisfying (3.9). Using $\tilde{D}_{C_2} = M - \tilde{N}_{C_2} N_P \tilde{D}_{C_1}$, for $j = 2$, (3.9) becomes $(M - \tilde{N}_{C_2} N_P \tilde{D}_{C_1}) D_P + \tilde{N}_{C_2} N_P = M D_P + \tilde{N}_{C_2} N_P (I_{ni} - \tilde{D}_{C_1} D_P) = I_{ni}$. But by (3.9) for $j = 1$, this last equality is $M D_P + (\tilde{N}_{C_2} N_P \tilde{N}_{C_1}) N_P = I_{ni}$, where M is $\mathscr{R}_{\mathscr{U}}$-unimodular, which implies that P is strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable since $M^{-1}(\tilde{N}_{C_2} N_P \tilde{N}_{C_1}) \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P.

Sufficiency: By (4.15), $C_j = C_S + (I_{ni} - Q_j N_P)^{-1} Q_j$ is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P. Since $P \in \mathscr{M}(\mathbb{R}_{sp}(s))$, for any $Q_j \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, the controller C_j is proper. Let k be an integer larger than $\| C_S N_P \|$, then $(I_{ni} + C_S N_P / k)^k$ is $\mathscr{R}_{\mathscr{U}}$-unimodular. By the binomial expansion [10]

$$(I_{ni} + C_S N_P / k)^k = I_{ni} + C_S N_P + \sum_{l=2}^{k} r_l (C_S N_P)^l, \tag{4.16}$$

where r_l are the binomial coefficients. Let

$$Q_2 = -C_S \sum_{l=2}^{k} r_l (N_P C_S)^{l-2}; \tag{4.17}$$

let $C_1 = C_S$, $C_2 = C_S + (I_{ni} - Q_2 N_P)^{-1} Q_2$. Then (3.10) holds since $I_{ni} + (I_{ni} - Q_2 N_P) C_S N_P = (I_{ni} + C_S N_P / k)^k$ is $\mathscr{R}_{\mathscr{U}}$-unimodular. Therefore, (C_1, C_2) is a reliable controller pair. □

Since this condition is necessary for the existence of a reliable controller pair (C_1, C_2), from here on, it is assumed that P is strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable.

Theorem 4.2. (All reliable controller pairs). *Let $P \in \mathbb{R}_p(s)^{no \times ni}$ be strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable. Let $C_S \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ be any $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P. Let (N_P, D_P) be the* RCF *of P satisfying* (4.13). *Then the set $S_2(P)$ of all reliable controller pairs (C_1, C_2) is:*

$$\begin{aligned} S_2(P) = \{(C_1, C_2) \mid\ & C_1 = C_S + (I_{ni} - Q_1 N_P)^{-1} Q_1,\ C_2 = C_S + (I_{ni} - Q_2 N_P)^{-1} Q_2, \\ & Q_1, Q_2 \in \mathscr{R}_{\mathscr{U}}^{ni \times no},\ I_{ni} + (I_{ni} - Q_2 N_P) C_S N_P - [(I_{ni} - Q_2 N_P) C_S + Q_2] N_P Q_1 N_P \\ & \textit{is } \mathscr{R}_{\mathscr{U}}\textit{-unimodular, and } \det(I_{ni} - Q_1 N_P) \in \mathscr{I}, \det(I_{ni} - Q_2 N_P) \in \mathscr{I}\}. \end{aligned} \tag{4.18}$$

If P is strictly proper, then $\det(I_{ni} - Q_2 N_P) \in \mathscr{I}$ *for all $Q_j \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, $j = 1, 2$. Furthermore, the controller C_j of the reliable controller pair $(C_1, C_2) \in S_2(P)$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if, in addition to satisfying the $\mathscr{R}_{\mathscr{U}}$-unimodularity condition in* (4.18), *$Q_j \in \mathscr{R}_{\mathscr{U}}^{ni \times no}$ is such that*

$$(I_{ni} - Q_j N_P) \textit{ is } \mathscr{R}_{\mathscr{U}}\textit{-unimodular, equivalently, } \det(I_{ni} - Q_j N_P) \in \mathscr{I}. \tag{4.19}$$

Proof. By (4.15), for C_j is an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P if and only if $C_j = C_S + (I_{ni} - Q_jN_P)^{-1}Q_j$, $Q\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$ satisfies $\det(I_{ni} - Q_jN_P)\in\mathscr{I}$ so that $C_j\in\mathscr{M}(\mathbb{R}_p(s))$. Then

$$(\tilde{D}_{C_j}, \tilde{N}_{C_j}) = ((I_{ni} - Q_jN_P), (I_{ni} - Q_jN_P)C_S + Q_j) \tag{4.20}$$

is an LCF of C_j. Since this LCF satisfies (3.9), by Lemma 3.3, (C_1, C_2) is a reliable controller pair if and only if for $j = 1, 2$, $Q_j\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is chosen such that (3.10) is satisfied; equivalently, $\tilde{D}_{C_2} + \tilde{N}_{C_2}N_P\tilde{D}_{C_1} = (I_{ni} - Q_2N_P) + [(I_{ni} - Q_2N_P)C_S + Q_2]N_P(I_{ni} - Q_1N_P) = I_{ni} + (I_{ni} - Q_2N_P)C_SN_P - [(I_{ni} - Q_2N_P)C_S + Q_2]N_PQ_1N_P$ is $\mathscr{R}_{\mathscr{U}}$-unimodular. Furthermore, $C_j = \tilde{D}_{C_j}^{-1}\tilde{N}_{C_j}$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if $\tilde{D}_{C_j}$ is $\mathscr{R}_{\mathscr{U}}$-unimodular, equivalently (4.19) holds. □

Corollary 4.3 (Special cases for reliable controller pairs). *Let $P\in\mathbb{R}_p(s)^{no\times ni}$ be strongly $\mathscr{R}_{\mathscr{U}}$-stabilizable.*

(i) *Suppose that one of the two controllers in the reliable controller pair (C_1, C_2) is $\mathscr{R}_{\mathscr{U}}$-stable. Without loss of generality, let $C_1 := C_S\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$ be an $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P. Let (N_P, D_P) be the* RCF *of P which satisfies* (4.13). *Then (C_S, C_2) is a reliable controller pair if and only if $C_2 = C_S + (I_{ni} - Q_2N_P)^{-1}Q_2$, where $Q_2\in\mathscr{R}_{\mathscr{U}}^{ni\times no}$ is such that*

$$I_{ni} + (I_{ni} - Q_2N_P)C_SN_P \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular and } \det(I_{ni} - Q_2N_P)\in\mathscr{I}. \tag{4.21}$$

(ii) *(C_1, C_2) is a reliable controller pair with $C_1 = C_2$ if and only if $C_1 = C_2$ is $\mathscr{R}_{\mathscr{U}}$-stable and both C_1 and $2C_1$ are $\mathscr{R}_{\mathscr{U}}$-stabilizing controllers for P.*

Proof. (i) If $C_1\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$, then the set $\boldsymbol{S}(P)$ can be characterized in terms of $C_S := C_1$ and (4.21) follows from (4.18) by taking $Q_1 = 0$.

(ii) By Theorem 4.2, (C_1, C_2) is a reliable controller pair if and only if it belongs to $\boldsymbol{S}_2(P)$ in (4.18). In addition, $C_1 = C_S + (I_{ni} - Q_1N_P)^{-1}Q_1 = C_2 = C_S + (I_{ni} - Q_2N_P)^{-1}Q_2$ if and only if $Q_1 = Q_2$. Using the LCF of C_j in (4.20), the $\mathscr{R}_{\mathscr{U}}$-unimodularity condition in (4.18) holds if and only if $\tilde{D}_{C_2} + \tilde{N}_{C_2}N_P\tilde{D}_{C_1}$ is $\mathscr{R}_{\mathscr{U}}$-unimodular, with $\tilde{D}_{C_2} = \tilde{D}_{C_1}$ since $C_1 = C_2$; equivalently, $(I_{ni} + \tilde{N}_{C_2}N_P)\tilde{D}_{C_2}$ is $\mathscr{R}_{\mathscr{U}}$-unimodular, which is satisfied if and only if both of the matrices in the product are $\mathscr{R}_{\mathscr{U}}$-unimodular. Now $\tilde{D}_{C_2}$ is $\mathscr{R}_{\mathscr{U}}$-unimodular if and only if C_2 is $\mathscr{R}_{\mathscr{U}}$-stable; using (3.9), $(I_{ni} + \tilde{N}_{C_2}N_P) = \tilde{D}_{C_2}D_P + 2\tilde{N}_{C_2}N_P$ is $\mathscr{R}_{\mathscr{U}}$-unimodular if and only if $2C_2$ is also an $\mathscr{R}_{\mathscr{U}}$-stabilizing controller for P. □

Corollary 4.4 (All reliable controller pairs for $\mathscr{R}_{\mathscr{U}}$-stable plants). *Let $P\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$.*

(i) *The set $\boldsymbol{S}_2(P)$ of all reliable controller pairs (C_1, C_2) is*

$$\begin{aligned}\boldsymbol{S}_2(P) = \{&(C_1, C_2) \mid C_1 = (I_{ni} - Q_1P)^{-1}Q_1,\ C_2 = (I_{ni} - Q_2P)^{-1}Q_2,\ Q_1, Q_2\in\mathscr{R}_{\mathscr{U}}^{ni\times no},\\ &I_{ni} - Q_2PQ_1P \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular and } \det(I_{ni} - Q_1P)\in\mathscr{I},\ \det(I_{ni} - Q_2P)\in\mathscr{I}\}.\end{aligned} \tag{4.22}$$

If P is strictly proper, then $\det(I_{ni} - Q_jP)\in\mathscr{I}$ for all $Q_j\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$, $j = 1, 2$. Furthermore, for $j = 1, 2$, the controller C_j of the reliable controller pair $(C_1, C_2)\in\boldsymbol{S}_2(P)$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if $Q_j\in\mathscr{R}_{\mathscr{U}}^{ni\times no}$ is such that $(I_{ni} - Q_jP)$ is $\mathscr{R}_{\mathscr{U}}$-unimodular, equivalently, $\det(I_{ni} - Q_jN_P)\in\mathscr{I}$.

(ii) *(C_1, C_2) is a reliable controller pair with $C_1 = C_2$ if and only if $C_1 = C_2 = (I_{ni} - Q_1P)^{-1}Q_1$, where $Q_1\in\mathscr{R}_{\mathscr{U}}^{ni\times no}$ is such that $(I_{ni} - Q_1P)$ and $(I_{ni} + Q_1P)$ are both $\mathscr{R}_{\mathscr{U}}$-unimodular.*

Proof. (i) If the plant is $\mathscr{R}_{\mathscr{U}}$-stable, then $C_S = 0$ is an $\mathscr{R}_{\mathscr{U}}$-stable $\mathscr{R}_{\mathscr{U}}$-stabilizing controller. With this choice of C_S, an RCF of $P\in\mathscr{M}(\mathscr{R}_{\mathscr{U}})$ which satisfies (4.13) is given by $(N_P, D_P) = (P, I_{ni})$. It then follows from Theorem 4.2 that (C_1, C_2) is a reliable controller pair if and only if $C_j = (I_{ni} - Q_jP)^{-1}Q_j$, where, for $j = 1, 2$, $Q_j\in\mathscr{R}_{\mathscr{U}}^{ni\times no}$ satisfy

$$I_{ni} - Q_2PQ_1P \text{ is } \mathscr{R}_{\mathscr{U}}\text{-unimodular} \tag{4.23}$$

and

$$\det(I_{ni} - Q_1P)\in\mathscr{I}, \qquad \det(I_{ni} - Q_2P)\in\mathscr{I}. \tag{4.24}$$

(ii) The two controllers of the reliable controller pair (C_1, C_2) are equal if and only if they belong to the set $S_2(P)$ in (4.22) with $Q_1 = Q_2$. Hence (4.23) becomes $(I_{ni} - Q_1PQ_1P) = (I_{ni} - Q_1P)(I_{ni} + Q_1P)$ is $\mathcal{R}_{\mathcal{U}}$-unimodular, equivalently, each matrix in the product is $\mathcal{R}_{\mathcal{U}}$-unimodular. Therefore, (4.24) holds and the two controllers $C_1 = C_2 = (I_{ni} - Q_1P)^{-1}Q_1$ are $\mathcal{R}_{\mathcal{U}}$-stable. □

Algorithm 4.5 (*Reliable controller pair design*). Let $P \in \mathbb{R}_p(s)^{no \times ni}$ be strongly $\mathcal{R}_{\mathcal{U}}$-stabilizable.

(1) Find any $\mathcal{R}_{\mathcal{U}}$-stable $\mathcal{R}_{\mathcal{U}}$-stabilizing controller $C_S \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ for P.

(2) Find the RCF (N_P, D_P) of P which satisfies (4.13).

(3) Choose any $Q_2 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$ such that

$$M_2 := I_{ni} + (I_{ni} - Q_2N_P)C_SN_P \text{ is } \mathcal{R}_{\mathcal{U}}\text{-unimodular and } \det(I_{ni} - Q_2N_P) \in \mathcal{I}. \tag{4.25}$$

If P is strictly proper, then $\det(I_{ni} - Q_2N_P) \in \mathcal{I}$ for *all* $Q_2 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$.

(4) Choose any $Q_1 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$ such that

$$M_1 := I_{ni} - M_2^{-1}[(I_{ni} - Q_2N_P)C_S + Q_2]N_PQ_1N_P = I_{ni} - M_2^{-1}(M_2 - I_{ni} + Q_2N_P)Q_1N_P$$
$$\text{is } \mathcal{R}_{\mathcal{U}}\text{-unimodular and } \det(I_{ni} - Q_1N_P) \in \mathcal{I}. \tag{4.26}$$

If P is strictly proper, then $\det(I_{ni} - Q_1N_P) \in \mathcal{I}$ for all $Q_1 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$.

(5) Let

$$C_1 = C_S + (I_{ni} - Q_1N_P)^{-1}Q_1, \qquad C_2 = C_S + (I_{ni} - Q_2N_P)^{-1}Q_2; \tag{4.27}$$

then (C_1, C_2) is a reliable controller pair.

Remark 4.6. (1) Algorithm 4.5 characterizes a subclass of reliable controller pairs. For any $Q_1, Q_2 \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ satisfying (4.26) and (4.25), respectively, $I_{ni} + (I_{ni} - Q_2N_P)C_SN_P - [(I_{ni} - Q_2N_P)C_S + Q_2]N_PQ_1N_P = M_2M_1$ is $\mathcal{R}_{\mathcal{U}}$-unimodular. Therefore, $\{(C_1, C_2) \mid \text{for } j = 1,2,\ C_j = C_S + (I_{ni} - Q_jN_P)^{-1}Q_j,\ Q_1 \text{ and } Q_2 \text{ satisfy (4.26) and (4.25)}\}$ is a subset of the set $S_2(P)$ of all reliable controller pairs in (4.18). The set of Q_2 satisfying the $\mathcal{R}_{\mathcal{U}}$-unimodularity condition in (4.25) is not empty; one method of choosing Q_2 was given in the proof of Theorem 4.1, i.e., let $Q_2 = -C_S\sum_{l=2}^{k} r_l(N_PC_S)^{l-2}$, whether r_l are the binomial coefficients in the binomial expansion of $(I_{ni} + C_SN_P/k)^k$, with the integer k chosen larger than $\| C_SN_P \|$. An obvious choice for Q_1 to satisfy (4.26) is the zero matrix implying $C_1 = C_S$ is $\mathcal{R}_{\mathcal{U}}$-stable; another way is to choose Q_1 so that $\| Q_1 \| < 1/\| N_PM_2^{-1}(M_2 - I_{ni} - Q_2N_P) \|$. The condition $\det(I_{ni} - Q_jN_P) \in \mathcal{I}$ is necessary and sufficient for C_j to be proper; if P is strictly proper, this condition holds for all $Q_j \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$. Choosing Q_j strictly proper is sufficient to ensure $\det(I_{ni} - Q_jN_P) \in \mathcal{I}$.

(2) Let P be $\mathcal{R}_{\mathcal{U}}$-stable. Following Corollary 4.4, Algorithm 4.5 is modified for $\mathcal{R}_{\mathcal{U}}$-stable plants as follows: (1) Let $C_S = 0$. (2) An RCF of $P \in \mathcal{M}(\mathcal{R}_{\mathcal{U}})$ which satisfies (4.13) is given by $(N_P, D_P) = (P, I_{ni})$. (3) Choose any $Q_2 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$ such that $\det(I_{ni} - Q_2P) \in \mathcal{I}$. If P is strictly proper, then $\det(I_{ni} - Q_2P) \in \mathcal{I}$ for *all* $Q_2 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$. (4) Choose any $Q_1 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$ such that $M_1 = I_{ni} - Q_2PQ_1P$ is $\mathcal{R}_{\mathcal{U}}$-unimodular and $\det(I_{ni} - Q_1P) \in \mathcal{I}$. If P is strictly proper, then $\det(I_{ni} - Q_1P) \in \mathcal{I}$ for *all* $Q_1 \in \mathcal{R}_{\mathcal{U}}^{ni \times no}$. (5) For $j = 1, 2$, let $C_j = (I_{ni} - Q_jP)^{-1}Q_j$; then (C_1, C_2) is a reliable controller pair.

Example 4.7. Let $P = (s+1)/(s-2)$. Let $\mathcal{U}$ be $\mathbb{C}_+$. Clearly, P is strongly $\mathcal{R}_{\mathcal{U}}$-stabilizable since it has no zeros in $\mathcal{U}$. An $\mathcal{R}_{\mathcal{U}}$-stable $\mathcal{R}_{\mathcal{U}}$-stabilizing controller for P is given by $C_S = 4/(s+1)$. The RCF (N_P, D_P) which satisfies (4.13) is $N_P = (s+1)/(s+2)$, $D_P = (s-2)/(s+2)$. Choose Q_2 satisfying (4.25) and Q_1 satisfying (4.26) as $Q_2 = 12(s+2)/(s+1)(s+10)$, $Q_1 = 2(s+2)/(s+20)$. From (4.27), the reliable controller pair corresponding to this choice of Q_1 and Q_2 is $C_1 = -2(s^2+s+38)/(s+1)(s-18)$, $C_2 = 16/(s-2)$. Neither one of the controllers in this reliable controller pair is $\mathcal{R}_{\mathcal{U}}$-stable. Using the same Q_2, an alternate solution is obtained by choosing $Q_1 = 0$. Then with the C_2 given above, (C_S, C_2) is a reliable controller pair.

5. Conclusions

For LTI, MIMO systems, a design method was developed to find two stabilizing controllers C_1 and C_2 such that the closed-loop system $\mathscr{S}(P, C_1, C_2)$ is stable both when the two controllers act together and when either C_1 or C_2 fails. All reliable controller pairs (C_1, C_2) are parametrized in Theorem 4.2 and a design method to achieve reliable controller pairs is given in Algorithm 4.5. The controller design method proposed here is only concerned with maintaining stability when one controller fails; other performance criteria might not be satisfied. Extensions of the reliable stabilization method to reliable decomposition of a given controller, which was briefly considered in [3] and [1], as well as maintaining other performance objectives after failure are problems to be studied in the future.

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