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DECOUPLING OF LINEAR MULTIVARIABLE PLANTS BY DYNAMICAL OUTPUT FEEDBACK

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Abstract

This paper presents an *algebraic* theory for decoupling linear multivariable feedback systems. A global parametrization of *all diagonal nonsingular* I/O maps and *all* D/O maps achievable by a stabilizing compensator or a given plant is given in the theorem.

Introduction

In the design theory of linear time-invariant (LT-I) multi-input multi-output (MIMO) systems, the characterization of all designs which can be achieved by a stabilizing controller for a given plant shows the limitations on achievable performance imposed by the plant and the constraints of linearity and stability. This paper presents a general algebraic design method for all diagonal I/O maps which can be achieved by a stabilizing two-input one output controller K for a given plant P. This method gives a decoupled closed loop system for which the (diagonal) I/O map can be specified independently of the D/O map.

The system $\Sigma(P,K)$ shown in Fig. 1 represents a more general case in that y_2, the output of interest, is not the same as z, the measured output; furthermore, the disturbance d is applied directly to the pseudo-state of P rather than being an additive input as for example in [Des. 1].

Algebraic Structure: [Bou. 1], [Lang. 1]

- $\mathbf{H}$: A principal ring (PID), (e.g., $\mathbf{R_U}$, the ring of proper rational functions analytic in $\mathbf{U}$).
- $\tilde{\mathbf{G}}$: The field of fractions over $\mathbf{H}$ (e.g., $\mathbf{R}(s)$).
- $\mathbf{I}$: A multiplicative subset of $\mathbf{H}$; equivalently, $\mathbf{I} \subset \mathbf{H}$, $0 \notin \mathbf{I}$, $1 \in \mathbf{I}$, and $x,y \in \mathbf{I}$ implies that $xy \in \mathbf{I}$ (e.g., $f \in \mathbf{I}$ if $f \in \mathbf{R_U}$ and $f(\infty) = 1$.)
- $\mathbf{G}$: $= \{n/d : n \in \mathbf{H}, d \in \mathbf{I}\}$, subring of $\tilde{\mathbf{G}}$ (e.g., $\mathbf{R}_p(s)$)
- $\mathbf{U(H)}$: $= \{m \in \mathbf{H} : m^{-1} \in \mathbf{H}\}$, the group of units in $\mathbf{H}$ (e.g., $f \in \mathbf{U(H)} \iff f \in \mathbf{R_U}$ and $f(s) \neq 0 \; \forall \, s \in \mathbf{U}$).

Problem Description an Assumptions

We consider the LT-I, MIMO system $\Sigma(P,K)$ in Figs. 1 and 2. Given a plant P, we design a controller K with two inputs and one output such that the resulting system is *stable*, K is *proper* and the I/O map $v \mapsto y_2$ is *nonsingular* and *decoupled*, i.e., *diagonal*. We assume:

(P) $P \in \mathbf{G}^{2n \times n}$ has a right-coprime factorization (r.c.f.)
$\begin{bmatrix} P^o \\ P^m \end{bmatrix} = \begin{bmatrix} N^o_{pr} \\ N^m_{pr} \end{bmatrix} D_{pr}^{-1}$ with D_{pr}, N^o_{pr}, $N^m_{pr} \in \mathbf{H}^{n \times n}$ det $D_{pr} \in \mathbf{I}$ and det $N^o_{pr} \not\equiv 0$.

(K) $K \in \mathbf{G}^{n \times 2n}$ has a left-coprime factorization (l.c.f.) $D_{cl}^{-1}[N_{rl} \vdots N_{fl}]$ with D_{cl}, N_{rl}, $N_{fl} \in \mathbf{H}^{n \times n}$, det $D_{cl} \in \mathbf{I}$ and det $(D_{cl} D_{pr} + N_{fl} N^m_{pr}) \in \mathbf{I}$.

Definition: The system $\Sigma(P,K)$ is called H-*stable* if and only if the map $H_{yu} : (v^T, u_1^T, u_2^T, d^T)^T \mapsto (y_1^T, y_2^T, z^T)$ has elements in $\mathbf{H}$.

Let

$$D_h := D_{cl} D_{pr} + N_{fl} N^m_{pr} \in \mathbf{H}^{n \times n} \tag{1}$$

$\Sigma(P,K)$ is H-stable if and only if det $D_h \in U(\mathbf{H})$ [Des. 1, corollary 3.1]; w.l.o.g. if and only if we can take $D_h = I$ [Vid. 1]. By (1), $\Sigma(P,K)$ H-stable implies that (N^m_{pr}, D_{pr}) are right-coprime.

The I/O Map $H_{y_2 v}$ and the D/O Map $H_{y_2 d}$

Definition (Δ_L): Let Δ_L be a diagonal matrix

$$\Delta_L = \mathrm{diag}[\Delta_{L1}, \Delta_{L2}, \ldots, \Delta_{Ln}] \in \mathbf{H}^{n \times n} \tag{2}$$

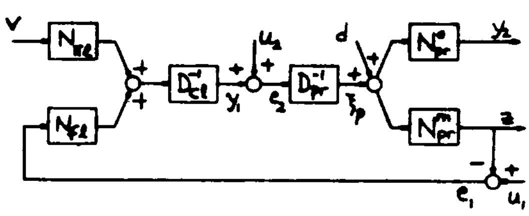

Fig. 1

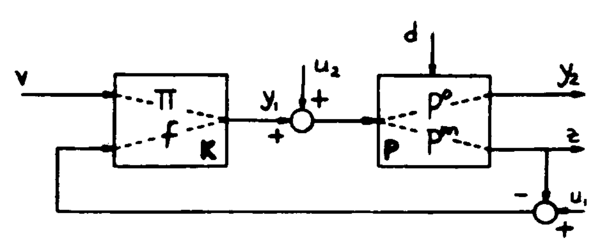

Fig. 2

where, for k = 1, ..., n, Δ_{Lk} is the g.c.d. over $\mathbf{H}$ of the elements of the k-th row of N_{pr}^o. Then

$$N_{pr}^o = \Delta_L \tilde{N}_{pr}^o \tag{3}$$

where Δ_L, $\tilde{N}_{pr}^o$ are not unique since each Δ_{Lk} is defined within a unit factor. (In $\mathbf{R_U}$, Δ_L "book-keeps" the plant zeros in $\mathbf{\bar{U}}$ that are common to all elements of the k-th row of N_{pr}^o.)

Definition (Δ_R): Let Δ_R be a diagonal matrix

$$\Delta_R = \mathrm{diag}[\Delta_{R1}, \Delta_{R2}, \ldots, \Delta_{Rn}] \in \mathbf{H}^{n\times n} \tag{4}$$

where, for j = 1, ..., n, Δ_{Rj} is a l.c.m. of $d_{1j}, \ldots, d_{nj}$ where the j-th column of $(\tilde{N}_{pr}^o)^{-1}$ is $\left[\frac{m_{1j}}{d_{1j}}, \ \cdots \ \frac{m_{nj}}{d_{nj}}\right]^T$, $d_{ij}, m_{ij} \in \mathbf{H}^{n\times n}$, i = 1, ..., n. Δ_R is defined within a unimodular factor.

For any $\Sigma(P,K)$ satisfying (P) and (K), the I/O map $H_{y_2 v} : v \mapsto y_2$ and the D/O map $H_{y_2 d} : d \mapsto y_2$ are given by

$$H_{y_2 v} = N_{pr}^o D_h^{-1} N_{rl} = \Delta_L \tilde{N}_{pr}^o N_{rl} \tag{5}$$

$$H_{y_2 d} = N_{pr}^o [I - D_h^{-1} N_{fl} N_{pr}^m] = N_{pr}^o D_{cl} D_{pr} \tag{6}$$

where we use (1), (2) and take $D_h = I$ since $\Sigma(P,K)$ is $\mathbf{H}$-stable.

Achievable Performance of $\Sigma(P,K)$

Let P be given and satisfy (P).

$\mathbf{H}_{y_2 v}$: = { $H_{y_2 v}$: for the given P, there exists a K satisfying (K) such that $\Sigma(P,K)$ is $\mathbf{H}$-stable with $H_{y_2 v}$ *diagonal* and nonsingular }

$\mathbf{H}_{y_2 d}$: = { $H_{y_2 d}$: for the given P, there exists a K satisfying (K) such that $\Sigma(P,K)$ is $\mathbf{H}$-stable with $H_{y_2 v}$ *diagonal* and *nonsingular*.}

Theorem: Consider $\Sigma(P,K)$ of Fig. 1: Let P and K satisfy (P) and (K). Let $P^m = D_{pl}^{-1} N_{pl}^m$ be a l.c.f. of P^m. Let Δ_L and Δ_R be defined by (2) and (4). then

i) the map $H_v \in \mathbf{H}^{n\times n}$ is an achievable diagonal, nonsingular I/O map of the $\mathbf{H}$-stable $\Sigma(P,K)$ if and only if

$$H_v \in \mathbf{H}_{y_2 v}(P) = \{\Delta_L \Delta_R Q_d : Q_d \in \mathbf{H}^{n\times n}, Q_d \text{ is diagonal, nonsingular.}\} \tag{7}$$

ii) the map $H_d \in \mathbf{H}^{n\times n}$ is an achievable D/O map of the $\mathbf{H}$-stable $\Sigma(P,K)$ if and only if

$$\begin{aligned} H_d &= \mathbf{H}_{y_2 d}(P) \\ &= \{N_{pr}^o [I - (U_{pr}^m + R D_{pl}) N_{pr}^m] \\ &= N_{pr}^o (V_{pr}^m - R N_{pl}^m) D_{pr} : R \in \mathbf{H}^{n\times n} \\ &\quad \text{s.t. } \det(V_{pr}^m - R N_{pl}^m) \in \mathbf{I}.\} \end{aligned} \tag{8}$$

and U_{pr}^m, $V_{pr}^m \in \mathbf{H}^{n\times n}$ are such that $U_{pr}^m N_{pr}^m + V_{pr}^m D_{pr} = I$.

Comment: Diagonalizing the I/O map is achieved by choosing N_{rl} ($= \tilde{N}_{pr}^o \Delta_R Q_d \in \mathbf{H}^{n\times n}$) and this choice is independent of that of $D_{cl}(= V_{pr}^m - R N_{pl}^m)$ and $N_{fl}(= U_{pr}^m + R D_{pl})$: thus this is a two-degrees-of-freedom design [Hor. 1]. These parameters specify a K that *stabilizes* and *decouples* P (with Q_d and R as above).

Example: We focus our attention on the diagonal I/O map of $\Sigma(P,K)$ and calculate only N_{rl}. Let $\mathbf{H} := \mathbb{R}(s, e^{-\tau s})$ = the entire ring of *proper rational* functions analytic in $\mathbb{C}_+$ with coefficients in $\mathbb{R}[e^{-\tau s}]$. P^o is strictly proper, is not $\mathbf{H}$-stable and has a simple zero at s = 3: $P^o(s, e^{-\tau s})$ =

$$\begin{bmatrix} \frac{e^{-s}}{s-1} & \vdots & \frac{1}{s-2} \\ & \vdots & \\ \frac{e^{-2s}}{s+1} & \vdots & \frac{e^{-s}}{s-1} \end{bmatrix} = N_{pr}^o D_{pr}^{-1} = \begin{bmatrix} \frac{e^{-s}}{s+2} & \vdots & \frac{s-1}{(s+1)^2} \\ & \vdots & \\ \frac{(s-1)e^{-2s}}{(s+1)(s+2)} & \vdots & \frac{(s-2)e^{-s}}{(s+1)^2} \end{bmatrix}$$

$\mathrm{diag}\left[\frac{s-1}{s+2}, \frac{(s-1)(s-2)}{(s+1)^{2-1}}\right]^{-1}$. Then Δ_L = $\mathrm{diag}\left[\frac{1}{s+2}, \frac{e^{-s}}{s+1}\right]$ and from $(\tilde{N}_{pr}^o)^{-1}$ ($\notin \mathbf{H}^{n\times n}$) we obtain $\Delta_R = \mathrm{diag}\left[\frac{(s-3)e^{-s}}{(s+1)^2}, \frac{(s-3)e^{-s}}{(s+1)^2}\right]$ and

$$N_{rl} = (\tilde{N}_{pr}^o)^{-1} \Delta_R Q_d = \begin{bmatrix} \frac{s-2}{s+1} & \vdots & \frac{-(s-1)(s+2)}{(s+1)^2} \\ & \vdots & \\ \frac{-(s-1)e^{-s}}{s+2} & \vdots & e^{-s} \end{bmatrix} Q_d$$

So, $H_{y_2 v} = \Delta_L \Delta_R Q_d = \mathrm{diag}\left[\frac{(s-3)e^{-s}}{(s+2)(s+1)^2}, \frac{(s-3)e^{-2s}}{(s+1)^3}\right] \cdot Q_d$, $Q_d \in \mathbf{H}^{n\times n}$. Note that each diagonal entry of Δ_R is equal to det $\tilde{N}_{pr}^o$; in fact in the *2x2* case, each diagonal entry of Δ_R is *always* equal to det $\tilde{N}_{pr}^o$ (modulo a unit factor). Consequently, $H_{y_2 v}$ has a zero of multiplicity *two* at s=3 and may have other $\mathbb{C}_+$-zeros due to Q_d.

Acknowledgement

Research sponsored by the National Science Foundation Grant ECS-8119763.

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