Let

$$x = \begin{bmatrix} x_p - x_c \\ x_c \end{bmatrix}; \quad w = \begin{bmatrix} w_p \\ v \end{bmatrix}.$$

We can get the closed-loop system in the form of (1) with 2 input and 3 output. If we choose an identity weight matrix W and the given matrix Q

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 100 \end{bmatrix}$$

the bound on the L_∞ norm is

$$\bar{\sigma}[Y] \operatorname{tr} QW^{-1} = 12.3327.$$

Using the continuous time algorithm given, the optimal $\hat{W}$ is

$$\hat{W} = \begin{bmatrix} 6.7853 & 7.8153 \\ 7.8153 & 125.1410 \end{bmatrix}.$$

Using $\hat{W}$, the bound is

$$\bar{\sigma}[Y] \operatorname{tr} Q\hat{W}^{-1} = 3.1112$$

an improvement of a factor of four. Discretizing the closed-loop system with sampling time $T = 0.1$ we can get the discrete closed-loop system in the form of (36). With the same Q matrix, if we choose identity weight matrix W_d, the bound on the L_∞ norm is

$$\bar{\sigma}[Y] \operatorname{tr} QW_d^{-1} = 1.2323.$$

If we use the weight matrix

$$\hat{W}_d = \begin{bmatrix} 6.7845 & 7.8125 \\ 7.8125 & 125.1410 \end{bmatrix}$$

obtained by the discrete algorithm, the bound becomes

$$\bar{\sigma}[Y] \operatorname{tr} Q\hat{W}_d^{-1} = 0.3109$$

a factor of four improvement. □

For this example, the initial weight matrix is identity. With the error tolerance 10^{-12}, the algorithm converges in seven iterations.

III. Conclusion

The optimal L_∞ bounds are given for disturbance rejection. The algorithm which produces these bounds is iterative and several examples show that the algorithm converges very quickly. These bounds are explicit in the output covariance matrix and may have a use in synthesizing convariance controllers to achieve a specified degree of disturbance rejection. This is an interesting problem for future work.

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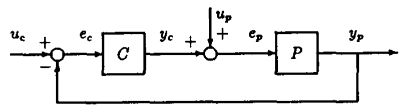

Fig. 1. The system $S(P, C)$.

Parameterization of Decoupling Controllers in the Unity-Feedback System

A. Nazli Gündeş

***Abstract*—This note studies the problem of decoupling in the linear, time invariant, multiinput–multioutput unity-feedback system. A parameterization of all stabilizing decoupling controllers and all achievable decoupled closed-loop transfer functions is obtained for full-row rank plants which do not have any coinciding poles and zeros in the undesirable region of the complex plane.**

I. Introduction

Decoupling is an important problem and has been studied in several papers. In the linear time-invariant (LTI), multiinput–multioutput (MIMO) unity-feedback system $S(P, C)$ (Fig. 1), decoupling is achieved if the closed loop, input–output transfer function H_{pc} from the command-input to the plant-output is diagonalized by using a stabilizing controller. A necessary condition for H_{pc} to be diagonal and nonsingular is that the plant's transfer function P must have full-row rank; a sufficient condition to achieve decoupling in the unity-feedback system is for the full-row rank plant to have no coinciding poles and zeros in the undesirable region of the complex plane [7], [2]–[4]. Conditions for the existence of decoupling controllers placed in the *feedback path* were obtained in [5]. The parameterization of all decoupling controllers and all achievable decoupled transfer functions in the two-degrees-of-freedom feedback system were obtained in [1]. Recently, a decoupling controller design algorithm was given in [6] for *square*, full-rank plants, which have no unstable pole and zero coincidences; the algorithm is based on interpolation conditions on polynomials.

In this note, we obtain the class of *all* decoupling controllers and the parameterization of *all* achievable diagonal, nonsingular, input–output transfer functions in the unity-feedback system $S(P, C)$ for full-row rank plants which have no coinciding undesirable poles and zeros. This parameterization is based on the Smith–McMillan form using a factorization approach [8], [3].

The algebraic tools are explained in Section II-A. Section II-B has several lemmas for the development of the main theorem, which is the parameterization of all decoupling controllers (Theorem 2.2.4). Section III contains concluding remarks.

Notation: We use $:=$ for "defined as," i.e., $a := b$ (or $b =: a$) means a is defined as b. Let $\mathscr{U}$ be a subset of the field $\mathbb{C}$ of complex numbers, where $\mathscr{U}$ is closed and symmetric about the real axis, $\pm\infty \in \mathscr{U}$ and $\mathbb{C} \setminus \mathscr{U}$ is nonempty. For continuous-time systems, $\mathscr{U} \supset \mathbb{C}_+ := \{s \in \mathbb{C} | \Re e(s) \geq 0\}$ and for discrete-time systems, $\mathscr{U} \supset \{s \in \mathbb{C} | |s| \geq 1\}$. Let $\mathscr{R}_{\mathscr{U}}$, $\mathbb{R}_p(s)$, $\mathbb{R}_{sp}(s)$, and $\mathbb{R}(s)$

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be the ring of proper rational functions which have no poles in $\mathscr{U}$, the ring of proper rational functions, the set of strictly proper rational functions, and the field of rational functions of s (with real coefficients). The group of units of $\mathscr{R}_{\mathscr{U}}$ is $\mathscr{J}$ and the set of nonstrictly proper elements of $\mathscr{R}_{\mathscr{U}}$ is $\mathscr{J} = \mathscr{R}_{\mathscr{U}} \setminus \mathbb{R}_{sp}(s)$. The set of matrices whose entries are in $\mathscr{R}_{\mathscr{U}}$ is denoted $\mathscr{M}(\mathscr{R}_{\mathscr{U}})$. A matrix M is called $\mathscr{R}_{\mathscr{U}}$-stable iff $M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$; $M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is $\mathscr{R}_{\mathscr{U}}$-unimodular iff $\det M \in \mathscr{J}$, where $\det M$ denotes the determinant of the matrix M. If $p, q \in \mathscr{R}_{\mathscr{U}}$, then $p \sim q$ iff $p = mq$ for some $m \in \mathscr{J}$. The identity matrix of size n is denoted I_n. The diagonal matrix, whose entries are $a_1, a_2, \cdots, a_m$ is denoted $\operatorname{diag}[a_1 \cdots a_m]$.

II. Main Results

A. The System $S(P, C)$

Consider the LTI, MIMO feedback system $S(P, C)$ shown in Fig. 1, where $P: e_p \mapsto y_p$ and $C: e_c \mapsto y_c$ represent the plant and the controller transfer functions. We assume that: 1) the plant and the controller have no hidden modes associated with eigenvalues of $\mathscr{U}$; 2) P and C are proper ($P \in \mathbb{R}_p(s)^{no \times ni}$ and $C \in \mathbb{R}_p(s)^{ni \times no}$); and 3) the system $S(P, C)$ is well posed, i.e., $(I_{no} + PC)^{-1} \in \mathscr{M}(\mathbb{R}_p(s))$.

The closed loop input–output transfer function $H_{pc}: u_c \mapsto y_p$ of $S(P, C)$ is

$$H_{pc} = PH_{cc} = PC(I_{no} + PC)^{-1}$$

where $H_{cc}: u_c \mapsto y_c$ is given by $H_{cc} = C(I_{no} + Pc)^{-1}$.

Definition 2.1.1: The system $S(P, C)$ is said to be *$\mathscr{R}_{\mathscr{U}}$-stable* iff $H_{yu}: \begin{bmatrix} u_c \\ u_p \end{bmatrix} \mapsto \begin{bmatrix} y_c \\ y_p \end{bmatrix} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$; $S(P, C)$ is said to be *decoupled* iff it is $\mathscr{R}_{\mathscr{U}}$-stable and the input-output transfer function $H_{pc}: u_c \mapsto y_p$ is *diagonal* and *nonsingular*.

The set $\mathscr{S}(P) := \{C | C \in \mathbb{R}_p(s)^{ni \times no}$ and $S(P, C)$ is $\mathscr{R}_{\mathscr{U}}$-stable$\}$ is called the *set of all $\mathscr{R}_{\mathscr{U}}$-stabilizing controllers* for P. The set $\mathscr{A}(P) := \{H_{pc}: u_c \mapsto y_p | C \in \mathscr{S}(P)\}$ is called the *set of all achievable input-output transfer functions* from the input u_c to the output y_p. The set $\mathscr{S}_{\mathscr{D}}(P) := \{C | C \in \mathscr{S}(P)$ and H_{pc} is diagonal and nonsingular$\}$ is called the *set of all decoupling controllers* for P. The set $\mathscr{A}_{\mathscr{D}}(P) := \{H_{pc} | C \in \mathscr{S}_{\mathscr{D}}(P)\}$ is called the *set of all achievable decoupled input–output transfer functions H_{pc}* from the input u_c to the output y_p. □

To achieve decoupling in the system $S(P, C)$, the plant's transfer function P must be full (normal) row rank (Lemma 2.1.2). If $P \in \mathbb{R}_p(s)^{no \times ni}$ has full-row rank (i.e., rank $P = n_o$), then a sufficient condition for the existence of decoupling controllers in the system $S(P, C)$ is that the full-row rank plant P does not have any $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros. In Section II-B, we parameterize the class of all decoupling controllers $\mathscr{S}_{\mathscr{D}}(P)$ and all achievable decoupled transfer functions $\mathscr{A}_{\mathscr{D}}(P)$ for plants which satisfy this sufficient condition. Coprime factorizations of the plant derived from its Smith–McMillan form are used in this parameterization; we discuss these briefly in Fact 2.1.3.

Lemma 2.1.2 (Necessary Condition for Decoupling): Let $P \in \mathbb{R}_p(s)^{no \times ni}$. If the system $S(P, C)$ is decoupled, then rank $P = n_o \leq n_i$.

Proof: If $S(P, C)$ is decoupled, then the $n_o \times n_o$ transfer function H_{pc} is nonsingular, where $H_{pc} = PH_{cc}$, therefore,

$$\operatorname{rank} H_{pc} = n_o = \operatorname{rank}(PH_{cc}) \leq \operatorname{rank} P \leq \min\{n_o, n_i\}.$$

Consequently, rank $P = n_o$, moreover, $n_o \leq n_i$. □

Facts 2.1.3 [8], [3]: Let $P \in \mathbb{R}_p(s)^{no \times ni}$ and let rank $P = n_o$. Under these assumptions, the following holds:

i) Smith–McMillan Form of P: There exist $\mathscr{R}_{\mathscr{U}}$-unimodular matrices $L \in \mathscr{R}_{\mathscr{U}}^{no \times no}$ and $R \in \mathscr{R}_{\mathscr{U}}^{ni \times ni}$ such that

$$\begin{aligned} P &= L[\Lambda \,\vdots\, 0_{no \times (ni - no)}] \operatorname{diag}\left[\Psi^{-1} \quad I_{(ni-no)}\right] R \\ &= L\Psi^{-1}[\Lambda \,\vdots\, 0_{no \times (ni-no)}]R \end{aligned} \tag{2.1}$$

$$\Lambda := \operatorname{diag}[\lambda_1 \cdots \lambda_{no}], \quad \Psi := \operatorname{diag}[\psi_1 \cdots \psi_{no}]. \tag{2.2}$$

For $j = 1, \cdots, n_o$, $\lambda_j \in \mathscr{R}_{\mathscr{U}}$, $\psi_j \in \mathscr{R}_{\mathscr{U}}$, λ_j divides λ_{j+1}, and ψ_{j+1} divides ψ_j. The pair (λ_j, ψ_j) is coprime, equivalently, there exist $u_j \in \mathscr{R}_{\mathscr{U}}$, $v_j \in \mathscr{R}_{\mathscr{U}}$ such that

$$u_j \lambda_j + v_j \psi_j = 1. \tag{2.3}$$

The factors λ_j and $\psi_j \in \mathscr{R}_{\mathscr{U}}$ are called the (numerator and denominator) invariant factors. The denominator invariant factors $\psi_1, \cdots, \psi_{no} \in \mathscr{J}$ if and only if $P \in \mathscr{M}(\mathbb{R}_p(s))$. By assumption, rank $P = n_o$, therefore, $\lambda_{no} \neq 0$.

ii) Coprime Factorizations of P and Bezout Identities: For $j = 1, \cdots, n_o$, let $u_j \in \mathscr{R}_{\mathscr{U}}$ and $v_j \in \mathscr{R}_{\mathscr{U}}$ satisfy (2.3). Let

$$U := \operatorname{diag}[u_1 \cdots u_{no}] \quad V := \operatorname{diag}[v_1 \cdots v_{no}] \tag{2.4}$$

therefore, $U\Lambda + V\Psi = \Lambda U + \Psi V = I_{no}$. Let

$$N_P := L[\Lambda \,\vdots\, O_{no \times (ni-no)}] \quad D_P := R^{-1} \operatorname{diag}[\Psi \quad I_{(ni-no)}] \tag{2.5}$$

$$\tilde{D}_P := \Psi L^{-1} \quad \tilde{N}_P := [\Lambda \,\vdots\, 0_{no \times (ni-no)}]R \tag{2.6}$$

$$U_P := \begin{bmatrix} U \\ 0_{(ni-no) \times no} \end{bmatrix} L^{-1}, \quad \tilde{U}_P := R^{-1} U_P L,$$

$$V_P := \operatorname{diag}[V \quad I_{(ni-no)}]R, \quad \tilde{V}_P := LV. \tag{2.7}$$

Then $P = N_P D_P^{-1}$ is a right-coprime factorization (rcf) and $P = \tilde{D}_P^{-1} \tilde{N}_P$ is a left-coprime factorization (lcf) of P, i.e., $N_P \in \mathscr{R}_{\mathscr{U}}^{no \times ni}$, $D_P \in \mathscr{R}_{\mathscr{U}}^{ni \times ni}$, $\tilde{N}_P \in \mathscr{R}_{\mathscr{U}}^{no \times ni}$, $\tilde{D}_P \in \mathscr{R}_{\mathscr{U}}^{no \times no}$, $\det D_P \sim \det \tilde{D}_P \in \mathscr{J}$ and

$$\begin{bmatrix} V_P & U_P \\ -\tilde{N}_P & \tilde{D}_P \end{bmatrix} \begin{bmatrix} D_P & -\tilde{U}_P \\ N_P & \tilde{V}_P \end{bmatrix} = \begin{bmatrix} I_{ni} & 0 \\ 0 & I_{no} \end{bmatrix} \tag{2.8}$$

$V_P \in \mathscr{R}_{\mathscr{U}}^{ni \times ni}$, $U_P \in \mathscr{R}_{\mathscr{U}}^{ni \times no}$, $\tilde{V}_P \in \mathscr{R}_{\mathscr{U}}^{no \times no}$, $\tilde{U}_P \in \mathscr{R}_{\mathscr{U}}^{ni \times no}$ are given by (2.7).

iii) All $\mathscr{R}_{\mathscr{U}}$-Stabilizing Controllers and All Achievable Input-Output Transfer Functions: The set $\mathscr{S}(P)$ of all $\mathscr{R}_{\mathscr{U}}$-stabilizing controllers is

$$\begin{aligned} \mathscr{S}(P) = \Bigg\{ C &= \left(V_P - \hat{Q}\tilde{N}_P\right)^{-1}\left(U_P + \hat{Q}\tilde{D}_P\right) \\ &= \left(\tilde{U}_P + D_P\hat{Q}\right)\left(\tilde{V}_P - N_P\hat{Q}\right)^{-1} \\ &= R^{-1}\begin{bmatrix} U + \Psi Q \\ Q_A \end{bmatrix}(V - \Lambda Q)^{-1}L^{-1} \Bigg| \\ \hat{Q} &= \begin{bmatrix} Q \\ Q_A \end{bmatrix} \in \mathscr{R}_{\mathscr{U}}^{ni \times no}, Q_A \in \mathscr{R}_{\mathscr{U}}^{(ni-no) \times no} \\ Q &\in \mathscr{R}_{\mathscr{U}}^{no \times no}, \det\left(\tilde{V}_P - N_P\hat{Q}\right) \sim \det\left(V_P - \hat{Q}\tilde{N}_P\right) \\ &\sim \det(V - \Lambda Q) \in \mathscr{J} \Bigg\}. \end{aligned} \tag{2.9}$$

The set $\mathscr{A}(P)$ of all achievable input–output transfer functions is

$$\mathscr{A}(P) = \Big\{H_{pc} = N_P\big(U_P + \hat{Q}\tilde{D}_P\big) = I_{no} - \big(\tilde{V}_P - N_P\hat{Q}\big)\tilde{D}_P$$
$$= L\Lambda(U + Q\Psi)L^{-1} = I_{no} - L(V - \Lambda Q)\Psi L^{-1} |$$
$$\hat{Q} = \begin{bmatrix} Q \\ Q_A \end{bmatrix} \in \mathscr{R}_{\mathscr{U}}^{ni\times no}, Q_A \in \mathscr{R}_{\mathscr{U}}^{(ni-no)\times no}$$
$$Q \in \mathscr{R}_{\mathscr{U}}^{no\times no}, \det\big(\tilde{V}_P - N_P\hat{Q}\big) \sim \det\big(V_P - \hat{Q}\tilde{V}_P\big)$$
$$\sim \det(V - \Lambda Q) \in \mathscr{I}\Big\}. \quad (2.10)$$

If P is strictly proper, then $\det(V_P - \hat{Q}\tilde{N}_P) \sim \det(\tilde{V}_P - N_P\hat{Q}) \in \mathscr{I}$ for *all* $\hat{Q} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, equivalently, $\det(V - \Lambda Q) \in \mathscr{I}$ for *all* $Q \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$.

iv) $\mathscr{U}$-poles and $\mathscr{U}$-zeros of P: Let $p_o \in \mathscr{U}$, then p_o is a $\mathscr{U}$-pole of P if and only if $\psi_1(p_o) = 0$. Let $z_o \in \mathscr{U}$, then z_o is a $\mathscr{U}$-zero of P if and only if $\lambda_{no}(z_o) = 0$. Therefore, the plant P does not have any $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros if and only if (λ_{no}, ψ_1) is a coprime pair or equivalently, there exist $\alpha \in \mathscr{R}_{\mathscr{U}}$, $\beta \in \mathscr{R}_{\mathscr{U}}$ such that for all $q \in \mathscr{R}_{\mathscr{U}}$

$$\alpha\lambda_{no} + \beta\psi_1 = (\alpha + q\psi_1)\lambda_{no} + (\beta - q\lambda_{no})\psi_1 = 1. \quad (2.11)$$

v) Bezout Identities When $\mathscr{U}$-Poles of P Do Not Coincide with its $\mathscr{U}$-Zeros: Let (λ_{no}, ψ_1) be a coprime pair (equivalently, (2.11) holds for some $\alpha \in \mathscr{R}_{\mathscr{U}}$, $\beta \in \mathscr{R}_{\mathscr{U}}$). For $j = 1,\cdots,n_o$, let

$$(u_j, v_j) = ((\alpha + q\psi_1)\lambda_{no}/\lambda_j, (\beta - q\lambda_{no})\psi_1/\psi_j) \quad (2.12)$$

the pair (u_j, v_j) satisfies (2.3) for all $q \in \mathscr{R}_{\mathscr{U}}$. The matrices U, V defined in (2.4) become

$$U = \text{diag}[(\alpha + q\psi_1)\lambda_{no}/\lambda_1 \cdots (\alpha + q\psi_1)]$$
$$= (\alpha + q\psi_1)\lambda_{no}\Lambda^{-1} \quad (2.13)$$
$$V = \text{diag}[(\beta - q\lambda_{no}) \cdots (\beta - q\lambda_{no})\psi_1/\psi_{no}]$$
$$= (\beta - q\lambda_{no})\psi_1\Psi^{-1}. \quad (2.14)$$

vi) All $\mathscr{R}_{\mathscr{U}}$-Stabilizing Controllers and All Achievable Input–Output Transfer Functions When $\mathscr{U}$-Poles of P Do Not Coincide with its $\mathscr{U}$-Zeros: Let (λ_{no}, ψ_1) be a coprime pair. Using U, V defined by (2.13) and (2.14) in (2.9) and (2.10), the sets $\mathscr{S}(P)$ and $\mathscr{A}(P)$ become

$$\mathscr{S}(P) = \Big\{C = R^{-1}\begin{bmatrix} \alpha\lambda_{no}I_{no} + \Psi Q\Lambda \\ Q_A\Lambda \end{bmatrix}$$
$$\cdot\Lambda^{-1}\Psi(\beta\psi_1 I_{no} - \Lambda Q\Psi)^{-1}L^{-1} |$$
$$Q_A \in \mathscr{R}_{\mathscr{U}}^{(ni-no)\times no}, Q \in \mathscr{R}_{\mathscr{U}}^{no\times no},$$
$$\det(\beta\psi_1 I_{no} - \Lambda Q\Psi) \in \mathscr{I}\Big\} \quad (2.15)$$

$$\mathscr{A}(P) = \big\{H_{pc} = L\Lambda(\alpha\lambda_{no}\Lambda^{-1} + Q\Psi)L^{-1}$$
$$= \alpha\lambda_{no}I_{no} + L\Lambda Q\Psi L^{-1}$$
$$= (1 - \beta\psi_1)I_{no} + L\Lambda Q\Psi L^{-1} | Q \in \mathscr{R}_{\mathscr{U}}^{no\times no},$$
$$\det(\beta\psi_1 I_{no} - \Lambda Q\Psi) \in \mathscr{I}\big\}. \quad (2.16)$$

If P is strictly proper, then $\det(\beta\psi_1 I_{no} - \Lambda Q\Psi) \in \mathscr{I}$ for *all* $Q \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. □

B. Parameterization of Decoupling Controllers

In this section, we assume that rank $P = n_o$ and that P does not have any $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros. Under these assumptions, it is possible to find decoupling controllers. In Theorem 2.2.4, we parameterize: 1) the class of *all* decoupling controllers for P; and 2) the class of all achievable decoupled input–output transfer functions H_{pc}.

Lemma 2.2.1: Let $A \in \mathscr{R}_{\mathscr{U}}^{no\times no}$ and let $\det A \neq 0$. Let $n_{ij}/d_{ij} \in \mathbb{R}(s)$ denote the (i, j)-entry of A^{-1}, where the pair (n_{ij}, d_{ij}) is coprime, $n_{ij}, d_{ij} \in \mathscr{R}_{\mathscr{U}}$, $d_{ij} \neq 0$. For $j = 1,\cdots,n_o$, let $\delta_j \in \mathscr{R}_{\mathscr{U}}$ be a least-common multiple (lcm) of all of the denominators $(d_{1j}, d_{2j},\cdots,d_{noj})$ in the jth column of A^{-1} and define $\Delta := \text{diag}[\delta_1 \cdots \delta_{no}]$. For $j = 1,\cdots,n_o$, let $\theta_j \in \mathscr{R}_{\mathscr{U}}$ be an lcm of all of the denominators $(d_{j1}, d_{j2},\cdots,d_{jno})$ in the jth row of A^{-1} and define $\Theta := \text{diag}[\theta_1 \cdots \theta_{no}]$. Under these assumptions:

i) $A^{-1}W_R \in \mathscr{R}_{\mathscr{U}}^{no\times no}$ for some diagonal, nonsingular, $\mathscr{R}_{\mathscr{U}}$-stable matrix W_R if and only if $W_R = \Delta Q_R$, where $Q_R \in \mathscr{R}_{\mathscr{U}}^{no\times no}$ is diagonal, nonsingular;

ii) $W_L A^{-1} \in \mathscr{R}_{\mathscr{U}}^{no\times no}$ for some diagonal, nonsingular, $\mathscr{R}_{\mathscr{U}}$-stable matrix W_L if and only if $W_L = Q_L\Theta$, where $Q_L \in \mathscr{R}_{\mathscr{U}}^{no\times no}$ is diagonal, nonsingular.

Proof: We only prove part i); the proof of ii) is similar.

Let $W_R = \text{diag}[w_1 \cdots w_{no}]$ be nonsingular and $\mathscr{R}_{\mathscr{U}}$-stable. Then $A^{-1}W_R$ is $\mathscr{R}_{\mathscr{U}}$-stable if and only if for each $j = 1,\cdots,n_o$, $n_{ij}d_{ij}^{-1}w_j \in \mathscr{R}_{\mathscr{U}}$, $i = 1,\cdots,n_o$. Since the pair (n_{ij}, d_{ij}) is coprime, $n_{ij}d_{ij}^{-1}w_j \in \mathscr{R}_{\mathscr{U}}$ for some $w_j \in \mathscr{R}_{\mathscr{U}}$ if and only if $d_{ij}^{-1}w_j \in \mathscr{R}_{\mathscr{U}}$, equivalently, w_j is a multiple of all d_{ij}, $i = 1,\cdots,n_o$ [8]. Therefore, $w_j = \delta_j q_j$ for some nonzero $q_j \in \mathscr{R}_{\mathscr{U}}$. □

Lemma 2.2.2: Let $P \in \mathbb{R}_p(s)^{no\times ni}$ and let rank $P = n_o$. Consider the Smith–McMillan form (2.1) of P. Let $\delta_{Lj} \in \mathscr{R}_{\mathscr{U}}$ be a greatest-common divisor (gcd) of the entries in the jth row of $L\Lambda$ and define

$$\Delta_L := \text{diag}[\delta_{L1} \cdots \delta_{Lno}], \qquad \hat{N} := \Delta_L^{-1}L\Lambda. \quad (2.17)$$

Let $n_{ij}/d_{ij} \in \mathbb{R}(s)$ denote the (i, j) entry of $\hat{N}^{-1} = \Lambda^{-1}L^{-1}\Delta_L$, where the pair (n_{ij}, d_{ij}) is coprime, $n_{ij}, d_{ij} \in \mathscr{R}_{\mathscr{U}}$, $d_{ij} \neq 0$. Let $\delta_{Rj} \in \mathscr{R}_{\mathscr{U}}$ be an lcm of all of the denominators $(d_{1j}, d_{2j},\cdots,d_{noj})$ in the jth column of $\hat{N}^{-1}$ and define

$$\Delta_R := \text{diag}[\delta_{R1} \cdots \delta_{Rno}]. \quad (2.18)$$

Let $\theta_{Rj} \in \mathscr{R}_{\mathscr{U}}$ be a gcd of the entries in the jth column of $\tilde{D}_P = \Psi L^{-1}$ and define

$$\Theta_R := \text{diag}[\theta_{R1} \cdots \theta_{Rno}], \quad \hat{D} := \tilde{D}_P\Theta_R^{-1} = \Psi L^{-1}\Theta_R^{-1}. \quad (2.19)$$

Let $x_{ij}/y_{ij} \in \mathbb{R}_p(s)$ denote the (i, j)-entry of $\hat{D}^{-1} = \Theta_R L\Psi^{-1}$, where the pair (x_{ij}, y_{ij}) is coprime, $x_{ij}, y_{ij} \in \mathscr{R}_{\mathscr{U}}$, $y_{ij} \in \mathscr{I}$. Let $\theta_{Lj} \in \mathscr{I}$ be an lcm of all of the denominators $(y_{j1}, y_{j2},\cdots,y_{jno})$ in the jth row of $\hat{D}^{-1}$ and define

$$\Theta_L := \text{diag}[\theta_{L1} \cdots \theta_{Lno}]. \quad (2.20)$$

With these definitions, if P does not have any $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros, then $(\Delta_L\Delta_R, \Theta_L\Theta_R)$ is right coprime and $(\Theta_L\Theta_R, \Delta_L\Delta_R)$ is left coprime. □

Comments: The entries $\delta_{Lj} \in \mathscr{R}_{\mathscr{U}}$ of the diagonal matrix Δ_L in (2.17) have the same $\mathscr{U}$-zeros as those common to every entry in the jth row of $L\Lambda$. So Δ_L extracts those $\mathscr{U}$-zeros of P which appear in every entry of some row of the numerator $N_P = L[\Lambda \,\vdots\, 0_{no\times(ni-no)}]$. Similarly, Θ_R extracts those $\mathscr{U}$-poles of P which appear in every entry of some column of the denominator $\tilde{D}_P = \Psi L^{-1}$. Since $\delta_{Lj} \neq 0$ and $\theta_{Rj} \neq 0$, the matrices Δ_L and Θ_R are nonsingular.

The matrix $\hat{N}^{-1} = \Lambda^{-1}L^{-1}\Delta_L$ is not necessarily $\mathscr{R}_{\mathscr{U}}$-stable or may not even be proper. But postmultiplying $\hat{N}^{-1}$ by the diagonal matrix Δ_R makes the product $\mathscr{R}_{\mathscr{U}}$-stable. Similarly, $\hat{D}^{-1} =$

$\Theta_R \tilde{D}_P^{-1} = \Theta_R L \Psi^{-1}$ is not necessarily $\mathscr{R}_{\mathscr{U}}$-stable; it has to be proper since $\tilde{D}_P^{-1}$ is proper. But premultiplying $\hat{D}^{-1}$ by the diagonal matrix Θ_L makes the product $\mathscr{R}_{\mathscr{U}}$-stable. Since $\delta_{Rj} \neq 0$ and $\theta_{Lj} \neq 0$, the diagonal matrices Δ_R and Θ_R are also nonsingular.

If an rcf $N_P D_P^{-1}$ other than the one in (2.5) or an lcf $\tilde{D}_P^{-1}\tilde{N}_P$ other than the one in (2.6) are used for the plant P, then the diagonal matrices obtained following the same procedure as described in Lemma 2.2.2 to obtain the matrices $\Delta_L, \Delta_R, \Theta_R, \Theta_L$ would have diagonal entries differing only by units in $\mathscr{R}_{\mathscr{U}}$ from those of the matrices defined by (2.17)–(2.20). It can be shown easily that the entries of the diagonal matrices $\Delta_L, \Delta_R, \Theta_R, \Theta_L$ are unique within unit multiples in $\mathscr{T}$ [4]. □

Proof of Lemma 2.2.2: Since λ_j divides λ_{no} for $j = 1, \cdots, n_o - 1$, the matrix $\Lambda^{-1}\lambda_{no}$ is $\mathscr{R}_{\mathscr{U}}$-stable. By (2.17), $\hat{N}^{-1}(\Delta_L^{-1}\lambda_{no}) = \Lambda^{-1}L^{-1}\lambda_{no} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, where $\Delta_L^{-1}\lambda_{no} = \hat{N}\Lambda^{-1}L^{-1}\lambda_{no} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is diagonal, nonsingular. By Lemma 2.2.1-i) $\hat{N}^{-1}(\Delta_L^{-1}\lambda_{no}) \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ if and only if $(\Delta_L^{-1}\lambda_{no}) = \Delta_R Q_R$ for some diagonal, nonsingular $Q_R \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, i.e.,

$$\lambda_{no} I_{no} = \Delta_L \Delta_R Q_R. \tag{2.21}$$

Similarly, $\Psi^{-1}\psi_1$ is $\mathscr{R}_{\mathscr{U}}$-stable. By (2.19), $(\psi_1 \Theta_R^{-1})\hat{D}^{-1} = \psi_1 L \Psi^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, where $\psi_1 \Theta_R^{-1} = \psi_1 L \Psi^{-1} \hat{D} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is diagonal, nonsingular. By Lemma 2.2.1-ii) $(\psi_1 \Theta_R^{-1})\hat{D}^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ if and only if $(\psi_1 \Theta_R^{-1}) = Q_L \Theta_L$ for some diagonal, nonsingular $Q_L \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, i.e.,

$$\psi_1 I_{no} = Q_L \Theta_L \Theta_R. \tag{2.22}$$

By assumption, P does not have $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros, equivalently, (λ_{no}, ψ_1) is coprime, i.e., (2.11) holds for some $\alpha, \beta \in \mathscr{R}_{\mathscr{U}}$. By (2.21), (2.22), and (2.11)

$$\alpha \Delta_L \Delta_R Q_R + \beta Q_L \Theta_L \Theta_R = I_{no} \tag{2.23}$$

where all of the matrices in (2.23) are diagonal, $\mathscr{R}_{\mathscr{U}}$-stable and hence, $(\Delta_L \Delta_R, \Theta_L \Theta_R)$ is right coprime and $(\Theta_L \Theta_R, \Delta_L \Delta_R)$ is left coprime. □

Corollary 2.2.3: Under the assumption of Lemma 2.2.2, if P does not have any $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros, then the matrix $\hat{N}^{-1} W \hat{D}^{-1}$ is $\mathscr{R}_{\mathscr{U}}$-stable for some *diagonal, nonsingular*, $\mathscr{R}_{\mathscr{U}}$-stable matrix W if and only if $W = \Delta_R Q_D \Theta_L$ for some *diagonal, nonsingular*, $\mathscr{R}_{\mathscr{U}}$-stable matrix Q_D.

Proof: ($\Leftarrow$) By definition of Δ_R, $\hat{N}^{-1}\Delta_R \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, by definition of Θ_L, $\Theta_L \hat{D}^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. Therefore, if $W = \Delta_R Q_D \Theta_L$ for some *diagonal, nonsingular*, $\mathscr{R}_{\mathscr{U}}$-stable matrix Q_D, then $\hat{N}^{-1} W \hat{D}^{-1} = \hat{N}^{-1}\Delta_R Q_D \Theta_L \hat{D}^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$.

($\Rightarrow$) If $\hat{N}^{-1} W \hat{D}^{-1} =: M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ then $\hat{N}^{-1} W = M \hat{D} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ for some diagonal, nonsingular $W \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. By Lemma 2.2.1-i) $W = \Delta_R Q_R$ for some diagonal, nonsingular $Q_R \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. Since $W \hat{D}^{-1} = \hat{N} M \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, by Lemma 2.2.1-ii) $W = Q_L \Theta_L$ for some diagonal, nonsingular $Q_L \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. Therefore, $W = \Delta_R Q_R = Q_L \Theta_L$. By Lemma 2.2.2, (Θ_L, Δ_R) is left coprime, therefore, $Q_R = \Delta_R^{-1} Q_L \Theta_L = Q_L \Delta_R^{-1} \Theta_L \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ if and only if $Q_L \Delta_R^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ [8], [3]. *Hence*, $W = Q_L \Theta_L = \Delta_R(\Delta_R^{-1} Q_L)\Theta_L = \Delta_R Q_D \Theta_L$, where $Q_D := \Delta_R^{-1} Q_L \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ is diagonal and nonsingular. □

Theorem 2.2.4 (All Decoupling Controllers, all Achievable Decoupled Maps): Let $P \in \mathbb{R}_p(s)^{ni \times no}$ and let rank $P = n_o$. Let P have no $\mathscr{U}$-poles coinciding with $\mathscr{U}$-zeros. Let $\Delta_L, \Delta_R, \Theta_R, \Theta_L$ be defined by (2.17)–(2.20). Under these assumptions:

i) the set $\mathscr{A}_{\mathscr{D}}(P)$ of all decoupled input–output transfer functions H_{pc} is given by

$$\mathscr{A}_{\mathscr{D}}(P) = \left\{ H_{pc} = \alpha \lambda_{no} I_{no} + \Delta_L \Delta_R Q_D \Theta_L \Theta_R \,\middle|\, Q_D = \operatorname{diag}[q_1 \cdots q_{no}],\ q_j \in \mathscr{R}_{\mathscr{U}},\ q_j(\infty) \neq \frac{\beta\psi_1}{\delta_{Lj}\delta_{Rj}\theta_{Lj}\theta_{Rj}}(\infty),\ j = 1, \cdots, n_o \right\} \tag{2.24}$$

ii) the set $\mathscr{S}_{\mathscr{D}}(P)$ of all decoupling controllers for P is given by

$$\mathscr{S}_{\mathscr{D}}(P) = \left\{ C = R^{-1}\begin{bmatrix} \alpha\lambda_{no}\Lambda^{-1} + \Psi \hat{N}^{-1}\Delta_R Q_D \Theta_L \hat{D}^{-1} \\ Q_A \end{bmatrix} \cdot \hat{D}\Theta_R(\beta\psi_1 I_{no} - \Delta_L \delta_R Q_D \Theta_L \Theta_R)^{-1} \,\middle|\, Q_A \in \mathscr{R}_{\mathscr{U}}^{(ni-no)\times no},\ Q_D = \operatorname{diag}[q_1 \cdots q_{no}],\ q_j \in \mathscr{R}_{\mathscr{U}},\ q_j(\infty) \neq \frac{\beta\psi_1}{\delta_{Lj}\delta_{Rj}\theta_{Lj}\theta_{Rj}}(\infty),\ j = 1, \cdots, n_o \right\}. \ \square \tag{2.25}$$

Comments: In (2.24) and (2.25), for $j = 1, \cdots, n_o$, $q_j \in \mathscr{R}_{\mathscr{U}}$ satisfies

$$q_j(\infty) \neq \frac{\beta\psi_1}{\delta_{Lj}\delta_{Rj}\theta_{Lj}\theta_{Rj}}(\infty). \tag{2.26}$$

Condition (2.26) guarantees that the decoupling controllers are proper. If P is strictly proper, (2.26) is satisfied for *any* $q_j \in \mathscr{R}_{\mathscr{U}}$.

To ensure the nonsingularity of the achieved decoupled transfer functions $H_{pc} = \alpha\lambda_{no} I_{no} + \Delta_L \Delta_R Q_D \Theta_L \Theta_R = (1 - \beta\psi_1) I_{no} + \Delta_L \Delta_R Q_D \Theta_L \Theta_R$, $q_j \in \mathscr{R}_{\mathscr{U}}$ must also satisfy

$$q_j \neq \frac{-\alpha\lambda_{no}}{\delta_{Lj}\delta_{Rj}\theta_{Lj}\theta_{Rj}} = \frac{\beta\psi_1 - 1}{\delta_{Lj}\delta_{Rj}\theta_{Lj}\theta_{Rj}}. \tag{2.27}$$

However, as shown in the proof of Lemma 2.2.2, (2.22) implies that $(\theta_{Lj}\theta_{Rj})$ is not coprime with ψ_1 except when $\psi_1 = 1$ (equivalently, P is $\mathscr{R}_{\mathscr{U}}$-stable). Therefore, (2.27) is automatically satisfied for *all* $q_j \in \mathscr{R}_{\mathscr{U}}$ except when $P \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$, in which case, the additional condition that $q_j \neq 0$ should be included in the parameterizations as shown in Corollary 2.2.5 below. □

Proof of Theorem 2.2.4: By assumption, (λ_{no}, ψ_1) is coprime, hence, by Fact 2.1.3-vi) $H_{pc} \in \mathscr{A}(P)$ is *diagonal* if and only if $H_{pc} = \alpha\lambda_{no} I_{no} + L\Lambda Q \Psi L^{-1} = \alpha\lambda_{no} I_{no} + \Delta_L \hat{N} Q \hat{D} \Theta_R$ is diagonal for some $Q \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. But $\alpha\lambda_{no} I_{no}$ is diagonal and Δ_L and Θ_R are diagonal and nonsingular, therefore, H_{pc} is diagonal if and only if $\hat{N} Q \hat{D} =: W$ is diagonal for some $Q \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ [i.e., $Q = \hat{N}^{-1} W \hat{D}^{-1} \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$ for some diagonal matrix $W \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$]. By Corollary 2.2.3, $W = \Delta_R Q_D \Theta_L$, hence, H_{pc} is diagonal if and only if Q is of the form

$$Q = \hat{N}^{-1}\Delta_R Q_D \Theta_L \hat{D}^{-1} \tag{2.28}$$

for some diagonal $Q_D \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. With $Q_D =: \operatorname{diag}[q_1 \cdots q_{no}]$, $H_{pc} = \alpha\lambda_{no} I_{no} + \Delta_L \Delta_R Q_D \Theta_L \Theta_R$ is *nonsingular* if and only if $\alpha\lambda_{no} + \delta_{Lj}\delta_{Rj} q_j \theta_{Lj}\theta_{Rj} = 1 - \beta\psi_1 + \delta_{Lj}\delta_{Rj} q_j \theta_{Lj}\theta_{Rj} \neq 0$. By (2.22), for $j = 1, \cdots, n_o$, there exist $Q_{Lj} \in \mathscr{R}_{\mathscr{U}}$ such that $\psi_1 = Q_{Lj}\theta_{Lj}\theta_{Rj}$, therefore, if $\psi_1 \neq 1$ [equivalently, $P \notin \mathscr{M}(\mathscr{R}_{\mathscr{U}})$], then H_{pc} is nonsingular for all $q_j \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$. If $\psi_1 = 1$ [equivalently, $P \in \mathscr{M}(\mathscr{R}_{\mathscr{U}})$], then H_{pc} is nonsingular if and only if $q_j \neq 0$ (see Corollary 2.2.5).

The parameterization (2.25) is obtained from (2.15) by letting Q be as in (2.28). Since L is $\mathscr{R}_{\mathscr{U}}$-unimodular, in the parameterization (2.15), $\det(\beta\psi_1 I_{no} - \Lambda Q \Psi) \in \mathscr{I}$ if and only if $\det(L(\beta\psi_1 I_{no} - \Lambda Q\Psi)L^{-1}) = \det(I_{no} - H_{pc}) = \det(\beta\psi_1 I_{no} - \Delta_L \Delta_R Q_D \Theta_L \Theta_R) \in \mathscr{I}$; equivalently $(\beta\psi_1 - \delta_{Lj}\delta_{Rj} q_j \theta_{Lj} \theta_{Rj})(\infty) \neq 0$. □

Corollary 2.2.5 (Decoupling Controllers for $\mathscr{R}_{\mathscr{U}}$-Stable Plants): Let $P \in \mathscr{R}_{\mathscr{U}}^{ni \times no}$ and let rank $P = n_o$. Then the sets $\mathscr{A}_{\mathscr{D}}(P)$ and $\mathscr{S}_{\mathscr{D}}(P)$ become

$$\mathscr{A}_{\mathscr{D}}(P) = \Big\{ H_{pc} = \Delta_L \Delta_R Q_D | Q_D = \operatorname{diag}[q_1 \cdots q_{no}]$$

$$q_j \in \mathscr{R}_{\mathscr{U}} \setminus \{0\}, q_j(\infty) \neq \frac{1}{\delta_{Lj}\delta_{Rj}}(\infty), j = 1, \cdots, n_o \Big\} \tag{2.29}$$

$$\mathscr{S}_{\mathscr{D}}(P) = \Bigg\{ C = R^{-1} \begin{bmatrix} \hat{N}^{-1}\Delta_R Q_D \\ Q_A L^{-1} \end{bmatrix} (I_{no} - \Delta_L \Delta_R Q_D)^{-1} |$$

$$Q_A \in \mathscr{R}_{\mathscr{U}}^{(ni-no)\times no}$$

$$Q_D = \operatorname{diag}[q_1 \cdots q_{no}], q_j \in \mathscr{R}_{\mathscr{U}} \setminus \{0\},$$

$$q_j(\infty) \neq \frac{1}{\delta_{Lj}\delta_{Rj}}(\infty), j = 1, \cdots, n_o \Bigg\}. \tag{2.30}$$

Proof: When P is $\mathscr{R}_{\mathscr{U}}$-stable, without loss of generality, $\Psi = I_{no}$ and $\psi_1 = 1$, in (2.11); $\alpha = 0$ and $\beta = 1$. Then $\Theta_R = I_{no} = \Theta_L$ and $\hat{D} = L = \tilde{D}_P$. The parameterizations (2.29) and (2.30) follow by substituting these values of $\Theta_L, \Theta_R, \hat{D}$ into (2.24) and (2.25), where the additional constraint that $q_j \neq 0$ is imposed to ensure that H_{pc} is nonsingular. □

III. Conclusions

For LTI, MIMO, full-row rank plants which do not have any undesirable poles coinciding with zeros, we parameterized the class of all controllers such that the unity-feedback system is (internally) stable and the closed-loop transfer function from the command-input u_c to the plant-output y_p is diagonal and nonsingular. If the plant had undesirable poles coinciding with zeros, this class of controllers could not be used, in that case, two-degrees-of-freedom stabilizing decoupling controllers would be more useful since any full-row rank plant (which does not have any undesirable hidden modes) *can* be decoupled using two-parameter controllers [1], [3].

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Vibrational Control of Linear Time Lag Systems with Arbitrarily Large but Bounded Delays

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***Abstract*—This note shows that vibrational stabilization can be effective for linear systems with large bounded delays. Theorems are given that define the procedures for the search of the stabilizing vibrations. Robust oscillatory stabilization insensitive to the delay size is also shown to take place for some classes of systems.**

I. Introduction

A number of practically important systems, such as chemical reactors [1] and combustion systems [2] are best described by including time delays in their states. Feedback stabilization of such systems [3], [4] is usually not an easy task, especially if the delays are significant and there are restrictions on sensing and actuation. In recent papers [5], [6] an open-loop vibrational control technique introduced in [7] was shown to be effective for a class of systems with small delays in the states. While the results of [5] and [6] demonstrated the viability of the technique as a possible alternative to feedback for time lag systems as well as provided the tools for the synthesis of fast periodic feedback for this class of systems, the restriction that the delay size be small limited their practical utility. The purpose of the present note is to remove this restriction on the delay size. This is not a trivial task and it partially motivated development of new averaging theorems for differential delay equations [8]. The techniques presented can also be used for the synthesis of fast periodic feedback controllers for systems with large bounded time lags. The present note gives the conditions for the existence of the stabilizing vibrations for a class of linear time lag systems with arbitrary fixed bounded delays and presents the procedure for the search of the parameters of the stabilizing vibrations (Section II). The calculation formula for the choice of the parameters of the stabilizing vibrations, and the conditions for the vibrational stabilization to be insensitive to the delay size are also given for specific classes of systems (Section III). The results are supported by the numerical examples (Section IV). Conclusions are given in Section V.

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