STABILITY UNDER SENSOR OR ACTUATOR FAILURES

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Abstract
We consider the standard linear, time-invariant, multi-input multi-output unity-feedback system with possible failures in the sensor or actuator-connections. We parametrize the set of all controllers such that the closed-loop system is stable when sensors or actuators fail. We consider two classes of failures: the failure of one connection and the failure of any number of connections provided that at least one connection does not fail. The general parametrization requires knowledge of the failure to update the stabilizing controller. For plants which can be decoupled, we also give a class of controllers which ensure stability under any sensor or actuator failures without updating for different perturbations.

1. Introduction
A multivariable feedback system is said to have integrity if it remains stable in the presence of arbitrary failures of the sensor or actuator-connections. The problem of integrity is a robust stability problem. Designing controllers such that the system remains stable for a class of failures is equivalent to simultaneous stabilization of the nominal plant and the plant multiplied by different failure-matrices.

In this paper we consider the stability of the standard linear, time-invariant, multi-input multi-output unity-feedback system under possible failures in either the sensor or actuator-connections. In the standard problem of integrity, the failure of a sensor or actuator means that the controllers which ensure stability under any sensor or actuator failures without updating for different perturbations.

Let \( P \) and \( C \) denote the transfer-functions of the plant and the controller. If we consider the possibility of failures in the sensor-connections, then the system we consider is \( S(F_s, P, C) \) (shown in Figure 1), where the outputs of \( P \) are multiplied by arbitrary stable transfer functions. The sub-block \( F_s \) represents a stable diagonal matrix, whose diagonal entries are nominally equal to 1 (meaning that the corresponding sensor-connection did not fail). The matrix \( F_s \) belongs to the class \( \mathcal{F}_{m\times m} \), which denotes the set of all possible failures of at most \( k \) of the sensor-connections, where the maximum number of failures \( k \) is between 1 and the number of outputs \( n_u \). Similarly, the system with possible actuator-connection failures is \( S(P, F_a, C) \) (shown in Figure 2), where \( F_a \) represents the actuator-connections. If the \( j \)-th actuator-connection fails, then the \( j \)-th output of the controller is multiplied by a stable transfer-function, which is different than 1. The diagonal matrix \( F_a \) belongs to the class \( \mathcal{F}_{m\times m} \), which denotes the set of all possible failures of at most \( m \) of the actuator-connections, where the maximum number of failures \( m \) is between 1 and the number of inputs \( n_u \). If \( k = n_u \) or \( m = n_u \), then the plant and the controller must be stable; in this case the problem becomes the complete integrity problem. Otherwise, the plant and the controller need not be stable and instead of complete integrity, we deal with \( k \)-sensor-integrity or \( m \)-actuator-integrity.

The purpose of this paper is to develop a controller design methodology such that the systems \( S(F_s, P, C) \) and \( S(P, F_a, C) \) are stable. We consider two classes of failures: 1) There exists at most one failure; 2) there exists at least one channel without failure. The controller characterizations in the general parametrizations (Theorems 3.1 and 3.2) are not independent of the failures. But for plants which can be decoupled in some sense, we also find a class of fixed controllers that ensure stability of \( S(F_s, P, C) \) and \( S(P, F_a, C) \) without updating the controller when the failure-matrices change (Propositions 3.3 and 3.4).

2. Preliminaries
Let \( \mathcal{U} \) be a subset of the field \( \mathbb{C} \) of complex numbers; \( \mathcal{U} \) is closed and symmetric about the real axis, \( \pm \in \mathcal{U} \) and \( \mathbb{C} \setminus \mathcal{U} \) is nonempty. Let \( \mathbb{R}_u \), \( \mathbb{R}_u(s) \), \( \mathbb{R}_u(p) \), \( \mathbb{R}(s) \) be the ring of proper rational functions which have no poles in \( \mathcal{U} \), the ring of proper rational functions, the set of strictly proper rational functions and the field of rational functions of \( s \) (with real coefficients), respectively. Let \( \mathcal{J} \) be the group of units of \( \mathbb{R}_u \) and let \( I := \mathbb{R}_u \setminus \mathbb{R}_u(p) \). The set of matrices whose entries are in \( \mathbb{R}_u \) is \( \mathcal{M}(\mathbb{R}_u) \).
is \( R_U \)-unimodular iff \( \det M \in \mathcal{J} \).

Let \( \mathcal{F}_m \) denote the class of sensor failures defined as follows: If \( \mathcal{F}_m \in \mathcal{F}_m \), then \( \mathcal{F}_m = \text{diag} \{ f_1, \ldots, f_m \} \), where, for \( j = 1, \ldots, n_0 \), \( f_j \in \mathcal{R}_U \) and at least \((n_0 - k)\) of the entries \( f_j \in 1; k \) is the maximum number of sensor failures and \( f_j = 0 \) if the \( j \)-th sensor is disconnected. We are interested in the classes \( \mathcal{F}_m \) (the arbitrary failure of at most one the \( n_0 \) sensors) and \( \mathcal{F}_{(n_0 - 1)} \) (arbitrary failures of at most \((n_0 - 1)\) of the \( n_0 \) sensors). Similarly, \( \mathcal{F}_{am} \) denotes the class of actuator-connection failures defined by \( \mathcal{F}_{am} = \{ \text{diag} \{ f_1, \ldots, f_m \} \} \), where, for \( j = 1, \ldots, n_1 \), \( f_j \in \mathcal{R}_U \) and at least \((n_1 - m) \) of the entries \( f_j = 1; m \) is the maximum number of actuator failures and \( f_j = 0 \) if the \( j \)-th actuator is disconnected. Again the classes of interest here are \( \mathcal{F}_m \) and \( \mathcal{F}_{(n_0 - 1)} \), defined similarly.

In \( S(\mathcal{F}_m, P, C) \), \( \{ y_P, y_C \} = H_S \{ u_P, u_C \} \) and in \( S(P, \mathcal{F}_m, C) \), \( \{ y_P, y_C \} = H_P \{ u_P, u_C \} \).

Assumptions: i) The plant \( P \in \mathbb{R}_p(s)^{nxm} \). ii) The controller \( C \in \mathbb{R}_p(s)^{nxn} \). iii) The systems \( S(\mathcal{F}_m, P, C) \text{ and } (P, \mathcal{F}_m, C) \) are well-posed; equivalently, \( H_S \in \mathcal{M}(\mathbb{R}_p(s)) \text{ and } H_P \in \mathcal{M}(\mathbb{R}_p(s)) \). iv) \( P \text{ and } C \) have no hidden-\( U \)-modes.

Let \( P = N_P D^2 \) denote any right-coprime-factorization (rcf) and \( P = D^2 \bar{N}_P \) denote any left-coprime-factorization (lcf) of \( P \in \mathbb{R}_p(s)^{nxm} \), where \( N_P \in \mathcal{R}_d(s)^{nxm} \), \( N_P = N_P \mathcal{R}_r(s)^{nxm} \), \( D_P \in \mathcal{R}_d(s)^{nxn} \), \( D_P = D_P \mathcal{R}_r(s)^{nxn} \); det \( D_P \in \mathcal{T} \) (equivalently, det \( D_P \in \mathcal{T} \)) if and only if \( P \in \mathcal{M}(\mathbb{R}_p(s)) \). There exist \( V_P, U_P, \bar{V}_P, \bar{U}_P \in \mathcal{R}(\mathbb{R}_p(s)) \) such that \( V_P D_P + D^2 U_P = I_{n_1}, D_P \bar{V}_P + U_P \bar{U}_P = I_{n_0}, V_P \bar{U}_P = U_P V_P \).

Definitions: a) 1) \( S(\mathcal{F}_m, P, C) \) is said to be \( R_U \)-stable iff \( H_S \in \mathcal{M}(\mathbb{R}_u(s)) \). ii) For \( k = 1, \ldots, n_0 \), \( S(\mathcal{F}_m, P, C) \) is said to have \( k \)-sensor-integrity iff it is \( R_U \)-stable for all \( \mathcal{F}_m \in \mathcal{F}_{(n_0 - 1)} \). iii) \( P \) is said to have no \( k \)-sensor-failure hidden-\( U \)-modes if for all \( \mathcal{F}_m \in \mathcal{F}_m \), \( \text{rank} \{ D_P \} = n_0 \), for all \( s \in \mathcal{U} \). iv) \( C \) is a controller with \( k \)-sensor-integrity if \( C \in \mathbb{R}_p(s)^{nxn} \) and \( (S(\mathcal{F}_m, P, C) \) has \( k \)-sensor-integrity); the set \( S_{sm}(P) = \{ C \mid C \in \mathbb{R}_p(s)^{nxm} \text{ and } (S(\mathcal{F}_m, P, C) \) has \( k \)-sensor-integrity \} \) is called the set of all controllers with \( k \)-sensor-integrity. b) 1) \( S(P, \mathcal{F}_m, C) \) is said to be \( R_U \)-stable iff \( H_P \in \mathcal{M}(\mathbb{R}_u(s)) \). ii) For \( m = 1, \ldots, n_1 \), \( S(P, \mathcal{F}_m, C) \) is said to have \( m \)-actuator-integrity iff it is \( R_U \)-stable for all \( \mathcal{F}_m \in \mathcal{F}_{(n_0 - 1)} \). iii) \( P \) is said to have no \( m \)-actuator-failure hidden-\( U \)-modes if for all \( \mathcal{F}_m \in \mathcal{F}_m \), \( \text{rank} \{ D_P \} = n_1 \), for all \( s \in \mathcal{U} \). iv) \( C \) is a controller with \( m \)-actuator-integrity if \( C \in \mathbb{R}_p(s)^{nxn} \) and \( S(P, \mathcal{F}_m, C) \) has \( m \)-actuator-integrity; the set \( S_{am}(P) = \{ C \mid C \in \mathbb{R}_p(s)^{nxm} \text{ and } (S(P, \mathcal{F}_m, C) \) has \( m \)-actuator-integrity \} \) is called the set of all controllers with \( m \)-actuator-integrity.

3. Main Results

Consider \( S(\mathcal{F}_m, P, C) \). If it has \( k \)-sensor-integrity, then \( P \) has no \( k \)-sensor-failure hidden-\( U \)-modes [3]. Let \( \mathcal{F}_m \in \mathcal{F}_m \); \( P \) has no 1-sensor-failure hidden-\( U \)-modes if and only if there is a \( R_U \)-unimodular matrix \( L_k \) such that \( L_k \bar{D}_P = \begin{bmatrix} 1 & d_{1,2} & \ldots & d_{1,n_0} \\ 0 & d_{2,2} & \ldots & d_{2,n_0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{n_0,n_0} \end{bmatrix} \), where \( \begin{bmatrix} d_{1,i} & d_{1,i+1} \\ \vdots & \vdots \\ \vdots & \vdots \\ d_{i-1,i} & d_{i,i+1} \end{bmatrix} \) is right-coprime for \( j = 1, \ldots, n_0 \).

For \( j = 2, \ldots, n_0 \), \( \ell = 1, \ldots, j \), there exist \( \bar{y}_{,l} \in \mathcal{R}_U \) such that \( \sum_{i=1}^{\ell} \bar{y}_{,l} d_{ij} = 1 \). Let \( \bar{Y}_l := \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix} \), where (\( d_{j,0}, \ldots, d_{n_0,n_0} \)) is coprime for \( j = 1, \ldots, n_0 \).

Let \( \mathcal{F}_m \in \mathcal{F}_{(n_0 - 1)} \); \( P \) has no \((n_0 - 1)\)-sensor-failure hidden-\( U \)-modes if and only if there is an \( R_U \)-unimodular matrix \( L_{(n_0-1)} \) such that \( L_{(n_0-1)} \bar{D}_P = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix} \), where (\( d_{j,0}, \ldots, d_{n_0,n_0} \)) is left-coprime for \( j = 1, \ldots, n_0 \).

Not since \( S(\mathcal{F}_m, P, C) \). If it has \( m \)-actuator-integrity, then \( P \) has no \( m \)-actuator-failure hidden-\( U \)-modes [3]. Let \( \mathcal{F}_m \in \mathcal{F}_m \); \( P \) has no 1_actuator-failure hidden-\( U \)-modes if and only if there is an \( R_U \)-unimodular matrix \( R_1 \) such that \( \bar{D}_P R_1 = \begin{bmatrix} d_{1,1} & d_{1,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{n_1,n_1} & d_{n_1,n_2} & \ldots & 0 \end{bmatrix} \), where (\( d_{j,0}, \ldots, d_{n_0,n_0} \)) is left-coprime for \( j = 1, \ldots, n_0 \).
If $P$ is strictly proper, then for any $Q \in \mathcal{R}_d^{\text{nix-no}}$, \( \det(D_0 + N_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \). □

Figure 3 shows the block-diagram of the $\mathcal{R}_d$-stable system $S(F_d, P, C)$, where $C \in S_{S_d}(P)$.  

### 3.2 Theorem (all controllers with m-actuator-integrality)

Consider $S(P, F_a, C)$. Let $D_0^{-1} N_0 = N_0 D_0^{-1}$ be an $\mathcal{R}_d$-stabilizing controller for the (nominal) plant $P$. If $F_a \in \mathcal{F}_{21}$, let $P$ have no 1-actuator-failure hidden-U-modes; let $\hat{Y}_0$ be $\hat{Y}_0$ and let $M_a$ be $M_a$. If $F_a \in \mathcal{F}_{21}$, let $P$ have no $(n_1 - 1)$-actuator-failure hidden-U-modes; let $\hat{Y}_0$ be $\hat{Y}_0$ and let $M_a$ be $M_a$. The set $S_{S_{S_d}}(P)$ of all controllers with m-actuator-integrality $(m = 1$ or $(n_1 - 1)$) is:

\[
S_{S_{S_d}}(P) = \mathcal{C} =\]

\[
\{ (I_{n_1} - D_0 Y_0) M_0^{-1} N_0 + (F_a + D_0 Y_0 (I_{n_1} - F_a))^{-1} D_0 Q \}
\]

\[
\{ D_0 + N_0 M_0^{-1} N_0 - N_0 (I_m - Y_0 M_0^{-1} (I_{n_1} - F_a)) D_0 P \}
\]

\[
\{ \hat{Y}_0 + (I_m - \hat{Y}_0 \hat{D}_0) \hat{D}_0 F_0 - Q \hat{N}_P F_0 \} \in \mathbb{I} \}
\]

If $P$ is strictly proper, then for any $Q \in \mathcal{R}_d^{\text{nix-no}}$, \( \det(D_0 + N_0 M_0^{-1} N_0 - N_0 (I_m - Y_0 M_0^{-1} (I_{n_1} - F_a)) D_0 P) \sim \det(\hat{Y}_0 + (I_m - \hat{Y}_0 \hat{D}_0) \hat{D}_0 F_0 - Q \hat{N}_P F_0) \in \mathbb{I} \). □

Figure 4 shows the block-diagram of the $\mathcal{R}_d$-stable system $S(P, F_a, C)$, where $C \in S_{S_{S_d}}(P)$.

In Proposition 3.3 below, assume that the transfer-function $H_{\text{fp}} : u_c \mapsto y_p$ can be made diagonal using an $\mathcal{R}_d$-stabilizing controller; i.e., there exists $C_{SP}$ such that $H_{\text{fp}} = P C_{SP} (I_m + P C_{SP})^{-1}$ is diagonal for the given (nominal) plant $P$. Then, $C_{SP} = (V_p - Q S_{SP} V_p)^{-1} (U_p + Q S_{SP}) = (U_p + D_p Q S_{SP}) (V_p - Q S_{SP} V_p)^{-1}$, where $S_{SP} \in \mathcal{R}_d^{\text{nix-no}}$ is such that $H_{\text{fp}} = N_0 (U_p + Q S_{SP} D_0)$ is diagonal. The class of plants for which the transfer-function $H_{\text{fp}}$ can be diagonalized is not empty; a sufficient condition for the existence of such controllers is that the plant $P$ is full-rank and does not have any U-poles coinciding with zeros (in this case the transfer-function $H_{\text{fp}}$ can be made diagonal and nonsingular). Let $C_{SP} = D^{\text{sp}} S_{SP}$ be any controller which diagonalizes $H_{\text{fp}}$; the parametrization of all such decoupling controllers is given in [5] for full-rank plants without pole-zero coincidences in $U$.

Similarly, in Proposition 3.4, assume that the transfer-function $H_{\text{fp}} : u_p \mapsto y_c$ can be made diagonal using an $\mathcal{R}_d$-stabilizing controller; i.e., there exists $C_{DP} \in S_{SP}(P)$ such that $H_{\text{fp}} = -C_{DP} (I_m + P C_{DP})^{-1}$ is diagonal for the given (nominal) plant $P$. Then $C_{DP} = (V_p - Q S_{DP} V_p)^{-1} (U_p + Q S_{DP}) = (U_p + D_p Q S_{DP}) (V_p - Q S_{DP} V_p)^{-1}$, where $S_{DP} \in \mathcal{R}_d^{\text{nix-no}}$ is such that $H_{\text{fp}} = - (U_p + D_p Q S_{DP}) \hat{N}_P$ is diag-

\[
\begin{align*}
&\det(D_0 + \hat{N}_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \\
&\sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \).
\end{align*}
\]


\[
\begin{align*}
&\det(D_0 + N_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \\
&\sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \).
\end{align*}
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\begin{align*}
&\det(D_0 + N_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \\
&\sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \).
\end{align*}
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\begin{align*}
&\det(D_0 + N_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \\
&\sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \).
\end{align*}
\]

\[
\begin{align*}
&\det(D_0 + N_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \\
&\sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \).
\end{align*}
\]

\[
\begin{align*}
&\det(D_0 + N_0 M_0^{-1} \hat{Y}_0 \hat{N}_P - Q(\hat{Y}_0 - \hat{D}_0 \hat{P}(I_m - F_0) M_0^{-1} \hat{Y}_0) \hat{N}_P) \\
&\sim \det(\hat{Y}_0 + F_0 D_0(I_m - \hat{D}_0 \hat{P}) + F_0 \hat{N}_P Q) \in \mathbb{I} \).
\end{align*}
\]
The class of plants for which the transfer-function $H_P$ can be diagonalized is not empty; a sufficient condition for the existence of such controllers is that the plant $P$ is full column-rank and does not have any $U$-poles coinciding with zeros (in this case the transfer-function $H_P$ can be made diagonal and nonsingular). Let $C_{AD} = D_{AD}^{-1} \tilde{N}_{AD} = N_{AD} D_{AD}^{-1}$ be any controller which diagonalizes $H_P$.

### 3.3. Proposition (controllers with $k$-sensor-integrity for decoupled plants):

Consider $S(F_s, P, C)$. Assume that the conditions of Theorem 3.1 hold. Suppose that there exists an $R_u$-stabilizing decoupling controller $C_{SD}$ for the (nominal) plant $P$, such that the transfer-function $H_P$ is diagonal. Under these conditions, a class of controllers with $k$-sensor-integrity ($k = 1$ or $(n_s - 1)$) is given by

$$
\{ C = (D_{SD} + N_{SD} Y_s \tilde{N}_{SP})^{-1} \tilde{N}_{SP} (I_{n_s} - Y_s D_P) \\
= (N_{SD} - D_P \tilde{N}_{SD} \tilde{Y}_s)(D_{SD} + N_P \tilde{N}_{SD} \tilde{Y}_s)^{-1} \}
$$

where $C_{SD} = D_{AD}^{-1} \tilde{N}_{AD}$ is any $R_u$-stabilizing decoupling controller for the (nominal) plant $P$ such that $\det(D_{SD} + N_{SD} Y_s \tilde{N}_{SP}) \approx \det(D_{SD} + N_P \tilde{N}_{SD} \tilde{Y}_s) \in I$.

Figure 5 shows the block-diagram of the $R_u$-stable system $S(F_s, P, C)$ using the controller in Proposition 3.3.

### 3.4. Proposition (controllers with $m$-actuator-integrity for decoupled plants):

Consider $S(P, F_A, C)$. Assume that the conditions of Theorem 3.2 hold. Suppose that there exists an $R_u$-stabilizing decoupling controller $C_{AD}$ for the (nominal) plant $P$, such that the transfer-function $H_P$ is diagonal. Under these conditions, a class of controllers with $m$-actuator-integrity ($m = 1$ or $(n_s - 1)$) is given by

$$
\{ C = (D_{AD} + Y_m N_{AD} \tilde{N}_{SP})^{-1} (\tilde{N}_{AD} - Y_m N_{AD} D_P) \\
= (I_{n_s} - D_P Y_m) N_{AD} (D_{AD} + N_P Y_m N_{AD})^{-1} \}
$$

where $C_{SD} = D_{AD}^{-1} \tilde{N}_{AD}$ is any $R_u$-stabilizing decoupling controller for the (nominal) plant $P$ such that $\det(D_{AD} + Y_m N_{AD} \tilde{N}_{SP}) \approx \det(D_{AD} + N_P Y_m N_{AD}) \in I$.

Figure 6 shows the block-diagram of the $R_u$-stable system $S(P, F_A, C)$ using the controller in Proposition 3.4.

### References


