Proceedings of the 28th Conference
on Decision and Control
Tampa, Florida • December 1989

# CONDITIONS FOR STABILITY OF FEEDBACK SYSTEMS UNDER SENSOR FAILURES

A. Nazli Gündeş
Department of Electrical Engineering and Computer Science
University of California, Davis, CA 95616

M. Güntekin Kabuli
Integrated Systems Inc.
2500 Mission College Blvd.
Santa Clara, CA 95054

## ABSTRACT

We derive conditions for the closed-loop stability of the linear, time-invariant, multiinput-multioutput unity-feedback system under sensor failures. We find compensators that achieve stability under sensor failures for a class of plants.

## 1. INTRODUCTION

For any given plant (which has no unstable hidden modes), there exists proper compensators such that the linear, time-invariant (lti), multiinput-multiioutput (MIMO) unity-feedback system is internally stable. In this paper, we find proper compensators for a class of plants allowing one or several of the sensor connections fail. We do not allow all of the sensor connections to fail; this would require that the plant is stable (see [Des.1, Fuj.1]).

**Notation:** $\mathcal{U}$ is a closed, nonempty subset of $\mathbb{C}$; $\mathcal{U}$ is symmetric about the real axis and $\mathbb{C}\setminus\mathcal{U}$ is nonempty. $\bar{\mathcal{U}} := \mathcal{U}\cup\{\infty\}$. $\mathcal{R}_{\mathcal{U}}$ denotes the ring of proper scalar rational functions of $s$ (with real coefficients) which have no poles in $\mathcal{U}$. $\mathcal{J}$ denotes the group of units of $\mathcal{R}_{\mathcal{U}}$. $\mathbb{R}_{\mathrm{p}}(s)$ denotes the ring of proper rational functions; $\mathbb{R}_{\mathrm{sp}}(s)$ denotes the set of strictly proper rational functions. $\mathcal{I}$ denotes the set of non-strictly proper elements of $\mathcal{R}_{\mathcal{U}}$. $\mathcal{M}(\mathcal{R}_{\mathcal{U}})$ denotes the set of matrices whose entries are in $\mathcal{R}_{\mathcal{U}}$. A matrix $A\in\mathcal{M}(\mathcal{R}_{\mathcal{U}})$ is $\mathcal{R}_{\mathcal{U}}$-unimodular iff $\det A\in\mathcal{J}$.

## 2. SYSTEM DESRIPTION AND ANALYSIS

Consider the lti, MIMO feedback system $S(F_S, P, C)$ (Figure 1), where $P: e\mapsto y$, $C:\hat{e}\mapsto\hat{y}$ and $(I_{no}-F_S): y\mapsto y_S$ represent the plant, , the compensator and the sensor connections, respectively. The entries of the diagonal matrix $F_S$ are 1s and 0s; the $j$-th entry is 1 if the $j$-th connection fails and 0 otherwise. Let $H_S: \begin{bmatrix} u\\ \hat{u}\end{bmatrix}\mapsto\begin{bmatrix} y\\ \hat{y}\end{bmatrix}$ denote the closed-loop input-output (I/O) map of $S(F_S, P, C)$.

**2.1 Assumptions:** **i)** $P\in\mathbb{R}_{\mathrm{p}}(s)^{no\times ni}$; **ii)** $C\in\mathbb{R}_{\mathrm{p}}(s)^{ni\times no}$; **iii)** $S(F_S,P,C)$ is well-posed, i.e., $H_S\in\mathcal{M}(\mathbb{R}_{\mathrm{p}}(s))$; **iv)** $P$ and $C$ have no hidden-modes associated with eigenvalues in $\bar{\mathcal{U}}$.

**2.2 Closed-loop I/O maps:** Let Assumptions 2.1 hold. Let $Q_S := C(I_{no}+(I_{no}-F_S)PC)^{-1}$. The I/O map $H_S$ is $H_S =: \begin{bmatrix} (I_{no}-PQ_S(I_{no}-F_S))P & PQ_S\\ -Q_S(I_{no}-F_S)P & Q_S\end{bmatrix}$.

**2.3 Definition ( $\mathcal{R}_{\mathcal{U}}$-stability ):** The system $S(F_S,P,C)$ is said to be $\mathcal{R}_{\mathcal{U}}$-stable iff $H_S\in\mathcal{M}(\mathcal{R}_{\mathcal{U}})$.

**2.4 Analysis:** Let $(N_P, D_P)$ be any right-coprime-fraction representation (rcfr) of $P$; i.e., let $N_P\in\mathcal{R}_{\mathcal{U}}^{no\times ni}$, $D_P\in\mathcal{R}_{\mathcal{U}}^{ni\times ni}$, $\det D_P\in\mathcal{I}$, $P=N_PD_P^{-1}$ and let $V_P$, $U_P\in\mathcal{M}(\mathcal{R}_{\mathcal{U}})$ be such that $V_PD_P+U_PN_P=I_{ni}$. Let $(\widetilde{D}_P,\widetilde{N}_P)$ be any left-coprime-fraction representation (lcfr) of $P$; i.e., let $\widetilde{N}_P\in\mathcal{R}_{\mathcal{U}}^{no\times ni}$, $\widetilde{D}_P\in\mathcal{R}_{\mathcal{U}}^{no\times no}$, $\det\widetilde{D}_P\in\mathcal{I}$, $P=\widetilde{D}_P^{-1}\widetilde{N}_P$ and let $\widetilde{V}_P$, $\widetilde{U}_P\in\mathcal{M}(\mathcal{R}_{\mathcal{U}})$ be such that $\widetilde{V}_P\widetilde{D}_P+\widetilde{U}_P\widetilde{N}_P=I_{no}$. Similarly, let $(N_C, D_C)$ be any rcfr and $(\widetilde{D}_C,\widetilde{N}_C)$ be any lcfr of $C$. Let $\xi_C$ denote the pseudo-state of $C$; using $D_C\xi_C=\hat{e}$, $N_C\xi_C=\hat{y}$, $y=Pe=\widetilde{D}_P^{-1}\widetilde{N}_Pe$, $y_S=(I-F_S)y$, $\hat{e}=\hat{u}-y_S$ and $e=u+\hat{y}$, the system $S(F_S,P,C)$ is described as:

$$\begin{bmatrix}\widetilde{D}_P & -\widetilde{N}_PN_C\\ (I-F_S) & D_C\end{bmatrix}\begin{bmatrix} y\\ \xi_C\end{bmatrix} = \begin{bmatrix}\widetilde{N}_P & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} u\\ \hat{u}\end{bmatrix},$$
$$\begin{bmatrix} I & 0\\ 0 & N_C\end{bmatrix}\begin{bmatrix} y\\ \xi_C\end{bmatrix} = \begin{bmatrix} y\\ \hat{y}\end{bmatrix}.$$

**2.5 Theorem ( $\mathcal{R}_{\mathcal{U}}$-stability under failures):** Let Assumptions 2.1 hold. Let $(\widetilde{D}_P,\widetilde{N}_P)$ be any lcfr of $P$ and $(N_C, D_C)$ be any rcfr of $C$. Then $S(F_S,P,C)$ is $\mathcal{R}_{\mathcal{U}}$-stable if and only if $D_{HS} := \begin{bmatrix}\widetilde{D}_P & -\widetilde{N}_PN_C\\ (I-F_S) & D_C\end{bmatrix}$ is $\mathcal{R}_{\mathcal{U}}$-unimodular. □

If $S(F_S,P,C)$ is $\mathcal{R}_{\mathcal{U}}$-stable, then $((I-F_S),\widetilde{D}_P)$ is a right-coprime (rc) pair since $D_{HS}$ is $\mathcal{R}_{\mathcal{U}}$-unimodular.

## 3. STABILIZABILITY UNDER FAILURES

We derive necessary conditions on the plant $P$ for the existence of stabilizing compensators in the presence of sensor failures. We consider two classes of sensor failures: **i)** only one but any one of the sensor connections may fail; this class is described by $\mathcal{F}_{S1} := \{F_S=\mathrm{diag}[f_1\ f_2\ldots f_{no}]\mid f_j=1 \text{ or } 0; f_jf_k=0, \text{for } k\neq j\}$; **ii)** any number of the sensor connections, but not all $n_o$ of them may fail; this class is described by $\mathcal{F}_{Sm} := \left\{F_S=\mathrm{diag}[f_1\ f_2\ldots f_{no}]\mid f_j=1 \text{ or } 0,\ \prod_{j=1}^{n_o}f_j=0\right\}$.

**3.1 Definition ( $\mathcal{R}_{\mathcal{U}}$-stabilizing compensator ):** A compensator $C$ is said to $\mathcal{R}_{\mathcal{U}}$-stabilize the plant $P\in\mathbb{R}_{\mathrm{p}}(s)^{no\times ni}$ iff $C\in\mathbb{R}_{\mathrm{p}}(s)^{ni\times no}$ and the system $S(F_S,P,C)$ is $\mathcal{R}_{\mathcal{U}}$-stable.

**3.2 Theorem ($\mathcal{R}_{\mathcal{U}}$-stability under failures):** Let $P\in\mathbb{R}_{\mathrm{p}}(s)^{no\times ni}$; let $(N_P, D_P)$ and $(\widetilde{D}_P,\widetilde{N}_P)$, be any rcfr and lcfr of $P$. Then the following necessary conditions hold: **i)** If there is a compensator which $\mathcal{R}_{\mathcal{U}}$-stabilizes $P$ for all $F_S\in\mathcal{F}_{S1}$, then there is an $\mathcal{R}_{\mathcal{U}}$-unimodular matrix $L_1\in\mathcal{R}_{\mathcal{U}}^{no\times no}$ such that

$$L_1\widetilde{D}_P = \begin{bmatrix} d_{1,1} & 0 & 0 & \ldots & 0 & 0\\ d_{2,1} & d_{2,2} & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ d_{no-1,1} & d_{no-1,2} & d_{no-1,3} & \ldots & d_{no-1,no-1} & 0\\ d_{no,1} & d_{no,2} & d_{no,3} & \ldots & d_{no,no-1} & 1\end{bmatrix}, \quad (3.1)$$

where, for $j=1,\ldots,n_o-1$, the pair

$$\left( d_{j,j}\ ,\ \begin{bmatrix} d_{j+1,j}\\ \vdots\\ d_{no,j}\end{bmatrix}\right) \text{ is right-coprime.} \quad (3.2)$$

**ii)** If there is a compensator which $\mathcal{R}_{\mathcal{U}}$-stabilizes $P$ for all $F_S\in\mathcal{F}_{Sm}$, then there is an $\mathcal{R}_{\mathcal{U}}$-unimodular matrix $L_m\in\mathcal{R}_{\mathcal{U}}^{no\times no}$ such that

$$L_m\widetilde{D}_P = \begin{bmatrix} d_{1,1} & 0 & 0 & \ldots & 0\\ d_{2,1} & 1 & 0 & \ldots & 0\\ d_{3,1} & 0 & 1 & \ldots & 0\\ \vdots & \vdots & & \ddots & 0\\ d_{no,1} & 0 & 0 & \ldots & 1\end{bmatrix}, \quad (3.3)$$

where, for $j=2,\ldots,n_o$, the pair

$$(\ d_{1,1}\ ,\ d_{j,1}\ ) \text{ is coprime.} \quad (3.4)$$

**3.3 Comments:** **i)** If there is a compensator which $\mathcal{R}_{\mathcal{U}}$-stabilizes $P=\widetilde{D}_P^{-1}\widetilde{N}_P$ for all $F_S\in\mathcal{F}_{S1}$, then an lcfr of $P$ is given by $(L_1\widetilde{D}_P, L_1\widetilde{N}_P)$, where (3.1)-(3.2) hold. Condition (3.2) implies that each *column* of the denominator matrix $L_1\widetilde{D}_P$ is full rank for all $s\in\bar{\mathcal{U}}$. The $n_o$-th diagonal entry of $L_1\widetilde{D}_P$ is $d_{no,no}=1$;

CH2642-7/89/0000-1688$1.00 

for $j = 1, \dots, n_o - 1$, $d_{j,j} \in \mathcal{I}$. For $j = 1, \dots, n_o$, there are $v_{j,k} \in \mathcal{R_U}$, where $k = j, \dots, n_o$, such that

$$\sum_{k=j}^{no} v_{j,k}\, d_{k,j} \;=\; v_{j,j}\, d_{j,j} \;+\; \sum_{k=j+1}^{no} v_{j,k}\, d_{k,j} \;=\; 1 \,. \tag{3.5}$$

Note that $d_{no,no} = 1$ implies that $v_{no,no} = 1$.

$$\text{Let } Y_1 := \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} & \dots & v_{1,no-1} & v_{1,no} \\ 0 & v_{2,2} & v_{2,3} & \dots & v_{2,no-1} & v_{2,no} \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & v_{no-1,no-1} & v_{no-1,no} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} L_1 \,.$$

Let $X_1 := I_{no} - Y_1 \widetilde{D}_P$ ; then,

$$Y_1\, \widetilde{D}_P \;+\; X_1\,(\, I_{no} \;-\; F_S\,) = I_{no} \;-\; X_1\, F_S \,, \tag{3.6}$$

where $(\, I_{no} \;-\; X_1\, F_S\,) \in \mathcal{M}(\mathcal{R_U})$ is $\mathcal{R_U}$-unimodular for all $F_S \in \mathcal{F}_{S1}$ since only one diagonal entry of $F_S$ may be equal to $1$.
**ii)** If there is a compensator which $\mathcal{R_U}$-stabilizes $P = \widetilde{D}_P^{-1}\widetilde{N}_P$ for all $F_S \in \mathcal{F}_{Sm}$, then an lcfr of $P$ is given by $(\, L_m\, \widetilde{D}_P \,,\, L_m\, \widetilde{N}_P \,)$, where (3.3)-(3.4) hold. Condition (3.4) implies that for $j = 2, \dots, n_o$, rank $\begin{bmatrix} d_{1,1} \\ d_{j,1} \end{bmatrix} = 1$ for all $s \in \bar{\mathcal{U}}$ (i.e., $d_{j,1}(s) \neq 0$, for all $s \in \bar{\mathcal{U}}$ where $d_{1,1}(s) = 0$). Note that $d_{1,1} \in \mathcal{I}$. For $j = 2, \dots, n_o$, there are $y_j$ , $y_{j,1} \in \mathcal{R_U}$ such that

$$y_j\; d_{1,1} \;+\; y_{j,1}\; d_{j,1} \;=\; 1 \,. \tag{3.7}$$

Let $Y_m :=$

$$\begin{bmatrix} y_2 & y_{2,1} & 0 & 0 & \dots & 0 & 0 \\ -d_{2,1}y_3 & 1 & -d_{2,1}y_{3,1} & 0 & \dots & 0 & 0 \\ -d_{3,1}y_4 & 0 & 1 & -d_{3,1}y_{4,1} & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots & \\ -d_{no-1,1}y_{no} & 0 & 0 & 0 & \dots & 1 & -d_{no-1,1}y_{no,1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} L_m$$

Let $X_m := I_{no} - Y_m \widetilde{D}_P$ ; then

$$Y_m\, \widetilde{D}_P \;+\; X_m\,(\, I_{no} \;-\; F_S\,) = I_{no} \;-\; X_m\, F_S \,, \tag{3.8}$$

where $(\, I_{no} \;-\; X_m\, F_S\,) \in \mathcal{M}(\mathcal{R_U})$ is $\mathcal{R_U}$-unimodular for all $F_S \in \mathcal{F}_{Sm}$ since $\prod_{j=1}^{no} f_j = 0$.

## 4. COMPENSATOR DESIGN

Let $P \in \mathbb{R}_{\mathrm{p}}(s)^{no \times ni}$ have rank $= n_o$ and have no $\mathcal{U}$-poles that coincide with $\mathcal{U}$-zeros. Then there exist $\mathcal{R_U}$-unimodular matrices $L \in \mathcal{R_U}^{no \times no}$, $R \in \mathcal{R_U}^{ni \times ni}$ such that (the Smith-form of $P$ is)

$$P \;=\; L\,\Lambda\,\Psi^{-1}\,R \;=\; L\,\tilde{\Psi}^{-1}\,\Lambda\,R \,, \tag{4.1}$$

where $\Lambda = \left[\mathrm{diag}[\lambda_1 \;\dots\; \lambda_{no}] \;\vdots\; 0_{no \times (ni-no)}\right]$, $\tilde{\Psi} = \mathrm{diag}[\psi_1 \;\dots\; \psi_{no}]$, and $\Psi = \mathrm{diag}\left[\tilde{\Psi} \,,\, I_{(ni-no)}\right]$ ; for $j = 1, \dots, n_o$, the pair $(\, \lambda_j \,,\, \psi_j\,)$ is coprime; for $j = 1, \dots, n_o - 1$, $\lambda_j$ divides $\lambda_{j+1}$ and $\psi_{j+1}$ divides $\psi_j$. Now rank $P = n_o$ implies that $\lambda_{no} \neq 0$. Furthermore, $P$ has no $\mathcal{U}$-poles coinciding with $\mathcal{U}$-zeros if and only if

$$(\; \lambda_{no} \;,\; \psi_1 \;) \;\text{ is a coprime pair;} \tag{4.2}$$

equivalently, there exist $\alpha$, $\beta \in \mathcal{R_U}$ such that, for all $q \in \mathcal{R_U}$,

$$\hat{\alpha}\; \lambda_{no} + \hat{\beta}\; \psi_1 \;:=\; (\alpha + q\; \psi_1)\lambda_{no} + (\beta - q\; \lambda_{no})\psi_1 = 1 \,. \tag{4.3}$$

An rcfr $(\, N_P \,,\, D_P\,)$ and an lcfr $(\, \widetilde{D}_P \,,\, \widetilde{N}_P\,)$ of $P$ is given by: $(\, N_P \,,\, D_P\,) := (\, L\Lambda \,,\, R^{-1}\Psi\,)$, $(\, \widetilde{D}_P \,,\, \widetilde{N}_P\,) := (\, \tilde{\Psi} L^{-1} \,,\, \Lambda R\,)$.

Let $U := \begin{bmatrix} \mathrm{diag}\left[\; \hat{\alpha}\lambda_{no}/\lambda_1 \quad \hat{\alpha}\lambda_{no}/\lambda_2 \;\dots\; \hat{\alpha}\lambda_{no}/\lambda_{no-1} \;\; \hat{\alpha} \;\right] \\ 0_{(ni-no) \times no} \end{bmatrix}$,
$\tilde{V} \;:=\; \mathrm{diag}\left[\; \hat{\beta} \quad \hat{\beta}\psi_1/\psi_2 \quad \dots \quad \hat{\beta}\psi_1/\psi_{no-1} \quad \hat{\beta}\psi_1/\psi_{no} \;\right]$,
$V := \mathrm{diag}\left[\tilde{V} \,,\, I_{(ni-no)}\right]$. A right-Bezout identity $V_P D_P + U_P N_P = I_{ni}$ for the rcfr $(\, N_P \,,\, D_P\,) = (\, L\Lambda \,,\, R^{-1}\Psi\,)$ is given by $U_P := U\,L^{-1}$, $V_P := V\,R$. Note that $N_P\,U_P = L\Lambda\,U\,L^{-1} = \hat{\alpha}\,\lambda_{no}\,I_{no}$.

**4.1 Corollary:** Under the assumptions of Theorem 3.2 and assuming that rank $P = n_o$ and that (4.2) holds, we have the following necessary conditions: **i)** If there is a compensator which $\mathcal{R_U}$-stabilizes $P$ for all $F_S \in \mathcal{F}_{S1}$, then by (3.1)-(3.2), the smallest invariant factor $\psi_{no}$ of $\widetilde{D}_P$ is 1 ( $\det \widetilde{D}_P \sim \prod_{j=1}^{no-1} \psi_j$ ). **ii)** If there is a compensator which $\mathcal{R_U}$-stabilizes $P$ for all $F_S \in \mathcal{F}_{S1}$, then by (3.3)-(3.4), the invariant factors $\psi_2, \dots, \psi_{no}$ of $\widetilde{D}_P$ are all 1 except for the largest one $\psi_1$ ( $\det \widetilde{D}_P \sim \psi_1$ ).

**4.2 Proposition ( $\mathcal{R_U}$-stabilizing compensator design):** Let $P \in \mathbb{R}_{\mathrm{p}}(s)^{no \times ni}$ ; let rank $P = n_o$ and let (4.2) hold. Let $(\, N_P \,,\, D_P\,)$ be any rcfr and $(\, \widetilde{D}_P \,,\, \widetilde{N}_P\,)$ be any lcfr of $P$.
**i)** Suppose that there is an $\mathcal{R_U}$-unimodular matrix $L_1 \in \mathcal{R_U}^{no \times no}$ such that (3.1)-(3.2) hold. Then $C = \widetilde{D}_C^{-1}\widetilde{N}_C = (\,VR \;+\; UL^{-1}Y_1\,\widetilde{N}_P\,)^{-1}UL^{-1}X_1$ is a compensator which $\mathcal{R_U}$-stabilizes $P$ for all $F_S \in \mathcal{F}_{S1}$, where $q \in \mathcal{R_U}$ is such that

$$\det(\, I_{no} - (\alpha + q\psi_1)\lambda_{no} X_1(\infty)\,) = \det(\, I_{no} - \hat{\alpha}\lambda_{no} X_1(\infty)\,) \neq 0 \,; \tag{4.4}$$

**ii)** Suppose that there exists an $\mathcal{R_U}$-unimodular matrix $L_m \in \mathcal{R_U}^{no \times no}$ such that (3.3)-(3.4) hold. Then $C = \widetilde{D}_C^{-1}\widetilde{N}_C = (\,VR \;+\; UL^{-1}Y_m\,\widetilde{N}_P\,)^{-1}UL^{-1}X_m$ is a compensator which $\mathcal{R_U}$-stabilizes $P$ for all $F_S \in \mathcal{F}_{Sm}$, where $q \in \mathcal{R_U}$ is such that

$$\det(\, I_{no} - (\alpha + q\psi_1)\lambda_{no} X_m(\infty)\,) = \det(\, I_{no} - \hat{\alpha}\lambda_{no} X_m(\infty)\,) \neq 0 \,.\square \tag{4.5}$$

Note that (4.4) and (4.5) hold automatically if $q \in \mathcal{R_U}$ is such that $q(\infty) = -\,\alpha(\infty)/\psi_1(\infty)$. If $P \in \mathcal{M}(\mathbb{R}_{\mathrm{sp}}(s))$, then $\lambda_{no} \in \mathbb{R}_{\mathrm{sp}}(s) \cap \mathcal{R_U}$ ; in this case, (4.4) and (4.5) hold *for all* $q \in \mathcal{R_U}$.

## 5. CONCLUSIONS

We considered the closed-loop stability of the unity-feedback system under two classes of sensor connection failures. The actuator-failure case is similar and omitted for brevity. If there exist compensators that $\mathcal{R_U}$-stabilize the given plant for all failures in these classes, then the denominator matrices of coprime factorizations of the plant must satisfy certain conditions. We found a set of compensators that $\mathcal{R_U}$-stabilize a class of MIMO plants under sensor failures.

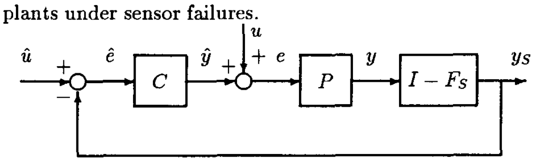


Figure 1. The system $S(\, F_S \,, P \,, C\,)$

## REFERENCES


[Des.1] C. A. Desoer, A. N. Gündeş, "Stability under sensor or actuator failures," *Proc. Conference on Decision and Control*, pp. 2148-2149, 1988.

[Fuj.1] M. Fujita, E. Shimemura, "Integrity against arbitrary feedback-loop failure in linear multivariable control systems," *Proc. 10th IFAC World Congress*, 1987.

[Vid.1] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.

[Vid.2] M. Vidyasagar, N. Viswanadham, "Algebraic design techniques for reliable stabilization," *IEEE Trans. Automatic Control*, AC-27, pp. 1085-1095, 1982.