Controller Design for Plants With Internal Delayed Feedback

A. N. Gündeş and Hitay Özbay

***Abstract*—A special class of retarded and neutral time delay systems is considered. These are plants with internal delayed feedback, and they may have finitely many or infinitely many unstable poles. Stabilizing controllers are obtained from a particular interpolation. A parametrization of all stabilizing integral-action controllers is obtained. Examples are given to illustrate this simple design procedure and its robustness properties for various uncertainties.**

***Index Terms*—Control design, delay systems, robust stability, stability analysis, uncertain system.**

I. Introduction

A simple stabilizing controller design method is proposed for a class of systems with delayed feedback. For a given feedforward transfer function

$$G(s)=\frac{n(s)}{d(s)}=\frac{\sum_{i=1}^{m_\tau} n_i(s)e^{-\tau_i s}}{d(s)} \quad \tau_i \geq 0 \tag{1}$$

with a monic polynomial $d(s)$, and $\deg n_i(s) \leq \deg d(s)$ for $i \in \{1,\dots,m_\tau\}$, the plant to be controlled is in the form

$$P(s)=\frac{G(s)}{1+G(s)W(s)} \tag{2}$$

where $W \in \mathscr{H}_\infty$ (see Fig. 1). Without loss of generality it is assumed that the time delays in $n(s)$ are ordered as $\tau_{i+1} > \tau_i$. In general, W is in the form

$$W(s)=\frac{\sum_{i=1}^{m_h} u_i(s)e^{-h_i s}}{v(s)} \quad h_i > 0 \tag{3}$$

where $v(s)$ is a Hurwitz polynomial, $\deg u_i(s) \leq \deg v(s)$ for $i \in \{1,\dots,m_h\}$. In many practical applications

$$W(s)=ke^{-hs} \tag{4}$$

where the feedback gain $k \neq 0$, and time delay $h > 0$ determine stability of the plant depending on the characteristics of the transfer function G. Clearly, the class of systems considered here as (2) corresponds to plants represented by

$$P(s)=\frac{v(s)\sum_{i=1}^{m_\tau} n_i(s)e^{-\tau_i s}}{d(s)v(s)+\sum_{\ell=1}^{m_\kappa} d_\ell(s)e^{-\kappa_\ell s}}. \tag{5}$$

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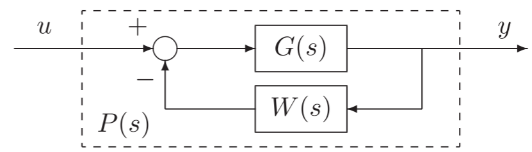

Fig. 1. Plant with internal feedback.

In (5) n_i's and d_ℓ's are polynomials satisfying certain degree conditions and τ_i's and κ_ℓ's are time delays. The plant model (2) appears in various physical systems, e.g., in structures where there is acoustic feedback. The so-called "feedback noise" appearing in amplified sound systems is the natural response of such a system. A similar structure appears in a model of cavity flow oscillations [23]. Another example for (2) is Kalecki's classical model of investment decisions and delivery of capital goods [17], [28]. Perhaps the most interesting example of the plant (2) is the AQM design for TCP flow control in computer networks [9], [21]. Many other application examples are in [12], [24], and references in.

Since the main problem is *feedback stabilization*, it is assumed that the plant is *unstable* (otherwise, the solution is trivial). Typically, coprime factorizations lead to stabilizing controllers [17]. The usual approach obtains factorizations by computing the locations of the finitely many right half plane poles and zeros, and this is possible due to software packages such as QPmR [27] and YALTA [1]. Obviously, controller design has to take into account the effects of numerical precision error in the root computation. In this article, a special coprime factorization is obtained directly from the unstable poles of $G(s)$ instead of the unstable poles of $P(s)$. Therefore, this approach eliminates the need for computation of roots for quasi-polynomials.

If G is strictly proper, then P is a retarded type delay system; then P can have at most finitely many unstable poles [18]. On the other hand, if G and W are biproper maps, then P is a neutral delay system, which may have infinitely many unstable poles, e.g., [8], [19]. Stabilization of plants in the form (2), alternatively (5), has been widely discussed, see [12], [19], and also the recent works [14], [17] for details and additional references. Usually, state-space-based methods require finding state feedback and observer gains (operators in the infinite dimensional space) [3], [4], [16], [22]. Algebraic methods are used for spectrum assignment [6], [11], [12]. In this article, we propose a simple interpolation-based method in line with the factorization approach used in some earlier works [10], [17], [19]. But the method here is much simpler in that interpolations are at the poles of G rather than the poles of P.

The article is organized as follows. Section II defines the problem. All stabilizing controllers and all integral-action controllers are derived in Section III. Stability robustness under uncertainty in the feedback gain and delay is analyzed via examples in Section IV. Conclusions are in Section V.

Notation: $\mathbb{C}$ denotes complex numbers. The closed right-half-plane (RHP) is $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \Re e(s) \geq 0\}$, the open left-half-plane is $\mathbb{C}_- = \{s \in \mathbb{C} \,|\, \Re e(s) < 0\}$. The instability region is the extended RHP, $\mathbb{C}_{+e} = \mathbb{C}_+ \cup \{\infty\}$. Real and positive real numbers are $\mathbb{R}$, $\mathbb{R}_+$;

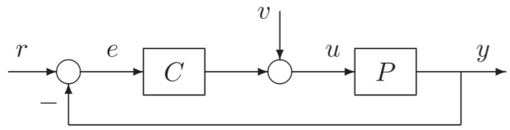

Fig. 2. Unity-feedback system $\mathcal{S}(C, P)$.

$\mathcal{R}_p$ denotes real proper rational functions of s. The space $\mathscr{H}_\infty$ is the set of all bounded analytic functions in $\mathbb{C}_+$. For $f \in \mathscr{H}_\infty$, the norm $\|\cdot\|$ is defined as $\|f\| := \operatorname{ess\,sup}_{s\in\mathbb{C}_+}|f(s)|$, where ess sup is the essential supremum. The polynomial degree is denoted by $\deg(d)$. We drop (s) in transfer functions such as $P(s)$.

II. Problem Definition

Consider a plant in the form (2) to be stabilized by a controller C via standard unity negative feedback (see Fig. 2).

The feedback system $\mathcal{S}(C, P)$ is stable, hence C stabilizes P, if $S := (1 + PC)^{-1}$, PS, CS are in $\mathscr{H}_\infty$. If P has finitely many poles and zeros in the extended RHP $\mathbb{C}_{+e}$, then stabilizing controllers can be derived by finding a function $S \in \mathscr{H}_\infty$ satisfying interpolation conditions $S(z_j) = 1$ and $S(p_i) = 0$ for all zeros z_j and poles p_i in $\mathbb{C}_{+e}$ (considering multiplicities as additional interpolation conditions) [5], [20]. For a large class of infinite dimensional plants, the parametrization of all stabilizing controllers are given in [25] using strongly coprime factorizations (solutions of a Bézout equation). For neutral time delay systems in the form (2), conditions under which the Bézout equation is solvable are discussed in [10], [19]. In this article, a particular interpolation-based stabilizing controller is given, considering a coprime factorization in the form

$$N(s) = \frac{n(s)}{\theta(s)},\ D(s) = \frac{d(s)}{\theta(s)},\ G(s) = D^{-1}N \tag{6}$$

$$P(s) = \frac{G}{1 + GW} = \frac{N(s)}{D(s) + N(s)W(s)} \tag{7}$$

n, d are as defined in (1) and θ is a Hurwitz polynomial, with $\deg(\theta) = \deg(d)$. It will be clear in Section III that the roots of θ appear in the set of closed-loop poles. There are many other important factors in the selection of θ as shown in the robust stability analysis of Section IV. Moreover, we assume that $\tau_1 = 0$. Otherwise, $P(s)$ can be written as $e^{-\tau_1 s}P_o(s)$, where P_o satisfies the above assumptions. Then, stabilizing controllers for P can be determined by finding a controller for P_o and using a predictor-based structure [13].

III. Controller Design Based on Interpolation

A. Design Based on RHP Poles

Let $p_1, \ldots p_\gamma \in \mathbb{C}_+$, with multiplicities $k_1, \ldots k_\gamma$, respectively, be the γ distinct RHP poles of $G(s)$, $\gamma \geq 1$. Define $\Phi(s) \in \mathscr{H}_\infty$ as

$$\Phi(s) := \prod_{i=1}^{\gamma}(1 - N(p_i)^{-1}N)^{k_i}. \tag{8}$$

Then the maps $Y := D^{-1}\Phi$ and $X_o := N^{-1}(1 - \Phi)$ are in $\mathscr{H}_\infty$, equivalently, $G\Phi \in \mathscr{H}_\infty$, and $G^{-1}(1 - \Phi) \in \mathscr{H}_\infty$. If $G(s)$ has $\mathbb{C}_+$-poles with nonzero imaginary parts, i.e., $\Re e(p_\ell) \geq 0$, $\Im m(p_\ell) \neq 0$ for some $\ell \in \{1, \ldots, \gamma\}$, then the terms in Φ corresponding to complex-conjugate pairs of poles $p_\ell, \overline{p}_\ell \in \mathbb{C}_+$ are: $(1 - N(p_\ell)^{-1}N)(1 - N(\overline{p}_\ell)^{-1}N) = 1 - 2\frac{\Re e(N(p_\ell))}{|N(p_\ell)|^2}N + \frac{1}{|N(p_\ell)|^2}N^2$.

Lemma 1: Let $G = D^{-1}N$, and $P = G/(1 + GW)$ as in (7). Then C is a stabilizing controller for P if and only if it is given by (9), where C_g is a stabilizing controller for G

$$C = C_g - W. \tag{9}$$

With Y, X_o, $X \in \mathscr{H}_\infty$ defined as (10), all stabilizing controllers C_g for G are given by (11)

$$Y := D^{-1}\Phi,\ X_o := N^{-1}(1 - \Phi),\ X := (X_o - YW) \tag{10}$$

$$C_g = (Y - NQ)^{-1}(X_o + DQ). \tag{11}$$

Then all stabilizing controllers C for P are given as follows:

$$C = \frac{X + (D + NW)Q}{Y - NQ} = \frac{N^{-1}(1 - \Phi) + DQ}{D^{-1}\Phi - NQ} - W \tag{12}$$

where $Q \in \mathscr{H}_\infty$, $Q(\infty) \neq Y(\infty)N^{-1}(\infty)$. □

Proof of Lemma 1: Let $X_gY_g^{-1}$, X_g, $Y_g \in \mathscr{H}_\infty$, be any coprime factorization of C_g. Then C_g stabilizes $G = D^{-1}N$ if and only if $X_gN + Y_gD = 1$, equivalently $(X_g - Y_gW)N + Y_g(D + NW) = 1$. This identity is equivalent to $(X_g - Y_gW)Y_g^{-1} = C_g - W$ stabilizes $(D + NW)^{-1}N = P$. The expression (11) of all controllers C_g for G is obtained from the coprime pair (X_o, Y) [25], [26]. The expression (12) of the controllers follows by using (11) in $C = C_g - W$. □

Proposition 1: Controllers based on RHP poles of G.

Let $\Phi(s) \in \mathscr{H}_\infty$ be as in (8), Y, X_o, $X \in \mathscr{H}_\infty$ as in (10).

1) The controller C_o in (13) stabilizes the plant P in (7)

$$C_o = Y^{-1}X = G^{-1}(\Phi^{-1} - 1) - W. \tag{13}$$

For $Q \in \mathscr{H}_\infty$ such that $Q(\infty) \neq Y(\infty)N(\infty)^{-1}$, all stabilizing controllers C are parameterized as follows:

$$C = \frac{X + QD(1 + GW)}{Y - QN} = G^{-1}\left(\frac{\Phi^{-1}}{1 - QND\Phi^{-1}} - 1\right) - W. \tag{14}$$

2) *Integral-Action Controllers:* Assume that $P(0) \neq 0$, i.e., $N(0) \neq 0$. Define $K \in \mathbb{R}$, X_I, $Y_I \in \mathscr{H}_\infty$ as follows:

$$K := (N^{-1}Y)(0) \tag{15}$$

$$X_I := X + KD(1 + GW)\quad Y_I := Y - KN. \tag{16}$$

The integral-action controller C_{Io} in the following stabilizes P:

$$C_{Io} = \frac{X_I}{Y_I} = G^{-1}\frac{[1 - \Phi + KND]}{\Phi - KND} - W. \tag{17}$$

For any arbitrary $\beta > 0$, let $F_\beta := \dfrac{s}{s + \beta}$. and for $Q_I \in \mathscr{H}_\infty$ such that $Q_I(\infty) \neq Y_I(\infty)N(\infty)^{-1}$, all stabilizing integral-action controllers C_I are parameterized as follows:

$$C_I = \frac{X_I + Q_IF_\beta D(1 + GW)}{Y_I - Q_IF_\beta N} = \frac{C_o + [K + Q_IF_\beta]D(1 + GW)Y^{-1}}{1 - [K + Q_IF_\beta]NY^{-1}}. \tag{18}$$

Proof of Proposition 1:

1) The controller C_o in (13) stabilizes P in (7) since $NX + (D + NW)Y = NN^{-1}(1 - \Phi) - NWY + (D + NW)Y = 1$. All stabilizing controllers follow from $N[X + QD(1 + GW)] + (D + NW)[Y - QN] = 1$. The condition $Q(\infty) \neq Y(\infty)N(\infty)^{-1}$ makes C proper; it is satisfied for all $Q \in \mathscr{H}_\infty$ if $G(s)$ is strictly proper.
2) The controller C_{Io} in (17) stabilizes P since $NX_I + (D + NW)Y_I = 1$. With $K \in \mathbb{R}$ as (15), $Y_I(0) = (Y - KN)(0) = 0$, and $(Y_I - Q_IF_\beta N)(0) = 0$ for all $Q_I \in \mathscr{H}_\infty$. Therefore, all C_I in (18) have integral-action. The condition $Q_I(\infty) \neq$

$Y(\infty)N(\infty)^{-1}$ makes C_I proper; it is satisfied for all $Q_I \in \mathscr{H}_\infty$ if $G(s)$ is strictly proper. □

Remark 1: 1) With the controller C_o given in (13), the sensitivity transfer function $S_o = (1+PC_o)^{-1}$ is

$$S_o = Y(D+NW) = \Phi(1+GW). \tag{19}$$

The integral-action controller C_{Io} in (17) gives the sensitivity transfer function $S_{Io} = (1+PC_{Io})^{-1}$

$$S_{Io} = (\Phi - KND)(1+GW). \tag{20}$$

With $K \in \mathbb{R}$ as (15), $S_{Io}(0) = 0$ implies that the closed-loop steady-state step reference tracking error is zero with C_{Io}. 2) If $G(s) \in \mathscr{H}_\infty$, then $\Phi = 1$ in (8). The polynomial θ can be chosen as $\theta = d$, which implies $D = 1, N = G, Y = 1, X_o = 0$. Proposition 1 is simplified as follows. The controller in (13) becomes $C_o = -W$, and (14) becomes

$$C = \frac{-W + Q(1+GW)}{1-QG}. \tag{21}$$

The integral-action controllers in (17) and (18) are simplified similarly for $G \in \mathscr{H}_\infty$. With $K = G(0)^{-1}$, (18) becomes

$$C_I = \frac{-W + (K+Q_IF_\beta)(1+GW)}{1-(K+Q_IF_\beta)G}. \tag{22}$$

B. Design Based on RHP Zeros

The main advantage of controller design based on interpolations at the unstable poles of G is that these are finitely many (the roots of the polynomial $d(s)$ in $\mathbb{C}_+$). The proposed design is applicable to the cases where $n(s)$ in (1) is a quasi-polynomial, which may have infinitely many roots in the RHP. On the other hand, when $n(s)$ is a polynomial, with low number of zeros in $\mathbb{C}_+$, it may be advantageous to use a dual design based on interpolation conditions at these zeros.

Let $z_1, \ldots z_\mu \in \mathbb{C}_+$, with multiplicities $m_1, \ldots m_\mu$, respectively, be the μ distinct RHP zeros of $G(s)$, where $\mu \geq 0$. In addition to these finite $\mathbb{C}_+$-zeros, $G(s)$ may have ρ zeros at infinity, $\rho \geq 0$. Define $\Psi(s) \in \mathscr{H}_\infty$ as

$$\Psi := (1-D)^\rho \prod_{i=1}^{\mu} (1 - D(z_i)^{-1}D)^{m_i}. \tag{23}$$

Then the maps $N^{-1}\Psi$ and $D^{-1}(1-\Psi)$ are in $\mathscr{H}_\infty$, equivalently, $G^{-1}\Psi \in \mathscr{H}_\infty$, and $G(1-\Psi) \in \mathscr{H}_\infty$. Note that $\Psi(\infty) = \prod_{i=1}^{\mu}(1 - D(z_i)^{-1})^{m_i}$ if $\rho = 0$, and $\Psi(\infty) = 0$ if $\rho \geq 1$. If $G(s)$ has $\mathbb{C}_+$-zeros with nonzero imaginary parts, i.e., $N(z_i) = 0, \Re e(z_i) \geq 0, \Im m(z_i) \neq 0$ for some $i \in \{1, \ldots, \mu\}$, then the terms in Ψ corresponding to complex-conjugate zeros $z_i, \overline{z}_i \in \mathbb{C}_+$ are: $(1 - D(z_i)^{-1}D)(1 - D(\overline{z}_i)^{-1}D) = 1 - 2\frac{\Re e(D(z_i))}{|D(z_i)|^2} D + \frac{1}{|D(z_i)|^2} D^2$.

Proposition 2: Controllers based on RHP zeros of G.

With $\Psi(s) \in \mathscr{H}_\infty$ defined in (23), define $\tilde{X}, \tilde{Y} \in \mathscr{H}_\infty$ as

$$\tilde{Y} := D^{-1}(1-\Psi), \quad \tilde{X} := N^{-1}\Psi - \tilde{Y}W. \tag{24}$$

1) The controller $\widetilde{C}_o$ in (25) stabilizes the plant P in (7)

$$\widetilde{C}_o = \tilde{Y}^{-1}\tilde{X} = G^{-1}[(1-\Psi)^{-1} - 1] - W. \tag{25}$$

For $Q \in \mathscr{H}_\infty$ such that $Q(\infty) \neq \tilde{Y}(\infty)N(\infty)^{-1}$, all stabilizing controllers $\widetilde{C}$ are parameterized as follows:

$$\begin{aligned}\widetilde{C} &= [\tilde{Y} - QN]^{-1}[\tilde{X} + QD(1+GW)] \\ &= G^{-1}\left(\frac{(1-\Psi)^{-1}}{1-QND(1-\Psi)^{-1}} - 1\right) - W.\end{aligned} \tag{26}$$

2) *Integral-Action Controllers:* Assume that $P(0) \neq 0$, i.e., $N(0) \neq 0$. Define $\tilde{K} \in \mathbb{R}$, $\tilde{X}_I$ $\tilde{Y}_I \in \mathscr{H}_\infty$ as follows:

$$\tilde{K} := (N^{-1}\tilde{Y})(0) \tag{27}$$

$$\tilde{X}_I := \tilde{X} + \tilde{K}D(1+GW) \quad \tilde{Y}_I := \tilde{Y} - \tilde{K}N. \tag{28}$$

The integral-action controller $\widetilde{C}_{Io}$ in the following stabilizes P:

$$\widetilde{C}_{Io} = \frac{\tilde{X}_I}{\tilde{Y}_I} = G^{-1}\frac{[\Psi + \tilde{K}ND]}{1 - \Psi - \tilde{K}ND} - W. \tag{29}$$

With F_β as in Proposition 1, and for $Q_I \in \mathscr{H}_\infty$ such that $Q_I(\infty) \neq \tilde{Y}(\infty)N(\infty)^{-1}$, all stabilizing integral-action controllers $\widetilde{C}_I$ are parameterized as in the following:

$$\begin{aligned}\widetilde{C}_I &= \frac{\tilde{X}_I + Q_IF_\beta D(1+GW)}{\tilde{Y}_I - Q_IF_\beta N} \\ &= \frac{\widetilde{C}_o + [\tilde{K} + Q_IF_\beta]D(1+GW)\tilde{Y}^{-1}}{1 - [\tilde{K} + Q_IF_\beta]N\tilde{Y}^{-1}}.\end{aligned} \tag{30}$$

Proof of Proposition 2: 1) The controller $\widetilde{C}_o$ in (13) stabilizes P in (7) since $N\tilde{X} + (D+NW)\tilde{Y} = N\Psi N^{-1} - NW\tilde{Y} + (D+NW)\tilde{Y} = 1$. All stabilizing controllers follow from $N[\tilde{X}+QD(1+GW)]+(D+NW)[\tilde{Y}-QN]=1$. The condition $Q(\infty) \neq \tilde{Y}(\infty)N(\infty)^{-1}$ makes $\widetilde{C}$ proper; it is satisfied for all $Q \in \mathscr{H}_\infty$ if $G(s)$ is strictly proper. 2) The controller $\widetilde{C}_{Io}$ in (29) stabilizes P since $N\tilde{X}_I + (D+NW)\tilde{Y}_I = 1$. With $\tilde{K} \in \mathbb{R}$ as (27), $\tilde{Y}_I(0) = (\tilde{Y} - \tilde{K}N)(0) = 0$, and $(\tilde{Y}_I - Q_IF_\beta N)(0) = 0$ for all $Q_I \in \mathscr{H}_\infty$. Therefore, all $\widetilde{C}_I$ in (30) have integral-action. The condition $Q_I(\infty) \neq \tilde{Y}(\infty)N(\infty)^{-1}$ makes $\widetilde{C}_I$ proper; it is satisfied for all $Q_I \in \mathscr{H}_\infty$ if $G(s)$ is strictly proper. □

Remark 2: 1) With the controller $\widetilde{C}_o$ given in (25), the sensitivity transfer function $S_o = (1+P\widetilde{C}_o)^{-1}$ is

$$S_o = \tilde{Y}(D+NW) = (1-\Psi)(1+GW). \tag{31}$$

The integral-action controller $\widetilde{C}_{Io}$ in (29) gives the sensitivity transfer function $S_{Io} = (1+P\widetilde{C}_{Io})^{-1}$

$$S_{Io} = (1 - \Psi - \tilde{K}ND)(1+GW). \tag{32}$$

With $\tilde{K} \in \mathbb{R}$ as in (27), $S_{Io}(0) = 0$ implies that the closed-loop steady-state step reference tracking error is zero.

2) If $G(s)^{-1} \in \mathscr{H}_\infty$, i.e., G has no zeros in $\mathbb{C}_+ \cup \{\infty\}$, then a coprime factorization of the plant P in (7) is

$$P = (G^{-1} + W)^{-1}. \tag{33}$$

In this case, $\Psi = 1$ in (23). Proposition 2 is simplified as follows: For $Q(\infty) \neq 0$ ($Q = 0$ is not a valid choice), the parametrization in (26) becomes

$$\widetilde{C} = -[Q^{-1} + G^{-1} + W]. \tag{34}$$

For $Q_I(\infty) \neq 0$, all integral-action controllers in (30) are

$$\widetilde{C}_I = -\left[\frac{(s+\beta)}{s}Q_I^{-1} + G^{-1} + W\right]. \tag{35}$$

IV. Examples With Performance and Robustness Analysis

A. Examples

Example 1: Consider the plant

$$P(s) = \frac{1}{s + ke^{-hs}},\ W = ke^{-hs},\ G(s) = \frac{1}{s},\ k \in \mathbb{R}. \tag{36}$$

1) First, apply the design procedure of Proposition 1. The only $\mathbb{C}_+$-pole of G is $p_1 = 0$. Choosing $\theta = (s+\alpha)$, $\alpha \in \mathbb{R}_+$, we have $\Phi = \frac{s}{s+\alpha}$. By (14), for any $Q \in \mathscr{H}_\infty$

$$C = (\alpha - ke^{-hs} + Q\frac{(s+ke^{-hs})}{(s+\alpha)})(1 - Q\frac{1}{(s+\alpha)})^{-1}. \tag{37}$$

With $K = N(0)^{-1} = \alpha$, by (18), for any $Q_I \in \mathscr{H}_\infty$

$$C_I = \frac{\frac{\alpha^2}{s} + 2\alpha - ke^{-hs} + Q_I\frac{(s+ke^{-hs})}{(s+\beta)}}{1 - Q_I\frac{1}{(s+\beta)}}. \tag{38}$$

2) For this G, the design procedure of Proposition 2 gives the same controllers $\widetilde{C} = C$, and $\widetilde{C}_I = C_I$. □

Example 2: With $W = ke^{-hs}$, $k \in \mathbb{R}$, consider the plant

$$P(s) = \frac{s-1}{s+3+ke^{-hs}(s-1)} \quad G(s) = \frac{s-1}{s+3}. \tag{39}$$

1) First, apply the design procedure of Proposition 1. In this case, $G \in \mathscr{H}_\infty$; therefore, $\Phi = 1$. Choose $\theta = (s+\alpha)$, $\alpha \in \mathbb{R}_+$. By (14), for any $Q \in \mathscr{H}_\infty$ satisfying $Q(\infty) \neq 1$

$$C = \frac{-ke^{-hs} + Q\frac{(s+3)}{(s+\alpha)}[\frac{(s+3)}{(s+\alpha)} + ke^{-hs}\frac{(s-1)}{(s+\alpha)}]}{1 - Q\frac{(s-1)(s+3)}{(s+\alpha)^2}}. \tag{40}$$

With $K = -\alpha^2/3$, by (17), an integral-action controller is

$$C_{Io} = \frac{-\alpha^2(s+3)^2}{s[(\alpha^2+3)s + 2\alpha(\alpha+3)]} - ke^{-hs}. \tag{41}$$

2) Now apply the design in Proposition 2 with the same θ. By (26), for any $Q \in \mathscr{H}_\infty$ satisfying $Q(\infty) \neq (\alpha+1)/4$

$$\widetilde{C} = \frac{\frac{3-\alpha}{\alpha+1} - ke^{-hs} + Q\frac{4}{(\alpha+1)}[\frac{(s+3)}{(s+\alpha)} + ke^{-hs}\frac{(s-1)}{(s+\alpha)}]}{1 - Q\frac{4(s-1)}{(\alpha+1)(s+\alpha)}}. \tag{42}$$

With $\tilde{K} = -\alpha(\alpha+1)/4$, by (29)

$$\widetilde{C}_{Io} = \frac{(3-\alpha^2-2\alpha)s - 4\alpha^2}{(\alpha+1)^2\, s} - ke^{-hs}. \tag{43}$$

If we choose $\alpha = 3$, then the controllers in (40) of Proposition 1 and in (42) of Proposition 2 are the same

$$C = \widetilde{C} = \frac{-W + Q(1+GW)}{1-QG}. \tag{44}$$

Similarly, for $\alpha = 3$, the integral-action controller C_{Io} in (41) becomes the same as $\widetilde{C}_{Io}$ in (43). □

Example 3: With $W = ke^{-hs}$, $k = 1$, consider the plant

$$P(s) = \frac{s-1}{s(s-2)+e^{-hs}(s-1)} \quad G(s) = \frac{s-1}{s(s-2)}. \tag{45}$$

1) First, apply the design in Proposition 1. Choose $\theta = (s^2 + 9s + 20)$; then $\Phi = s(s-2)(s+29)(s-31)/\theta^2$. By (13)

$$C_o(s) = \frac{2(11\,s^2 + 519\,s - 200)}{(s+29)(s-31)} - e^{-hs}. \tag{46}$$

With C_o in (46), the transfer function from r to y (complementary sensitivity) is $T(s) = \frac{(s-1)}{\theta^2}[2(11\,s^2 + 519\,s - 200) - e^{-sh}(s+29)(s-31)]$. With $K = 899$, by (17)

$$C_{Io}(s) = \frac{921\,s^2 - 760\,s - 400}{s(s-901)} - e^{-hs}. \tag{47}$$

With the integral-action controller C_{Io} in (47), $T(s) = \frac{(s-1)}{\theta^2}[921\,s^2 - 760\,s - 400 - e^{-hs}s(s-901)]$.

2) Now apply the design in Proposition 2 with the same θ. Then $\Psi = (s-1)(11s+20)(31s-20)/\theta^2$. By (25)

$$\widetilde{C}_o(s) = \frac{(11s+20)(31s-20)}{(s^2-321s-580)} - e^{-hs}. \tag{48}$$

With $\widetilde{C}_o$ in (48), $T(s) = \frac{(s-1)}{\theta^2}[(11s+20)(31s-20) - e^{-hs}(s^2 - 321s - 580)]$. With $\tilde{K} = 580$, the integral-action controllers $\widetilde{C}_{Io}$ (29) and C_{Io} (17) are the same here for this choice of θ. Therefore, the complementary sensitivity $T(s)$ and the step response are the same with both controllers. □

B. Performance Analysis

Using the controller parametrization given by (14), for $Q \in \mathscr{H}_\infty$ the closed loop sensitivity function is $S = (1+PC)^{-1} = (D + NW)(Y - NQ)$, where $Y = D^{-1}\Phi \in \mathscr{H}_\infty$. Given a sensitivity shaping function W_s, say in the form

$$W_s(s) = \gamma\frac{(s+\delta)}{(s+\tau)} \tag{49}$$

with fixed corner frequencies $\tau \gg \delta > 0$, we want to find the smallest $\gamma > 0$ such that there exists $Q \in \mathscr{H}_\infty$ resulting in $|S(j\omega)| \leq |W_s(j\omega)|$ for all ω. For this purpose, first find a low order minimum phase W_G such that $|W_G(j\omega)| \geq |D(j\omega) + N(j\omega)W(j\omega)|$, $\forall\,\omega$. Then define $W_1(s)$ as in (50) and solve the one-block problem (51)

$$W_1(s) = W_G(s)\frac{(s+\tau)}{(s+\delta)} \tag{50}$$

$$\gamma_o = \inf_{Q \in \mathscr{H}_\infty} \|W_1(Y - NQ)\|_\infty. \tag{51}$$

When $N(s) = n(s)/\theta(s)$ is finite dimensional, so is $Y = D^{-1}\Phi$ and hence, the problem above can be solved easily. In case $n(s)$ is a quasi-polynomial with finitely many roots in the RHP, an inner–outer factorization of N can be done, and the one block problem can still be solved using the Nevanlinna–Pick interpolation (see e.g., [17]).

Consider P in Example 3; taking $\theta(s) = (s+4)(s+5)$ as before, we have $N(s) = \frac{(s-1)}{\theta}$, $D(s) = \frac{s(s-2)}{\theta}$, $Y(s) = \frac{(s+29)(s-31)}{\theta}$. Let $W(s) = e^{-hs}$, $h = 0.15$ sec. Fig. 3 shows $|W_G(j\omega)|$ and $|D(j\omega) + N(j\omega)e^{-jh\omega}|$ for $W_G(s) = 1.04\,\frac{(s+0.35)}{(s+6.5)}$.

Now fix $\delta = 10^{-4}$ and define $W_s = \gamma(s+\delta)/(s+\tau)$ as (49), and $W_1(s) = W_G(s)(s+\tau)/(s+10^{-4})$ as (50). Since N has a single RHP zero at $s = 1$, the one-block problem (51) to minimize the weighted sensitivity γ_o is $\gamma_o = |W_1(1)Y(1)| = |\frac{W_1(1)}{D(1)}| \approx 5.616(\tau+1)$. This means that the smallest achievable weighted sensitivity is less than $|W_s(j\omega)| = 5.616\,(\tau+1)\,\frac{|j\omega+10^{-4}|}{|j\omega+\tau|}$, which is shown in Fig. 4 for different τ values:

For $\tau = 0.5$ the optimal Q solving the one-block problem (51) is given as Q_o in (52). Since N is strictly proper we used a filter $(0.001s + 1)^{-1}$ to make Q_o proper

$$Q_o(s) \approx \frac{9.1(s+15.66)(s^2 - 0.9914\,s + 1.103)}{(0.001s+1)(s+0.5)(s+0.35)}. \tag{52}$$

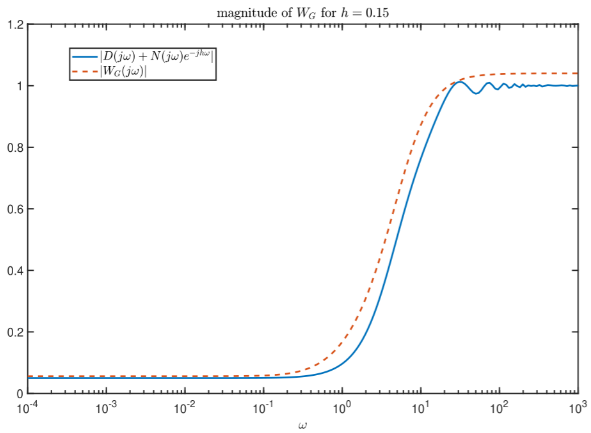

Fig. 3. Magnitude of W_G for $h = 0.15$ s.

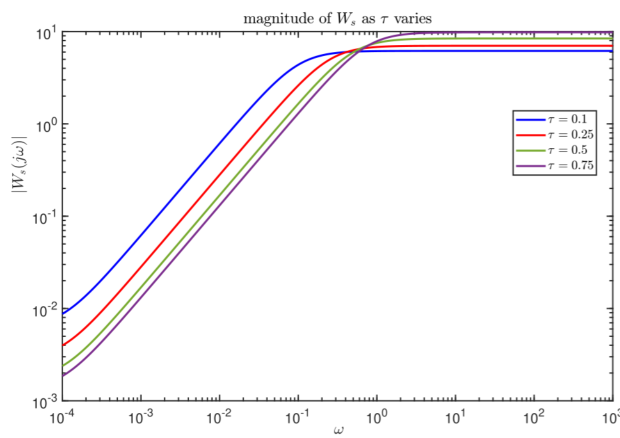

Fig. 4. Magnitude of W_s for $\tau = 0.1,\ 0.25,\ 0.5,\ 0.75$.

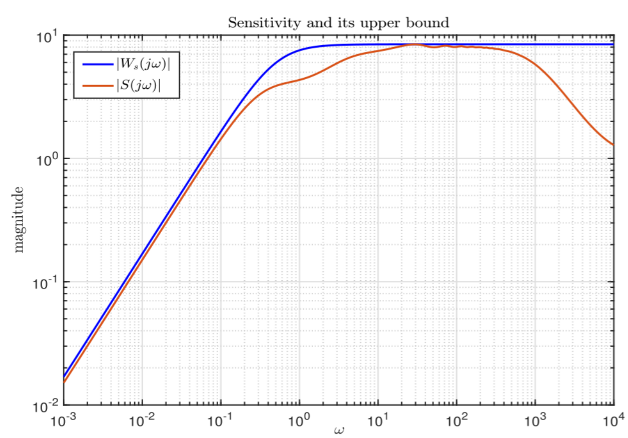

Fig. 5. Magnitudes of W_s and S for $\tau = 0.5$.

The resulting controller is $C(s) = C_g(s) - e^{-hs}$, where

$$C_g(s) = \frac{9121.2(s+7.119)(s-0.4884)(s+0.122)}{(s-8116)(s+7.178)(s+10^{-4})}. \tag{53}$$

The sensitivity magnitude and the upper bound specified as $|W_s(j\omega)|$ are shown in Fig. 5.

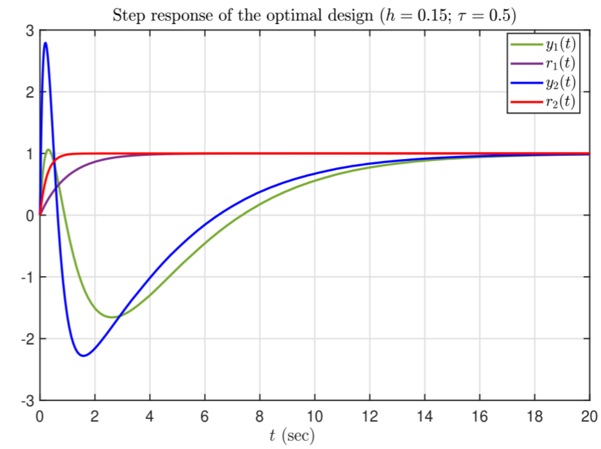

Fig. 6. Tracking performances for $\eta = 1$ and $\eta = 0.25$.

To demonstrate the tracking performance of this design, consider a reference input $R(s) = \frac{1}{s(\eta s+1)}$, where η determines the speed of the desired response. The system outputs and the reference signals corresponding to $\eta = \eta_1 = 1$ and $\eta = \eta_2 = 0.25$ are shown in Fig. 6. As the reference signal gets faster the overshoot in the response gets larger.

C. Robustness Analysis

From (14), for the plant $P = N(D + NW)^{-1}$, all stabilizing controllers are parameterized as $C = [X + (D + NW)Q][Y - NQ]^{-1}$, where X and Y as in (10) are derived from Φ in (8). We now investigate stability robustness when this controller is applied to the uncertain plant $P_\Delta = \frac{N}{D+NW_\Delta}$, $W_\Delta \in \mathscr{H}_\infty$, i.e., the case where there is mismatch in W (uncertainty in the delayed feedback within the plant). The characteristic equation of the feedback system $\mathcal{S}(C, P_\Delta)$ is $M = N(X + (D + NW)Q) + (D + NW_\Delta)(Y - NQ) = 1 + (W_\Delta - W)N(Y - NQ)$. Therefore, the system is robustly stable if

$$\|(W_\Delta - W)N(Y - NQ)\|_\infty < 1. \tag{54}$$

The inequality (54) can be used to determine the largest tolerable uncertainty in W. In other words, it is possible to find $Q \in \mathscr{H}_\infty$ satisfying (54) if the uncertainty level is small enough. To illustrate this point, reconsider Examples 2 and 3. First, consider the neutral time delay system in (45) of Example 2, with $k \geq 0, h \geq 0$. Here $G(s) = \frac{s-1}{s+3}$ and $W(s) = ke^{-hs}$. The plant in (45) has infinitely (resp. finitely) many poles in $\mathbb{C}_+$ if $k > 1$ (resp. $k < 1$) [19]. Assume $W_\Delta(s) = ke^{-h_\delta s}$, where $h_\delta = h + \delta$ with δ being the delay uncertainty. An upper bound $\delta_h \geq \delta$ is assumed to be known. Recall that $N = G$ and $Y = 1$. A conservative, yet simple uncertainty bound can be determined as follows: $|(W_\Delta(j\omega) - W(j\omega))N(j\omega)| < |V(j\omega)|, \forall\omega$, where

$$V(s) = k\,\delta_h \frac{1.25\,(s+\epsilon)}{(0.5\,\delta_h\, s+1)} \frac{(s+1)}{(s+3)} \tag{55}$$

with arbitrarily small $\epsilon > 0$. Since $N(s)$ has only one zero in $\mathbb{C}_+$, $z = 1$, (54) is satisfied if $|V(1)| < 1$, equivalently

$$k\,\frac{1.25(1+\epsilon)\,\delta_h}{\delta_h + 2} < 1. \tag{56}$$

The inequality (56) gives a relationship between k and the largest allowable δ_h. Note that for $k < 0.8$ we can have δ_h arbitrarily large. For $k > 0.8$ the largest allowable δ_h is inversely proportional to k.

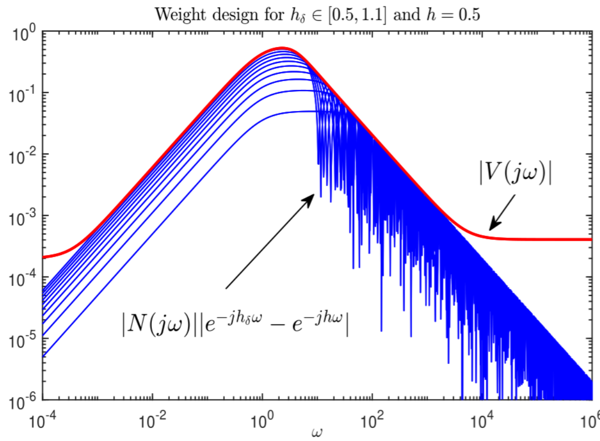

Fig. 7. Weight selection for the allowable delay uncertainty of $\delta_h = 0.6$ and $\epsilon = 0.0002$.

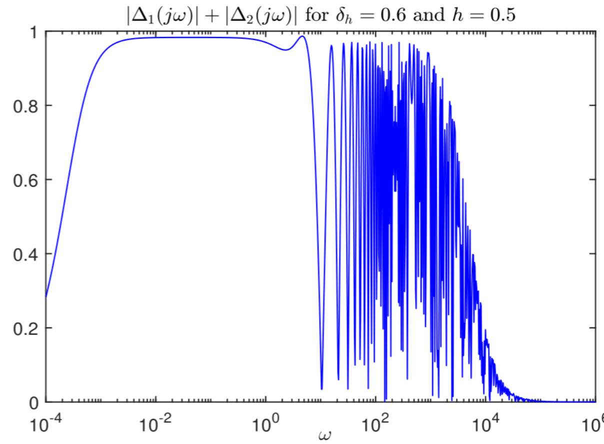

Fig. 8. Stability robustness verification with $\delta_h = 0.6$ and $\epsilon = 0.0002$.

Now consider $P(s)$ in Example 3; $W(s) = e^{-hs}$, $N(s) = \frac{s-1}{\theta(s)}$, $D(s) = \frac{s(s-2)}{\theta(s)}$. For unknown $h_\delta > 0$, let $W_\Delta(s) = e^{-h_\delta s}$. Define a weight $V(s)$ such that $V, V^{-1} \in \mathscr{H}_\infty$ and $|V(j\omega)| \geq |e^{-jh_\delta\omega} - e^{-jh\omega}|\,|N(j\omega)|, \forall\omega$. If $Q \in \mathscr{H}_\infty$ is designed to satisfy (57), then we have robust stability

$$\|V\,(Y - NQ)\|_\infty < 1. \tag{57}$$

The left-hand side of the inequality (57) defines a finite dimensional one-block $\mathscr{H}_\infty$ control problem, and it can be solved easily. For the specific example considered here, we can choose (see [17], p. 84)

$$V(s) = f(\delta_h s)\,\frac{(\epsilon\delta_h s + 1)(s+1)}{\theta(s)} \tag{58}$$

where $\delta_h = \max|h_\delta - h|$ (the largest possible delay uncertainty), and with $\epsilon > 0$ as an arbitrarily small number

$$f(\delta_h s) = \frac{(2.028)(\delta_h s + \epsilon)(\delta_h s + 2.5)}{(\delta_h s)^2 + (2+\sqrt{3})(\delta_h s) + 5.056}. \tag{59}$$

The selection of the weight $V(s)$ is illustrated in Fig. 7, where we used $\epsilon = 0.0002$ and $\theta(s) = (s+1)^2$. Since $N(1) = 0$, the one-block problem has a solution if and only if $|V(1)Y(1)| < 1$. Note that $Y(1) = 1/D(1)$; so, we can compute the left-hand side of the inequality as $V(1)Y(1) = V(1)/D(1) = -2(1+\epsilon\delta_h)f(\delta_h)$. Thus, the largest allowable delay uncertainty $\delta_{\max}$ is the largest value of δ_h satisfying $|f(\delta_h)| < (2(1+\epsilon\delta_h))^{-1}$. [17, Fig. 5.5] shows that $f(0.6) = 0.492 < 0.5$; so, $\delta_{\max} \approx 0.6$ when $\epsilon \searrow 0$. In other words, given a nominal $W(s) = e^{-hs}$, the controller designed stabilizes all plants P_Δ where $W_\Delta(s) = e^{-h_\delta s}$ provided that $|h_\delta - h| < 0.6$. By design, $F = V(Y - NQ)$ satisfies $\|F\|_\infty < 1$. In fact, for $V(s)$ defined in (58) above, with $\epsilon = 0.0002$, we have $F = V(1)/D(1) = -0.9860$ is a constant. There is still room to increase delay uncertainty, but the extra margin is needed in the controller implementation (as discussed below).

Now using (14), the robustly stabilizing optimal controller can be expressed as in (60); when applied to a plant $P_\Delta = N(D + NW_\Delta)^{-1}$, the resulting sensitivity function is in the following:

$$C_{\text{opt}}(s) = C_F(s) - W = \frac{F^{-1}(s)V(s) - D(s)}{N(s)} - W \tag{60}$$

$$(1 + P_\Delta C_{\text{opt}})^{-1} = \frac{V^{-1}F(D + NW_\Delta)}{1 + V^{-1}FN(W_\Delta - W)}. \tag{61}$$

Since $\|FV^{-1}N(W_\Delta - W)\|_\infty \leq \|F\|_\infty < 1$, we have robust stability. However, the controller in (60) is improper because of dividing by $N(s)$ in $C_F(s)$. In order to circumvent this problem, the optimal Q (which is improper) $Q_{\text{opt}}(s) = \frac{(Y - V^{-1}F)}{N}$ is multiplied by a factor of $(\varepsilon s + 1)^{-1}$ with $\varepsilon \searrow 0$, to define $Q_\varepsilon(s) = \frac{1}{\varepsilon s + 1}Q_{\text{opt}}(s)$. Now Q_ε is used in the controller parametrization, which leads to an approximately optimal controller C_ε

$$C_\varepsilon = \frac{X + (D + NW)Q_\varepsilon}{Y - NQ_\varepsilon}. \tag{62}$$

With plant P_Δ and controller C_ε the sensitivity function is

$$(1 + P_\Delta C_\varepsilon)^{-1} = \frac{(Y - NQ_\varepsilon)(D + NW_\Delta)}{1 + \Delta_1 + \Delta_2} \tag{63}$$

where $\Delta_1 = (\varepsilon s + 1)^{-1}V^{-1}FN(W_\Delta - W)$ and $\Delta_2 = (\varepsilon s + 1)^{-1}\varepsilon sYN(W_\Delta - W)$. When $\varepsilon = 0$ we have $\Delta_2 = 0$ and $\Delta_1 = FV^{-1}N(W_\Delta - W)$, with $\|\Delta_1\|_\infty \leq \|F\|_\infty < 1$. In order to guarantee robust stability, in this case we choose ε small enough so that $\|\Delta_1\|_\infty + \|\Delta_2\|_\infty < 1$. For example, with $\varepsilon = \epsilon = 0.0002$, the graph of $|\Delta_1(j\omega)| + |\Delta_2(j\omega)|$ is in Fig. 8. In this example $\|\Delta_2\|_\infty < 8 \times 10^{-4}$, so the graph shown in Fig. 8 is mainly the magnitude of Δ_1.

Remark 3: Another method of obtaining a parameterization of all stabilizing controllers for plants as (7) is to find a coprime factorization of P based on its $\mathbb{C}_+$-poles (this requires using a quasi-polynomial root computation tool, such as YALTA or QPmR), as done in [17] and many other works. However, in that approach uncertainty in W will lead to uncertainty in pole locations and hence, uncertainty in the coprime factors. Although robustness to coprime factor uncertainty can also be analyzed using standard $\mathscr{H}_\infty$ techniques, it is not possible to find an explicit bound on the coprime factor uncertainty directly from $(W_\Delta - W)$ since an intermediate root computation is involved.

V. Conclusion

A simple interpolation-based controller design method is proposed for retarded and neutral time delay systems (plants with internal delayed feedback) in the form $P = G(1 + GW)^{-1}$. All stabilizing controllers are obtained from C_o, which is constructed from the unstable poles of G, whereas traditional designs use unstable poles of the plant P, whose computations introduce an extra layer of complexity. A stabilizing integral-action controller C_I is obtained from C_o, and a parametrization of all stabilizing integral-action controllers is obtained by using C_I . Performance and robustness issues are discussed with examples in order to illustrate the impact of the selection of the free controller design parameters.

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