Controller Design for Diagonal Decoupling and Integral Action

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***Abstract*—In the standard linear, time-invariant feedback system, controllers that achieve diagonal decoupling and closed-loop stability exist if and only if the plant satisfies the diagonal denominator condition or has no coinciding poles and zeros in the region of instability. A simple and systematic decoupling design procedure is presented under each of these conditions. The closed-loop poles can be placed at any desired points, and free parameters are included for satisfying additional design objectives. The designs are also extended to provide integral action in order to track step inputs with zero steady-state error.**

***Index Terms*—Control design, decoupling control, integral-action, tracking controller.**

I. Introduction

The removal of the interaction between inputs and outputs while achieving internal stabilization of the closed-loop system is a very important controller design objective for linear time-invariant (LTI), multi-input multi-output (MIMO) systems. The complete elimination of this coupling results in a diagonal and nonsingular complementary sensitivity transfer function. Research on diagonal decoupling has a very long history. This problem has been studied under different solvability conditions using various state-feedback and output-feedback approaches, (e.g., [4], [8], [15], [17]). and in some cases with configurations that may include a precompensator in the feedback loop [3].

Decoupling controller designs that also achieve internal stability were proposed using two-parameter compensation schemes (e.g., [1], [9]). The problem becomes challenging when the output being decoupled is the one used in feedback, and the controller is expected to achieve both internal stability and diagonal decoupling. In this one-degree-of-freedom configuration, diagonally decoupling controllers exist for LTI, MIMO plants with no right-half plane (RHP) pole-zero coincidences [7], [10], [14]. Various conditions were also explored for plants that do not satisfy this well-known sufficient condition [6], [11], [13], [16].

This article establishes that the decoupling problem can be solved if and only if at least one of two conditions holds; if the plant RHP pole-zero coincidences, then it can be decoupled if and only if it satisfies the *diagonal denominator condition*. The significance of this article is the simple and systematic design methods that provide explicit controller parameterizations for diagonal decoupling with internal stabilization. Additionally, integral action can be included in the designed controllers. There is no need to obtain coprime factorizations or to solve additional Bézout identities for the proposed controller designs. The closed-loop poles are assigned as desired, and additional free parameters are included to be used for satisfying other design objectives.

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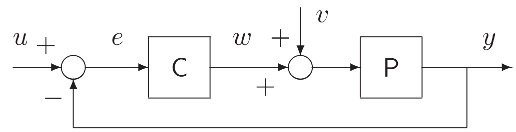

Fig. 1. Unity-feedback system $\mathscr{S}(P, C)$.

The problem is formally stated in Section II. The two decoupling conditions and the main Theorem 1 are stated in Section III. The constructive proof of existence of decoupling controllers is given by Propositions 1 and 2, which develop the new decoupling controller design method without using coprime factorization computations. In Section IV, these design procedures are extended to include integral action in addition to internal stability and decoupling, so that the steady-state errors for constant reference inputs go to zero asymptotically. Several examples of the proposed decoupling and integral-action designs are given in Section V.

The following standard notation is used: The region of instability is the extended closed RHP, $\mathbb{C}_+ \cup \{\infty\} = \{ s \in \mathbb{C} | \mathscr{R}e(s) \geq 0 \} \cup \{\infty\}$; the open left-half plane (OLHP) is $\mathbb{C}_-$. Real and positive real numbers are $\mathbb{R}$ and $\mathbb{R}_+$; $\mathbf{R_p}$ denotes real proper rational functions; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$. The set of matrices with entries in $\mathbf{S}$ is $\mathcal{M}(\mathbf{S})$. The identity matrix is I. For simplicity, we drop (s) in transfer matrices such as $P(s)$. Let $M := \{1, \ldots, m\}$. A diagonal matrix, with diagonal entries D_j, $j \in M$, is denoted by diag $\left[D_j \right]_{j \in M}$ or diag $\left[D_1, \ldots, D_m \right]$. Transmission zeros and blocking zeros of an MIMO plant are simply called zeros here; we are only interested in the zeros in $\mathbb{C}_+ \cup \{\infty\}$. The degree of the polynomial $p(s)$ is denoted by $\deg(p)$. A polynomial that has all roots in $\mathbb{C}_-$ is called a Hurwitz polynomial (HP).

II. Problem Description

The standard LTI, MIMO unity-feedback system $\mathscr{S}(P, C)$ is in Fig. 1; $P \in \mathbf{R_p}^{m \times m}$, and $C \in \mathbf{R_p}^{m \times m}$ are the plant's and the controller's transfer functions, and rank $P = m$. The objective is to design simple stabilizing controllers that achieve diagonal decoupling, and asymptotic tracking of step-input references with zero steady-state error. The closed-loop transfer function $\mathbf{H}$ from (u, v) to (w, y) is:

$$\mathbf{H} = \begin{bmatrix} C(I + PC)^{-1} & -C(I + PC)^{-1}P \\ PC(I + PC)^{-1} & (I + PC)^{-1}P \end{bmatrix}. \tag{1}$$

Let H_{yu} denote the input–output (complementary sensitivity) transfer function from u to y. Let H_{eu} denote the input-error (sensitivity) transfer function from u to e.

Definition 1: i) The system $\mathscr{S}(P, C)$ is stable if the closed-loop transfer function $\mathbf{H} \in \mathcal{M}(\mathbf{S})$. The controller C is called a stabilizing controller if C is proper and $\mathbf{H} \in \mathcal{M}(\mathbf{S})$. ii) The stable $\mathscr{S}(P, C)$ is diagonally decoupled if H_{yu} is diagonal and nonsingular. A stabilizing controller is called a decoupling controller if $\mathscr{S}(P, C)$ is diagonally decoupled. iii) The system $\mathscr{S}(P, C)$ has integral action if it is stable, and $H_{eu}(0) = 0$. A stabilizing controller C is an integral-action controller if it has poles at $s = 0$. iv) The controller C is a decoupling controller with integral action if C stabilizes P, has poles at $s = 0$, and the stable system $\mathscr{S}(P, C)$ is both diagonally decoupled and has integral action. □

For stable $\mathscr{S}(P, C)$, the steady-state error $e(t)$ for constant inputs applied at $u(t)$ goes to zero asymptotically if and only if the system has integral action, i.e., $H_{eu}(0) = 0$. The controller is designed with poles at zero to achieve integral action (see the internal model principle [5], [12]). A necessary condition for diagonal decoupling is that $P \in \mathbf{R_p}^{m\times m}$ is full rank m since the rank of nonsingular $H_{yu} = PC(I + PC)^{-1}$ is m. When the design includes integral action, it is also assumed that P has no zeros at $s = 0$.

III. Decoupling Controller Design

Let $P \in \mathbf{R_p}^{m\times m}$, rank$P = m$. If there exist controllers such that the system $\mathscr{S}(P, C)$ is diagonally decoupled, then for some diagonal, nonsingular $\Psi \in \mathbf{S}^{m\times m}$, $H_{yu} = PC(I + PC)^{-1} = \Psi$ and $\mathbf{H} \in \mathcal{M}(\mathbf{S})$ in (1)

$$\mathbf{H} = \begin{bmatrix} P^{-1}\Psi & -P^{-1}\Psi P \\ \Psi & (I-\Psi)P \end{bmatrix}. \tag{2}$$

All decoupling controllers $C \in \mathbf{R_p}^{m\times m}$ are expressed as

$$C = P^{-1}\Psi\,(I-\Psi)^{-1} \tag{3}$$

Since Ψ is designed so that $P^{-1}\Psi \in \mathcal{M}(\mathbf{S})$, the controllers in (3) are proper if $(I - \Psi)^{-1} \in \mathcal{M}(\mathbf{R_p})$. The closed-loop poles are assigned by choosing the denominators freely in the diagonal matrix Ψ. Write the entries of $P \in \mathbf{R_p}^{m\times m}$ in polynomial factored form

$$P =: \left[\frac{x_{ij}}{y^s{}_{ij}y^u{}_{ij}}\right]_{i,j\in M} \tag{4}$$

x_{ij} is the numerator, $(y^s{}_{ij}y^u{}_{ij})$ is the monic denominator polynomial of the ijth entry of P. The roots of $y^u{}_{ij}$ are the $\mathbb{C}_+$–poles of P. All roots of $y^s{}_{ij}$ are in $\mathbb{C}_-$. For $i \in M$, define y_i as the monic least-common-multiple of all $y^u{}_{ij}$ in the ith row. Let φ_i be any monic HP such that $\deg(\varphi_i) = \deg(y_i)$. Define $Y_i \in \mathbf{S}$ and the diagonal $Y \in \mathbf{S}^{m\times m}$ as (5)

$$y_i = \mathrm{lcm}\, y^u{}_{ij},\ \ j \in M;\ Y_i = \frac{y_i}{\varphi_i},\ \ Y = \mathrm{diag}\left[Y_i\right]_{i\in M}. \tag{5}$$

Therefore, $(y^u{}_{ij})^{-1}y_i \in \mathbf{S}$, $Y_i\frac{x_{ij}}{y_{ij}} \in \mathbf{S} \Longrightarrow YP \in \mathbf{S}^{m\times m}$. The terms $Y_i(\infty) = 1$. If all m entries in the ith row of P are stable, then $Y_i = 1$. Write the entries of P^{-1} in polynomial factored form

$$P^{-1} =: \left[\frac{n_{ij}}{d^s{}_{ij}d^u{}_{ij}}\right]_{i,j\in M} \tag{6}$$

n_{ij} is the numerator, $(d^s{}_{ij}d^u{}_{ij})$ is the monic denominator polynomial of the (possibly improper) ijth entry of P^{-1}. All roots of $d^u{}_{ij}$ are in $\mathbb{C}_+$; all roots of $d^s{}_{ij}$ are in $\mathbb{C}_-$.

Since rank $P = m$, $(YP)^{-1}$ exists, and it may be proper or improper. Write the entries of $(YP)^{-1}$ in factored form

$$(YP)^{-1} =: \left[\frac{e_{ij}}{f^s{}_{ij}f^u{}_{ij}}\right]_{i,j\in M} = \left[\frac{n_{ij}}{d^s{}_{ij}d^u{}_{ij}}\,\frac{\varphi_j}{y_j}\right]_{i,j\in M} \tag{7}$$

e_{ij} is the numerator, $(f^s{}_{ij}f^u{}_{ij})$ is the monic denominator polynomial of the ijth entry of $(YP)^{-1}$. All roots of $f^u{}_{ij}$ are in $\mathbb{C}_+$ and all roots of $f^s{}_{ij}$ are in $\mathbb{C}_-$. For $j \in M$, define f_j as the monic least-common-multiple of all $f^u{}_{ij}$ in the jth column; therefore, $(f^u{}_{ij})^{-1}f_j \in \mathbf{S}$

$$f_j = \mathrm{lcm}\, f^u{}_{ij},\quad i \in M. \tag{8}$$

The expression for P^{-1} in (6) is a special case of (7) when $P \in \mathcal{M}(\mathbf{S})$, since $Y = I$ in that case.

An important sufficient condition for existence of decoupling controllers is obtained in terms of the $\mathbb{C}_+$-zeros of f_j as stated in Condition 1 below: The system $\mathscr{S}(P, C)$ can be decoupled if y_j has no zeros coinciding with the $\mathbb{C}_+$-zeros of the corresponding f_j. This sufficient existence condition is equivalent to each (y_j, f_j) pair being coprime. Furthermore, if P does not satisfy Condition 1, then the only way it can be decoupled is if P has no coinciding RHP poles and zeros, which is stated as Condition 2.

Condition 1. Diagonal denominator condition: For $\ell = 1, \ldots, \mu_j$, let $z_{j\ell} \in \mathbb{C}_+$ be the μ_j distinct roots of f_j, where $0 \le \mu_j$. The multiplicity of $z_{j\ell}$ is $m_{j\ell}$. For $j \in M$, if $y_j(z_{j\ell}) \ne 0$ for all $z_{j1}, z_{j2}, \ldots z_{j\mu_j} \in \mathbb{C}_+$ then P satisfies the *diagonal denominator condition*. □

If P does not satisfy the *diagonal denominator condition*, i.e., for some $j \in M$, $y_j(z_{j\ell}) = 0$ for any of the $z_{j1}, z_{j2}, \ldots z_{j\mu_j} \in \mathbb{C}_+$, then this $z_{j\ell} \in \mathbb{C}_+$ is also a pole of P. Define $\gamma \in \mathbf{S}$ as the monic least-common-multiple of all $y^u{}_{ij}$ in all entries of P in (4)

$$\gamma := \mathrm{lcm}\, y^u{}_{ij} = \mathrm{lcm}\, y_i,\quad i, j \in M. \tag{9}$$

Define λ as the monic least-common-multiple of all $d^u{}_{ij}$ in all entries of P^{-1} in (6)

$$\lambda = \mathrm{lcm}\, d^u{}_{ij},\quad i, j \in M. \tag{10}$$

Therefore, $(d^u{}_{ij})^{-1}\lambda \in \mathbf{S}$. If P^{-1} has no finite $\mathbb{C}_+$-poles, then all $d^u{}_{ij} = 1$, which implies $\lambda = 1$.

Condition 2. No RHP pole-zero coincidence condition: For $\ell = 1, \ldots, \mu$, let $z_\ell \in \mathbb{C}_+$ be the μ distinct roots of λ where $0 < \mu$, where the multiplicity of z_ℓ is m_ℓ. If $\gamma(z_\ell) \ne 0$ for $z_1, z_2, \ldots z_\mu \in \mathbb{C}_+$, then P satisfies the *no RHP pole-zero coincidence condition*. □

Theorem 1: Necessary and sufficient conditions for decoupling: Let $P \in \mathbf{R_p}^{m\times m}$, rank$P = m$. There exist decoupling controllers if and only if P satisfies at least one of the two conditions: Condition 1 (*diagonal denominator condition*) or Condition 2 (*no RHP pole-zero coincidence condition*). □

Proof of Theorem 1: Necessity: Suppose that Conditions 1 and 2 both fail. Let $z \in \mathbb{C}_+$ be a common zero of y_j and f_j for some j. Then, P has a pole at $s = z$ that appears in the jth row, and $s = z$ is also a pole that appears in some column of P^{-1}. By (2), for closed-loop stability, $P^{-1}\Psi \in \mathcal{M}(\mathbf{S})$ and $P^{-1}\Psi P = (YP)^{-1}\Psi(YP) \in \mathcal{M}(\mathbf{S})$ imply either $\psi_j(z) = 0$ or that every entry of Ψ is the same, which requires that P has no coinciding RHP poles and zeros. But also by (2), $(I - \Psi)P \in \mathcal{M}(\mathbf{S})$ implies $(1 - \psi_j(s))$ must have a zero at $s = z$ to cancel the pole of P in the jth row. This is a contradiction since $\psi_j(z_j) = 0$. Therefore, the transfer function $H_{yu} = \Psi$ of the stable closed-loop system cannot be diagonal when both conditions fail. ■

The sufficiency of each condition in Theorem 1 is proved by explicit construction. Propositions 1 and 2 give detailed construction of a complete set of decoupling controllers for P satisfying the *diagonal denominator condition* and the *no RHP pole-zero coincidence condition*, respectively. These designs also allow placing the closed-loop poles at any desired locations in the OLHP.

Proposition 1: Decoupling controller design procedure for P satisfying the diagonal denominator condition.

Step 1: For each entry of $(YP)^{-1}$, define the integers $\rho_{ij} \ge 0$, and ρ_j as the largest of all ρ_{ij} in the jth column as (11). If the ijth entry of $(YP)^{-1}$ is proper, then $\rho_{ij} = 0$

$$\rho_{ij} = \max\{0, \deg(e_{ij}) - \deg(f^s{}_{ij} f^u{}_{ij})\},$$
$$\rho_j = \max_{i=1,\dots m} \rho_{ij}, \quad i, j \in M. \tag{11}$$

For $j \in M$, let ξ_j be any monic HP, $\deg(\xi_j) = \rho_j + \deg(f_j)$. Define $F_j \in \mathbf{S}$ and the diagonal $F \in \mathbf{S}^{m\times m}$ as (12)

$$F_j = \frac{f_j}{\xi_j}, \quad F = \text{diag}\left[F_j\right]_{j\in M}. \tag{12}$$

By (12), $\dfrac{e_{ij}}{f^u{}_{ij}} F_j \in \mathbf{S} \Longrightarrow (YP)^{-1}F \in \mathbf{S}^{m\times m}$. If $\rho_j \neq 0$, then F_j has ρ_j zeros at infinity.

Step 2: Define $\theta_j \in \mathbf{S}$ as (13); $Y_j(\infty) = 1 \Longrightarrow \theta_j(\infty) = \prod_{\ell=1}^{\mu_j}(1 - Y_j(z_{j\ell})^{-1})^{m_{j\ell}}$ if $\rho_j = 0$; $\theta_j(\infty) = 0$ if $\rho_j \neq 0$

$$\theta_j := (1 - Y_j)^{\rho_j} \prod_{\ell=1}^{\mu_j} (1 - Y_j(z_{j\ell})^{-1} Y_j)^{m_{j\ell}}, \; j \in M. \tag{13}$$

Step 3: Define $\Psi = \text{diag}\left[\psi_j\right]_{j\in M} \in \mathbf{S}^{m\times m}$, where $\psi_j \in \mathbf{S}$ have one of three possible values as (14)–(16)

i) If $Y_j = 1$, then with $q_j \in \mathbf{S}$

$$\psi_j = F_j q_j, \quad q_j \neq 0, \; q_j(\infty) \neq F_j(\infty)^{-1} \tag{14}$$

and $\psi_j(1-\psi_j)^{-1} = F_j q_j (1 - F_j q_j)^{-1}$

ii) If $Y_j \neq 1$, but $F_j = 1$, then with $q_j \in \mathbf{S}$,

$$\psi_j = (1 - q_j Y_j), \quad q_j(\infty) \neq 0 \tag{15}$$

and $\psi_j(1-\psi_j)^{-1} = (1 - q_j Y_j)(q_j Y_j)^{-1} = (Y_j^{-1} q_j^{-1} - 1)$.

iii) If $Y_j \neq 1$, and $F_j \neq 1$, then with $q_j \in \mathbf{S}$

$$\psi_j = (\theta_j + q_j F_j Y_j), \; q_j(\infty) \neq (1 - \theta_j(\infty)) F_j(\infty)^{-1} \tag{16}$$

$\psi_j(1-\psi_j)^{-1} = (\theta_j + q_j F_j Y_j)(1 - \theta_j - q_j F_j Y_j)^{-1}$.

Then, $C = P^{-1}\Psi(I - \Psi)^{-1}$ in (17) is a decoupling controller:

$$C = P^{-1}\text{diag}\left[\psi_j (1-\psi_j)^{-1}\right]_{j\in M}. \tag{17}$$

The corresponding diagonal input–output transfer function is $H_{yu} = \Psi$. The closed-loop poles are the roots of the polynomials φ_j in Y_j and ξ_j in F_j, and the poles of q_j. □

Remarks 1: Justification of Proposition 1: For all three cases (14)–(16), ψ_j is constructed so that $(YP)^{-1}F \in \mathcal{M}(\mathbf{S}) \Longrightarrow (YP)^{-1}\Psi \in \mathcal{M}(\mathbf{S})$. Also, $(I - \Psi)Y^{-1} \in \mathcal{M}(\mathbf{S})$ by construction. With C as in (17), the stability of $\mathbf{H}$ in (1) is shown as follows: Since the diagonal Ψ and Y commute, and $YP \in \mathcal{M}(\mathbf{S})$, $(YP)^{-1}\Psi \in \mathcal{M}(\mathbf{S})$, we have $H_{wu} = P^{-1}\Psi = (YP)^{-1}\Psi Y \in \mathcal{M}(\mathbf{S})$, $H_{wv} = -H_{wu}P = -(YP)^{-1}\Psi YP \in \mathcal{M}(\mathbf{S})$, $H_{yu} = PH_{wu} = \Psi \in \mathcal{M}(\mathbf{S})$, $H_{yv} = (I - \Psi)P = (I - \Psi)Y^{-1}YP \in \mathcal{M}(\mathbf{S})$. In (14)–(16), the constraints on $q_j(\infty)$ ensure that $(1 - \psi_j)^{-1} \in \mathbf{R_p}$, and hence, C in (17) is proper. Therefore, C is a stabilizing controller, and the corresponding input–output transfer function $H_{yu} = \Psi$ is diagonal and nonsingular. □

Remarks 2: **Special Case:** *Decoupling design for stable plants:* If $P \in \mathbf{S}^{m\times m}$, rank $P = m$, then $y^u{}_{ij} = 1 = y_i$, $Y_i = 1$, $Y = I$. Therefore, $(YP)^{-1} = P^{-1}$, i.e., $d^u{}_{ij} = f^u{}_{ij}$ in (6) and (7); hence, the *diagonal denominator condition* holds for stable plants. The decoupling controller design in Proposition 1 is simplified by choosing $\psi_j = F_j q_j$ as in (14) of part (i) as follows: Let $Q = \text{diag}[\,q_j\,]_{j\in M} \in \mathbf{S}^{m\times m}$ be any stable, nonsingular, diagonal matrix, with $q_j \in \mathbf{S}$, $q_j \neq 0$, $q_j(\infty) \neq F_j(\infty)^{-1}$. Let $\Psi = \text{diag}[\,F_j q_j\,]_{j\in M} = FQ$. The decoupling controllers in (17) become

$$C = P^{-1}FQ(I - FQ)^{-1}. \tag{18}$$

The corresponding diagonal input–output transfer function is $H_{yu} = \Psi = \text{diag}[\,F_j q_j\,]_{j\in M}$, where $q_j \in \mathbf{S}$ satisfies $q_j \neq 0$, and $q_j(\infty) \neq F_j(\infty)^{-1}$ to ensure propernes of C. The closed-loop poles are the roots of the polynomial ξ_j in F_j and the poles of q_j. □

Remarks 3: **Special Case:** *Decoupling design for inverse-stable plants*: If $P \notin \mathcal{M}(\mathbf{S})$, but $P^{-1} \in \mathbf{S}^{m\times m}$, then $(YP)^{-1} \in \mathcal{M}(\mathbf{S})$. Therefore, $F_j = 1$, and $F = I$ in (12); hence, the *diagonal denominator condition* holds for inverse-stable plants. The decoupling controller design in Proposition 1 is simplified by choosing $\psi_j = (1 - q_j Y_j)$ as in (15) of part (ii) as follows: Let $Q = \text{diag}\left[q_j\right]_{j\in M} \in \mathbf{S}^{m\times m}$ be any stable, nonsingular, diagonal matrix, with $q_j \in \mathbf{S}$ satisfying $q_j(\infty) \neq 0$. Let $\Psi = \text{diag}[\;(1 - q_j Y_j)\;]_{j\in M}$. The decoupling controllers in (17) become

$$C = P^{-1}(I - QY)(QY)^{-1} = P^{-1}(Y^{-1}Q^{-1} - I). \tag{19}$$

The diagonal input–output transfer function is $H_{yu} = \Psi = (I - QY) = \text{diag}[1 - q_j Y_j]_{j\in M}$, where $q_j(\infty) \neq 0$ so that C is proper. The the closed-loop poles are the roots of the polynomial φ_j in Y_j and the poles of q_j. □

The diagonal decoupling controller design in Proposition 1 only applies to plants satisfying the assumption $y_j(z_{j\ell}) \neq 0$ for the $\mathbb{C}_+$-zeros $z_{j\ell} \in \mathbb{C}_+$ of f_j, for $\ell = 1, \dots, \mu_j$. If at least one y_j has zeros that coincide with $z_{j\ell}$ of the corresponding f_j, it is still possible to decouple the system $\mathscr{S}(P, C)$ if the plant's $\mathbb{C}_+$-zeros do not coincide with its poles. Proposition 2 gives a procedure similar to the method in Proposition 1. Although decoupling for the case of plants with noncoincident $\mathbb{C}_+$-zeros and poles has been studied extensively (e.g., [6], [7], [10]), the procedure given here provides a much simpler and complete design without computing coprime factorizations.

Proposition 2: Decoupling controller design procedure for P satisfying the no RHP pole-zero coincidence condition.

Let $P \in \mathbf{R_p}^{m\times m}$, $P \notin \mathcal{M}(\mathbf{S})$, $P^{-1} \notin \mathcal{M}(\mathbf{S})$, rank$P = m$. With $Y_i \in \mathbf{S}$ as in (5), suppose that, for at least one y_j, $j \in M$, $y_j(z_{j\ell}) = 0$ for at least one of the $\mathbb{C}_+$-zeros $z_{j\ell} \in \mathbb{C}_+$ of f_j, $\ell = 1, \dots, \mu_j$. If $\gamma(z_\ell) \neq 0$ at each $\mathbb{C}_+$-zero z_ℓ of λ, then:

Step 1: Let φ be a monic HP such that $\deg(\varphi) = \deg(\gamma)$; define $\Gamma \in \mathbf{S}$

$$\Gamma := \frac{\gamma}{\varphi}. \tag{20}$$

Step 2: Define the integers $\tilde{\rho}_{ij} \geq 0$ for each entry of P^{-1}, and ρ as the largest of all $\tilde{\rho}_{ij}$. If P^{-1} is proper, then $\rho = 0$.

$$\tilde{\rho}_{ij} = \max\{0, \deg(n_{ij}) - \deg(d^u{}_{ij} d^s{}_{ij})\},$$
$$\rho = \max_{i,j=1,\dots m} \tilde{\rho}_{ij}, \quad i, j \in M. \tag{21}$$

Let ξ be any monic HP such that $\deg(\xi) = \rho + \deg(\lambda)$. Define $\Lambda \in \mathbf{S}$ as in (22); then $\Lambda P^{-1} \in \mathbf{S}^{m\times m}$

$$\Lambda = \frac{\lambda}{\xi}. \tag{22}$$

Step 3: Define $\theta \in \mathbf{S}$ as (23); $\Gamma(\infty) = 1 \Longrightarrow \theta(\infty) = \prod_{\ell=1}^{\mu}(1 - \Gamma(z_\ell)^{-1})^{m_\ell}$ if $\rho = 0$; $\theta(\infty) = 0$ if $\rho \neq 0$.

$$\theta := (1 - \Gamma)^{\rho} \prod_{\ell=1}^{\mu} (1 - \Gamma(z_\ell)^{-1}\Gamma)^{m_\ell}. \tag{23}$$

Step 4: Define $\psi \in \mathbf{S}$ as

$$\psi = (\theta + q\Lambda\Gamma), \; q \in \mathbf{S}, \; q(\infty) \neq (1 - \theta(\infty))\Lambda(\infty)^{-1}. \tag{24}$$

Then, C in (25) is a decoupling controller

$$C = \psi(I - \psi)^{-1}P^{-1} = (\theta + q\Lambda\Gamma)(1 - \theta - q\Lambda\Gamma)^{-1}P^{-1}. \tag{25}$$

The corresponding diagonal transfer function is $H_{yu} = \psi I$. The closed-loop poles are the roots of the polynomials φ, ξ, and the poles of q. □

Remarks 4: Justification of Proposition 2: With ψ as in (24), and C given by (25), $\Lambda P^{-1} \in \mathcal{M}(\mathbf{S})$ implies $\psi P^{-1} \in \mathcal{M}(\mathbf{S})$ since the $\mathbb{C}_+$-zeros of θ and of Λ are the same. Furthermore, $(1-\psi) = R\Gamma$ for some $R \in \mathbf{S}$ implies $(1-\psi)P \in \mathcal{M}(\mathbf{S})$. Then, in $\mathbf{H}$ of (1), $H_{wu} = \psi P^{-1} \in \mathcal{M}(\mathbf{S})$, $H_{wv} = -H_{wu}P = -\psi I \in \mathcal{M}(\mathbf{S})$, $H_{yu} = PH_{wu} = \psi I \in \mathcal{M}(\mathbf{S})$, $H_{yv} = (I-\psi)P \in \mathcal{M}(\mathbf{S})$. In (24), the constraint on $q(\infty)$ ensures that $(1-\psi)^{-1} \in \mathbf{R_p}$, and hence, the decoupling controller C in (17) is proper. □

Remarks 5: Plants that satisfy both sufficient conditions: Some plants with no coinciding $\mathbb{C}_+$-poles and zeros also satisfy the assumptions of Proposition 1. The controller design procedure first checks if *Condition 1* holds, and applies Proposition 1 to such plants. The plant in Example 3 is an example of a plant satisfying both conditions. □

IV. Integral-Action Decoupling Controllers

The necessary condition for existence of integral-action controllers is that P has no zeros at $s = 0$. Under this assumption, the proposed decoupling controller designs are extended to include integral action in Propositions 3 and 4.

Proposition 3: Integral-action decoupling controllers for P satisfying the diagonal denominator condition.

Under the assumption of Proposition 1, let P have no zeros at $s = 0$. The decoupling controller in (17) becomes also an integral-action controller $C_I = P^{-1}\Psi_I(I-\Psi_I)^{-1}$ if the entries $\psi_{jI} \in \mathbf{S}$ of $\Psi_I = \text{diag}\left[\psi_{jI}\right]_{j\in M} \in \mathbf{S}^{m\times m}$ has one of three possible values

i) If $Y_j = 1$, then for any $\alpha_j \in \mathbb{R}_+$

$$\psi_{jI} = F_j\left(F_j(0)^{-1} + \frac{s}{s+\alpha_j}\hat{q}_j\right) \tag{26}$$

$\hat{q}_j \in \mathbf{S}$, $\hat{q}_j(\infty) \neq F_j(\infty)^{-1} - F_j(0)^{-1}$.

ii) If $Y_j \neq 1$, but $F_j = 1$, then for any $\alpha_j \in \mathbb{R}_+$

$$\psi_{jI} = \left(1 - \frac{s}{s+\alpha_j}\hat{q}_j Y_j\right),\ \hat{q}_j \in \mathbf{S},\ \hat{q}_j(\infty) \neq 0 \tag{27}$$

$\psi_{jI}(1-\psi_{jI})^{-1} = (\frac{s+\alpha_j}{s}Y_j^{-1}\hat{q}_j^{-1} - 1)$.

iii) If $Y_j \neq 1$, and $F_j \neq 1$, then for any $\alpha_j \in \mathbb{R}_+$

$$\psi_{jI} = \theta_j + F_jF_j(0)^{-1}(1-\theta_j) + \frac{s}{s+\alpha_j}\hat{q}_jF_jY_j \tag{28}$$

$\hat{q}_j \in \mathbf{S},\ \hat{q}_j(\infty) \neq (1-\theta_j(\infty))(F_j(\infty)^{-1} - F_j(0)^{-1})$.

With ψ_{jI} as in (26)–(28) of cases (i)–(iii), an integral-action decoupling controller C_I is

$$C_I = P^{-1}\text{diag}\left[\psi_{jI}(1-\psi_{jI})^{-1}\right]_{j\in M}. \tag{29}$$

For case (ii), $\hat{q}_j = 0$ is not a possible choice for ψ_{jI} in (26) and (27). If all entries are as in case (iii) for all $j \in M$, then by (29), $\hat{q}_j = 0$ gives the "nominal" integral-action decoupling controller C_I^o as (30)

$$C_I^o = P^{-1}\text{diag}\left[(1-\theta_j)^{-1}(1-F_jF_j(0)^{-1})^{-1} - 1\right]_{j\in M}. \tag{30}$$

The decoupled closed-loop transfer function with integral action is $H_{yuI} = \Psi_I$, and the dc-gain of the input-error transfer function is $H_{euI}(0) = I - \Psi_I(0) = 0$. □

Remarks 6: Justification of integral action for Proposition 3: For ψ_{jI} as in (26)–(28), $\Psi_I(0) = I$. Therefore, C_I in (29) has poles at $s = 0$. In (26)–(28), the constraints on $\hat{q}_j(\infty)$ ensure that $(1-\psi_{jI})^{-1} \in \mathbf{R_p}$; C in (17) is a proper decoupling controller with integral action, and $H_{euI}(0) = I - \Psi(0) = 0$, i.e., the system $\mathscr{S}(P, C)$ is decoupled and has integral action. □

Remarks 7: **Special Case:** *Integral-action decoupling controllers for stable plants*: If $P \in \mathbf{S}^{m\times m}$ has no zeros at $s = 0$, i.e., rank$P(0) = m$, then the integral-action decoupling controller design in Proposition 3 is simplified, and the decoupling C in (18) is also an integral-action controller if the diagonal $Q = \text{diag}\,[\,q_j\,]_{j\in M} \in \mathbf{S}^{m\times m}$ satisfies (31) for any $\alpha \in \mathbb{R}_+$ and any diagonal $Q_I = \text{diag}[\,\hat{q}_j\,]_{j\in M} \in \mathbf{S}^{m\times m}$

$$Q = \text{diag}\left[F_j(0)^{-1} + \frac{s}{s+\alpha}\hat{q}_j\right]_{j\in M} \tag{31}$$

$\hat{q}_j \in \mathbf{S}$, $\hat{q}_j(\infty) \neq F_j(\infty)^{-1} - F_j(0)^{-1}$, for $j \in M$. With $Q \in \mathbf{S}^{m\times m}$ as in (31), the integral-action decoupling controller C_I is given by (29), with ψ_{jI} as in (26). For $\hat{q}_j = 0$ in (31), the "nominal" integral-action decoupling controller C_I^o is

$$C_I^o = P^{-1}\text{diag}\,\left[F_jF_j(0)^{-1}(1-F_jF_j(0)^{-1})^{-1}\right]_{j\in M}. \tag{32}$$

The diagonal input–output transfer function H_{yuI} is

$$H_{yuI} = \text{diag}\left[F_j\,(\,F_j(0)^{-1} + \frac{s}{s+\alpha}\hat{q}_j\,)\right]_{j\in M} \tag{33}$$

$\hat{q}_j \in \mathbf{S}$ satisfy $\hat{q}_j(\infty) \neq F_j(\infty)^{-1} - F_j(0)^{-1}$. □

Remarks 8: **Special Case:** *Integral-action decoupling design for inverse-stable plants*: If $P \notin \mathcal{M}(\mathbf{S})$, but $P^{-1} \in \mathbf{S}^{m\times m}$, then P has no zeros at $s = 0$, i.e., rank$P(0) = m$. The integral-action decoupling controller design in Proposition 3 is simplified, and the decoupling C in (19) is also an integral-action controller if the diagonal $Q \in \mathbf{S}^{m\times m}$ satisfies (34) for any $\alpha \in \mathbb{R}_+$

$$Q = \frac{s}{s+\alpha}Q_I = \frac{s}{s+\alpha}\text{diag}\,[\,\hat{q}_j\,]_{j\in M},\ \hat{q}_j(\infty) \neq 0\,. \tag{34}$$

For diagonal $Q_I \in \mathbf{S}^{m\times m}$ as in (34), the integral-action decoupling controller C_I and the diagonal input–output transfer function H_{yuI} are

$$C_I = P^{-1}[\frac{s+\alpha}{s}(Q_IY)^{-1} - I],\ H_{yuI} = I - \frac{s}{s+\alpha}Q_IY. \tag{35}$$

□

Proposition 4: Integral-action decoupling controllers for P satisfying the no RHP pole-zero coincidence condition.

Under the assumption of Proposition 2, let P have no zeros at $s = 0$. For any $\alpha \in \mathbb{R}_+$, let $\psi_I \in \mathbf{S}$ be:

$$\psi_I = \theta + \Lambda\Lambda(0)^{-1}(1-\theta) + \frac{s}{s+\alpha}\hat{q}\Lambda\Gamma \tag{36}$$

$\hat{q} \in \mathbf{S}, \hat{q}(\infty) \neq (1-\theta(\infty))(\Lambda(\infty)^{-1} - \Lambda(0)^{-1})$. Then, with ψ_I as in (36), C_I in (37) is a decoupling integral-action controller

$$C_I = \psi_I\,(1-\psi_I)^{-1}P^{-1}. \tag{37}$$

For $\hat{q} = 0$, the integral-action decoupling controller becomes

$$C_I^o = \left[\,(1-\theta)^{-1}(1-\Lambda\Lambda(0)^{-1})^{-1} - 1\right]P^{-1}. \tag{38}$$

Then, $H_{yuI} = \psi_I I$, and $H_{euI} = (1-\psi_I)I = [(1-\theta)(1-\Lambda\Lambda(0)^{-1}) - \frac{s}{s+\alpha}\hat{q}\Lambda\Gamma]I$. □

Remarks 9: Justification of integral action for Proposition 4: For ψ_I as in (36), $(1-\psi_I(0)) = 0$ implies C_I in (37) has poles at $s = 0$, and $H_{euI}(0) = (1-\psi_I(0))I = 0$. The constraint on $\hat{q}(\infty)$ ensures that $(1-\psi_I)^{-1} \in \mathcal{M}(\mathbf{R_p})$; therefore, C in (37) is proper. □

V. EXAMPLES

Example 1: P satisfies Condition 1; has one coinciding RHP pole-zero: The plant $P = \begin{bmatrix} \frac{s-1}{s-2} & \frac{s-1}{(s-2)(s+4)} \\ \frac{s+3}{s-1} & \frac{2}{s+4} \end{bmatrix}$ has $\mathbb{C}_+$-poles at 1,2, and $\mathbb{C}_+$-zeros at 1,5, and infinity. Then, $y_1 = (s-2)$, $y_2 = (s-1)$. Define $h_1 = (s+4)$, $h_2 = (s-5)$. Choose $\varphi_1 = (s+3)$, $\varphi_2 = (s+5)$; $Y = \text{diag}\left[\frac{y_1}{\varphi_1}, \frac{y_2}{\varphi_2}\right]$, $(YP)^{-1} = \begin{bmatrix} \frac{2\varphi_1}{h_2} & \frac{-\varphi_2}{h_2} \\ \frac{-\varphi_1^2 h_1}{y_2 h_2} & \frac{\varphi_2 h_1}{h_2} \end{bmatrix}$. The zeros of $f_1 = y_2 h_2$ are $z_{11} = 1, z_{12} = 5$; the zero of $f_2 = h_2$ is $z_{21} = 5$. With $\rho_1 = \rho_2 = 1$, ξ_1 is a third order and ξ_2 is a second-order HP; $F_1 = \frac{f_1}{\xi_1}$ and $F_2 = \frac{f_2}{\xi_2}$. By (13), $\theta_1 = (1-Y_1)(1-Y_1(1)^{-1}Y_1)(1-Y_1(5)^{-1}Y_1) = \frac{-125\, y_2 f_2}{3\varphi_1^3}$, $\theta_2 = (1-Y_2)(1-Y_2(5)^{-1}Y_2) = \frac{-9f_2}{\varphi_2^2}$. **a)** By (16), $F_j(\infty) = 0$ for $j = 1,2$ implies $\psi_j = (\theta_j + q_j F_j Y_j)$ for any $q_j \in \mathbf{S}$. Let $g_1 := (3s^2 + 158s - 353)$, $g_2 := (5s^2 + 18s + 297)$. Choose $q_1 = q_2 = 0$; $H_{yu} = \text{diag}\left[\theta_1, \theta_2\right]$. By (17), $C = P^{-1}\text{diag}\left[\frac{-125\, y_2 f_2}{y_1 g_1}, \frac{-9f_2}{y_2(s+20)}\right] = \begin{bmatrix} \frac{-250y_2}{g_1} & \frac{9}{s+20} \\ \frac{125\varphi_1 h_1}{g_1} & \frac{-9h_1}{s+20} \end{bmatrix}$.

b) Choose $\xi_1 = \varphi_1^3$, $\xi_2 = \varphi_2^2$, $\hat{q}_1 = \hat{q}_2 = 0$; let $g_3 := (544s^3 + 1521s^2 + 34938s - 2187)$, $g_4 := (14s^2 + 185s + 125)$. By (30), $C_I^o = \begin{bmatrix} \frac{-2y_2 g_3}{s g_1 g_2} & \frac{g_4}{s(s+15)(s+20)} \\ \frac{\varphi_1 g_3 h_1}{s g_1 g_2} & \frac{-g_4 h_1}{s(s+15)(s+20)} \end{bmatrix}$ is an integral-action controller.

Example 2: P satisfies Condition 1, has two coinciding RHP poles zeros: Define $h_1 = (s-1)$, $h_2 = (s-2)$, $h_3 = (s+5)$, $h_4 = (s+7)$. The plant $P = \begin{bmatrix} \frac{-h_1 h_2}{(s+2)(s+4)^2} & \frac{h_1 h_4}{h_3(s+4)^2} & \frac{-h_1 h_2 h_4}{(s+2)(s+4)^2} \\ \frac{(s+2)^2}{h_2(s+4)} & \frac{-(s+2)^2}{h_2(s+4)h_3} & \frac{s+2}{s+4} \\ 0 & \frac{s+4}{h_1 h_3} & 0 \end{bmatrix}$ has coinciding $\mathbb{C}_+$-poles and zeros at $s = 1$ and $s = 2$. With $y_1 = 1$, $y_2 = h_2, y_3 = h_1$, $\varphi_1 = 1$, choose $\varphi_2 = (s+2), \varphi_3 = (s+4)$; then $Y = \text{diag}\left[1, \frac{y_2}{\varphi_2}, \frac{y_3}{\varphi_3}\right], (YP)^{-1} = \begin{bmatrix} \frac{\varphi_2}{h_1} & \frac{h_4}{\varphi_3} & 0 \\ 0 & 0 & h_3 \\ \frac{-\varphi_2^2}{h_1 h_2} & \frac{-1}{\varphi_3} & \frac{\varphi_2}{h_2} \end{bmatrix}$. By Proposition 1, $f_1 = h_1 h_2$, $f_2 = 1$, $f_3 = h_2$. The $\mathbb{C}_+$-zero of f_3 is $z_1 = 2$; $Y_3(z_1) \neq 0$. By (13), $\theta_3 = (1-Y_3)(1-Y_3(2)^{-1}Y_3) = \frac{-25h_2}{\varphi_3^2}$. From (14)–(16), $\psi_1 = F_1 q_1 = \frac{h_1 h_2}{\xi_1} q_1$, $\psi_2 = (1 - Y_2 q_2) = (1 - \frac{h_2}{\varphi_2} q_2)$, $\psi_3 = (\frac{-25h_2}{\varphi_3^2} + \frac{h_1 h_2}{\varphi_3 \xi_3} q_3)$; $q_1, q_2, q_3 \in \mathbf{S}$, $q_1(\infty) \neq 0$, $q_2(\infty) \neq 0$; ξ_1, ξ_3 are second-order HPs.

a) Choose $\xi_1 = \varphi_3(s+5)$, $q_1 = 0.5$, $q_2 = 1$, $q_3 = 0$. Define $g_1 := (s+19)$, $g_2 := (s+34)$, $g_3 := (33s^2 + 464s + 128)$. By (17), $C = P^{-1}\text{diag}\left[\frac{h_1 h_2}{\varphi_2 g_1}, \frac{4}{h_2}, \frac{-25h_2}{h_1 g_2}\right] = \begin{bmatrix} \frac{h_2}{g_1} & \frac{4h_4}{\varphi_2\varphi_3} & 0 \\ 0 & 0 & \frac{-25h_2 h_3}{\varphi_3 g_2} \\ \frac{-\varphi_2}{g_1} & \frac{-4}{\varphi_2\varphi_3} & \frac{-25\varphi_2}{\varphi_3 g_2} \end{bmatrix}$ is a decoupling controller.

b) In (26)–(28), choose $\xi_3 = \varphi_3^2$, $\hat{q}_1 = \hat{q}_3 = 0$, $\hat{q}_2 = 1$, $\alpha_2 = 4$. Then, $\psi_{1I} = \frac{10h_1 h_2}{\xi_1}$, $\psi_{2I} = \frac{8(s+1)}{\varphi_2\varphi_3}$, $\psi_{3I} = \frac{-h_2 g_3}{\varphi_3^4}$, $C_I = P^{-1}\,\text{diag}\left[\frac{-10h_1 h_2}{3s(3s-13)}, \frac{8(s+1)}{s h_2}, \frac{-h_2 g_3}{s h_1 g_2(s+16)}\right]$

$$= \begin{bmatrix} \frac{-10\varphi_2 h_2}{3s(3s-13)} & \frac{8h_4(s+1)}{s\varphi_2\varphi_3} & 0 \\ 0 & 0 & \frac{-h_2 g_3 h_3}{s\varphi_3 g_2(s+16)} \\ \frac{10\varphi_2^2}{3s(3s-13)} & \frac{-8(s+1)}{s\varphi_2\varphi_3} & \frac{-\varphi_2 g_3}{s\varphi_3 g_2(s+16)} \end{bmatrix}.$$ □

Example 3: P satisfies both Conditions 1 and 2: The plant $P = \begin{bmatrix} \frac{s-1}{s-2} & \frac{s-1}{(s-2)(s+4)} \\ \frac{s+3}{s-4} & \frac{2}{s+4} \end{bmatrix}$ has $\mathbb{C}_+$-poles at 2,4, and none coinciding with the $\mathbb{C}_+$-zeros at $1, 11$, and infinity. Define $h_1 := (s-1)$, $h_2 := (s-11)$. With $y_1 = (s-2)$, $y_2 = (s-4)$, choosing $\varphi_1 = (s+3)$, $\varphi_2 = (s+5)$, $Y = \text{diag}\left[\frac{y_1}{\varphi_1}, \frac{y_2}{\varphi_2}\right]$, $(YP)^{-1} = \begin{bmatrix} \frac{2y_2\varphi_1}{h_1 h_2} & \frac{-\varphi_2}{h_2} \\ \frac{-\varphi_1^2(s+4)}{h_1 h_2} & \frac{\varphi_2(s+4)}{h_2} \end{bmatrix}$. The zeros of $f_1 = h_1 h_2$ are $z_{11} = 1, z_{12} = 11$; the zero of $f_2 = h_2$ is $z_{21} = 11$. With $\rho_1 = \rho_2 = 1$, choosing $\xi_1 = \varphi_1^3$ and $\xi_2 = \varphi_2^2$, $F_1 = \frac{f_1}{\xi_1}$, $F_2 = \frac{f_2}{\xi_2}$. By (13), $\theta_1 = (1-Y_1)(1-Y_1(1)^{-1}Y_1)(1-Y_1(11)^{-1}Y_1) = \frac{-125h_1 h_2}{9\varphi_1^3}$, $\theta_2 = (1-Y_2)(1-Y_2(11)^{-1}Y_2) = \frac{-81h_2}{7\varphi_2^2}$.

a) By (16), $F_j(\infty) = 0$ implies $\psi_j = (\theta_j + q_j F_j Y_j)$ for any $q_j \in \mathbf{S}$. Let $q_1 = q_2 = 0$, $g_1 := (9\tilde{}s^2 + 224s - 809)$, $g_2 := (7s + 179)$, $g_3 := (11\tilde{}s^2 + 72s + 621)$; by (17), $C = P^{-1}\text{diag}\left[\frac{-125h_1 h_2}{g_1 y_1}, \frac{-81h_2}{g_2 y_2}\right] = \begin{bmatrix} \frac{-250\, y_2}{g_1} & \frac{81}{g_2} \\ \frac{125\varphi_1(s+4)}{g_1} & \frac{-81(s+4)}{g_2} \end{bmatrix}$.

b) Choosing $\hat{q}_1 = \hat{q}_2 = 0$, define $g_4 = (1132s^3 + 6813s^2 + 71064s - 6561)$, $g_5 = (1066s^2 + 12685s + 4375)$, $g_6 = (11s + 135)$. By (30), $C_I^o = P^{-1}\text{diag}\left[\frac{-g_4 h_1 h_2}{s g_1 g_3 y_1}, \frac{-g_5 h_2}{s g_2 g_6 y_2}\right] = \begin{bmatrix} \frac{-2\, g_4 y_2}{s g_1 g_3} & \frac{g_5}{s g_2 g_6} \\ \frac{\varphi_1 g_4(s+4)}{s g_1 g_3} & \frac{g_5(s+4)}{s g_2 g_6} \end{bmatrix}$ is an integral-action controller. □

Example 4: P satisfies Condition 2 but not Condition 1: Define $h_1 = (s+1)$, $h_2 = (s-1)$, $h_3 = (s^2 - 6s + 3)$, $h_4 = (s-3)$, $g_1 = (s-4)$, $g_2 = (s+4)$. The plant $P = \begin{bmatrix} \frac{(s+2)}{h_1 h_2} & \frac{1}{h_2} \\ \frac{-(s+2)}{h_2 h_4} & \frac{h_3}{h_2 h_4} \end{bmatrix}$ has no coincident $\mathbb{C}_+$-poles and zeros. Then, $y_1 = h_2$, $y_2 = y_1 h_4$. Choose $\varphi_1 = (s+2)$, $\varphi_2 = \varphi_1 g_2$; then $Y = \text{diag}\left[\frac{y_1}{\varphi_1}, \frac{y_2}{\varphi_2}\right]$, $(YP)^{-1} = \begin{bmatrix} \frac{h_1 h_3}{g_1 y_1} & \frac{-h_1 g_2}{g_1 y_1} \\ \frac{h_1\varphi_1}{g_1 y_1} & \frac{\varphi_2}{g_1 y_1} \end{bmatrix}$. The $\mathbb{C}_+$-zeros of $f_1 = f_2 = g_1 y_1$ are $z_{11} = z_{21} = 1, z_{12} = z_{22} = 4$. Since $y_1(z_{11}) = 0$ and $y_2(z_{21}) = 0$, P does not satisfy *Condition 1* required for the decoupling controller design in Proposition 1. Check the alternate *Condition 2:*$P^{-1} = \begin{bmatrix} \frac{h_1 h_3}{g_1\varphi_1} & \frac{-h_1 h_4}{g_1\varphi_1} \\ \frac{h_1}{g_1} & \frac{h_4}{g_1} \end{bmatrix}$. Then, $\lambda = g_1$ has only one $\mathbb{C}_+$-zero at $z_1 = 4$. With $\gamma = y_2$, P satisfies the *no RHP pole-zero coincidence condition* since $\gamma(4) \neq 0$. Let $\varphi = \varphi_2$, and since $\rho = 1$, choose ξ as any second-order HP; for example, $\xi = \varphi$. Then, $\Gamma = \frac{\gamma}{\varphi}$, $\Lambda = \frac{\lambda}{\xi}$. By (23), $\theta = (1-\Gamma)(1-\Gamma(4)^{-1}\Gamma) = \frac{-25\lambda(2s+1)(3s-2)}{\varphi^2}$. a) Following Proposition 2, choosing $q = 0$, by (25), $C = \theta(1-\theta)^{-1}P^{-1} = \frac{-25\lambda(2s+1)(3s-2)}{\gamma(s^2+166s+88)}P^{-1}$, and $H_{yu} = \psi I = \theta I = \frac{-25\lambda(2s+1)(3s-2)}{(s+2)^2(s+4)^2} I$. b) Since P has no zeros at $s = 0$, integral-action decoupling controllers are obtained by Proposition 4. For $\hat{q} = 0$, by (37), C_I becomes $C_I^o = \frac{-(152s^4+1199s^3-146s^2-208s+128)}{\gamma s(s+8)(s^2+166s+88)} \begin{bmatrix} \frac{h_1 h_3}{\varphi_1} & \frac{-h_1 h_4}{\varphi_1} \\ h_1 & h_4 \end{bmatrix}$, and $H_{yuI} = \frac{-\lambda(152s^4+1199s^3-146s^2-208s+128)}{(s+2)^3(s+4)^3} I$; then $H_{euI} = \frac{s\gamma(s+8)(s^2+166s+88)}{(s+2)^3(s+4)^3} I$ is zero at $s = 0$. □

Example 5: P cannot be decoupled: The plant $P = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} \end{bmatrix}$, which was considered in [11], [15], [16], has a coinciding $\mathbb{C}_+$-pole and zero at $s = 1$. With $y_1 = 1$, $y_2 = (s-1)$, choosing $\varphi_2 = (s+5)$, $Y = \text{diag}\left[1, \frac{s-1}{s+5}\right]$, $(YP)^{-1} = \begin{bmatrix} \frac{s(s+1)}{s-1} & \frac{-(s+1)(s+5)}{s-1} \\ \frac{-(s+2)}{s-1} & \frac{(s+2)(s+5)}{s-1} \end{bmatrix}$. The only $\mathbb{C}_+$-zero of $f_1 = f_2 = (s-1)$ is $z_{11} = z_{21} = 1$. The *diagonal denominator condition* is not satisfied since $y_2(1) = 0$. Since both *Condition 1 and 2* fail, by Theorem 1, there are no decoupling controllers for P. □

VI. Conclusion

Simple decoupling controller design procedures were proposed for all plant classes that can be diagonally decoupled. The designs include the option of integral action in the controllers, which implies that steady-state errors for constant reference inputs go to zero asymptotically. The controllers are derived from the inverse of the plant's transfer function, without using coprime factorizations. With minor modifications, the designs can be applied to non-square full row-rank plants by using right inverses.

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