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Brief paper

Controller redesign for delay margin improvement[☆]

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ABSTRACT

Two important design objectives in feedback control are steady-state error minimization and delay margin maximization. For many practical systems it is not possible to have infinite delay margin and zero steady state error for unit step reference input. This paper proposes a re-design method for controllers initially designed to satisfy the steady-state error requirement. The objective is to make structural changes in the controller so that a lower bound of the delay margin is improved without affecting the steady-state error. The order of the new controller is $(\nu+1)$ higher than the order of the original controller, where ν is the number of unstable poles of the plant.

1. Introduction

In feedback control theory, one of the most important stability robustness measures is the *delay margin (DM)*. Classical techniques, such as lead–lag and PID controller designs, try to meet a given desired phase margin requirement, Dorf and Bishop (2017); but these designs do not directly guarantee the amount of delay uncertainty that can be tolerated. In order to tackle this issue directly, many studies were devoted to the *delay margin optimization* problem in recent years, (Middleton & Miller, 2007; Qi, Zhu, & Chen, 2017) and (Zhu, Qi, Ma, & Chen, 2018). More precisely, it would be desirable to compute the largest possible *DM* that can be obtained over all stabilizing controllers for a given nominal plant. Finding the *optimal controller* maximizing the *DM* is still an open problem for the general class of unstable plants with multiple poles in the right half-plane (Zhu et al., 2018). Therefore, recent publications on this topic consider some special class of plants, or investigate upper and lower bounds of the largest achievable *DM*, see e.g., Ju and Zhang (2016), Middleton and Miller (2007) and Qi et al. (2017).

Typically, there are many controller design requirements other than *DM* optimization. These additional design specifications are related to tracking performance, disturbance rejection /attenuation, or sensitivity shaping, (Doyle, Francis, & Tannenbaum, 1992). This paper considers the *delay margin improvement* problem, for an initially designed stabilizing controller achieving a desired steady state error for a specified reference input. This is quite similar to modification of an initially designed lead–lag controller in order to improve the phase margin. For stable plants, lag controllers typically increase the phase margin and decrease the crossover frequency Dorf and Bishop (2017). For stable systems, the *DM* can be improved by lag controllers since the *DM* is the ratio of phase margin over crossover frequency, Özbay (1999), However, *DM* improvement is not as simple for unstable plants.

If the nominal plant is stable, then the largest achievable *DM* is infinity, and it is obtained by a small gain controller (e.g., zero controller gives infinite *DM* for stable plants). On the other hand, such a controller has poor tracking performance, with large steady-state tracking error for a unit step reference signal. Therefore, it makes sense to design a stabilizing integral action controller first so that the steady-state error is zero when unit step reference input is applied, and then to modify the controller structure to improve the delay margin without changing the steady-state tracking performance.

The paper is organized as follows: In Section 2, following the definition of delay margin (*DM*), its computation and lower bounds are discussed. The trade-off with *DM* maximization and tracking error minimization is also illustrated. The main results presented in Section 3 expand and provide more detailed discussions of the preliminary results that were presented in Gündeş and Özbay (2019). A tuning method is proposed so that a previously designed controller that satisfies the steady-state requirements is modified in order to improve the *DM* without changing the controller poles at $s = 0$. Three examples are given

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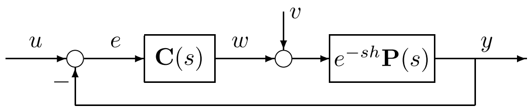

Fig. 1. The feedback system $\mathscr{S}(e^{-sh}\mathbf{P}, \mathbf{C})$.

in Section 4. Conclusions are in Section 5. Detailed proofs are given in the Appendix.

Notation: The closed right half-plane (RHP) is $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \Re e(s) \geq 0\}$, and the open left half-plane (LHP) is $\mathbb{C}_- = \{s \in \mathbb{C} \mid \Re e(s) < 0\}$. The region of instability $\mathcal{U}$ is the extended closed RHP, i.e., $\mathcal{U} = \mathbb{C}_+ \cup \{\infty\}$. Real and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively; $\mathcal{R}_p$ denotes real proper rational functions of s; $\mathcal{S} \subset \mathcal{R}_p$ is the stable subset with no poles in $\mathcal{U}$. The space $\mathcal{H}_\infty$ is the set of all bounded analytic functions in $\mathbb{C}_+$. A matrix-valued function H is in $\mathcal{M}(\mathcal{H}_\infty)$ if all its entries are in $\mathcal{H}_\infty$. For $f \in \mathcal{H}_\infty$, the norm $\|\cdot\|$ is defined as $\|f\| := \text{ess}\sup_{s\in\mathbb{C}_+}|f(s)|$, where ess sup denotes the essential supremum. The degree of the polynomial d is denoted by $\deg(d)$. For simplicity, we drop (s) in transfer functions such as $\mathbf{P}(s)$ when this causes no confusion.

2. Problem definition and preliminaries

Consider the feedback system $\mathscr{S}(e^{-sh}\mathbf{P}, \mathbf{C})$ in Fig. 1. The rational transfer functions $\mathbf{P} \in \mathcal{R}_p$ and $\mathbf{C} \in \mathcal{R}_p$ represent a given nominal plant (without time delays) and the controller. The input–output map from u to y (complementary sensitivity function) is denoted by H_{yu}; the input-error map from u to e (sensitivity function) is denoted by H_{eu}. With u, v, w, y as inputs and outputs, the closed-loop map $\mathbf{H}$ is given by (1):

$$\begin{aligned} H_{yu} &= e^{-sh}\mathbf{PC}(1+e^{-sh}\mathbf{PC})^{-1}\ , \\ H_{eu} &= (1+e^{-sh}\mathbf{PC})^{-1} = I - H_{yu}\ , \\ \mathbf{H} &= \begin{bmatrix} \mathbf{C}H_{eu} & -\mathbf{C}H_{eu}e^{-sh}\mathbf{P} \\ H_{yu} & H_{eu}e^{-sh}\mathbf{P} \end{bmatrix} \end{aligned} \tag{1}$$

Definition 1. **(a)** The feedback system $\mathscr{S}(e^{-sh}\mathbf{P}, \mathbf{C})$ shown in Fig. 1 is stable if $\mathbf{H}$ is in $\mathcal{M}(\mathcal{H}_\infty)$.
(b) The controller $\mathbf{C} \in \mathcal{R}_p$ is a stabilizing controller for $e^{-sh}\mathbf{P}$ if $\mathscr{S}(e^{-sh}\mathbf{P}, \mathbf{C})$ is stable.
(c) The system $\mathscr{S}(e^{-sh}\mathbf{P}, \mathbf{C})$ is stable and has integral-action if the closed-loop map $\mathbf{H}$ is stable, and the (input-error) transfer function H_{eu} has zeros at $s = 0$.
(d) The controller $\mathbf{C}$ is an integral-action controller if $\mathbf{C}$ stabilizes $e^{-sh}\mathbf{P}$ and $\mathbf{C}$ has at least one pole at $s = 0$.
(e) Let $\mathbf{C} \in \mathcal{R}_p$ be a stabilizing controller for the delay-free plant $\mathbf{P}$. The minimum time-delay $h_m > 0$ such that the closed-loop system $\mathscr{S}(e^{-sh_m}\mathbf{P}, \mathbf{C})$ becomes unstable is called the delay margin (DM). □

An initial controller $\mathbf{C}_o(s)$ is designed to stabilize the delay-free feedback system $\mathscr{S}(\mathbf{P}, \mathbf{C}_o)$. The input–output transfer function H_{yu} with the controller $\mathbf{C}_o$ defined as

$$H_o := \mathbf{PC}_o(1+\mathbf{PC}_o)^{-1}\ . \tag{2}$$

The following assumption is used throughout:

Assumption. The stabilizing controller $\mathbf{C}_o$ is designed so that the open-loop system is strictly proper, i.e. $G_o(\infty) = (\mathbf{PC}_o)(\infty) = 0$. Consequently, the closed-loop transfer function H_o is also strictly proper.

This assumption ensures that the characteristic equation of the feedback system is a retarded quasi-polynomial. Hence, there is no possibility of neutral chain of poles asymptotically approaching a vertical line in the complex plane. By continuity, the feedback system $\mathscr{S}(e^{-hs}\mathbf{P}, \mathbf{C}_o)$ with delayed plant is stable for all $h \in [0, h_m)$ for some $h_m > 0$. The largest possible h_m satisfying this property is the *delay margin (DM)* of the feedback system $\mathscr{S}(\mathbf{P}, \mathbf{C}_o)$. When $\mathbf{P}$ and $\mathbf{C}_o$ are rational transfer functions, the *DM* can be computed from the Nyquist plot of G_o, (Özbay, 1999). The Matlab command `allmargin` gives a "vector of delay margins", computed as $[\phi_1/\omega_1, \ldots, \phi_k/\omega_k]$, where ω_i's are the gain crossover frequencies, i.e., $|G_o(j\omega_i)| = 1$, and $\phi_i = \pi + \angle G_o(j\omega_i)$, for $i = 1, \ldots, k$, with $\angle G_o(j\omega_i)$ denoting the phase. Then, the *DM* of the feedback system $\mathscr{S}(\mathbf{P}, \mathbf{C}_o)$ is

$$DM = \min\{\phi_1/\omega_1, \ldots, \phi_k/\omega_k\}.$$

For example, the feedback system with plant and controller given as in (3) has three crossover frequencies, and the *DM* is 0.1363 s (we take the smallest of the three values given by the `allmargin` command of Matlab):

$$\mathbf{P}(s) = \frac{10(s^2+0.4s+1)}{s\,(s^2+2.4s+36)}\ , \quad \mathbf{C}_o(s) = \frac{2}{s}\ . \tag{3}$$

Although the *DM* can be numerically computed as precisely as desired, sometimes it is helpful to estimate its lower and upper bounds for controller design purposes, (Zhu et al., 2018). In this paper, we use such a lower bound and modify the controller $\mathbf{C}_o$ by introducing extra design parameters that improve the lower bound.

Assuming that $\mathscr{S}(\mathbf{P}, \mathbf{C}_o)$ is stable, a sufficient condition for stability of $\mathscr{S}(e^{-hs}\mathbf{P}, \mathbf{C}_o)$ for all $h \in [0, h_m)$ can be derived from the small gain theorem as

$$\|(1-e^{-hs})H_o\| < 1\ , \quad \forall\ h \in [0, h_m)\ . \tag{4}$$

It can be shown that (4) holds if (see Qi et al. (2017))

$$|\psi_{h_m}(\omega)\ H_o(j\omega)| < 1\ , \quad \forall\ \omega \in \mathbb{R}\ , \tag{5}$$

where $\psi_{h_m}(\omega) = 2\sin(\omega h_m/2)$ for $\omega \in [0, \pi/h_m)$ and $\psi_{h_m}(\omega) = 2$ for $\omega \geq \pi/h_m$. Clearly, the largest h_m for which (5) holds gives a lower bound of the actual *DM*, and it can be computed graphically by finding the largest h_m satisfying

$$|\psi_{h_m}(\omega)| < \frac{1}{|H_o(j\omega)|}\ , \quad \forall\ \omega \in \mathbb{R}\ . \tag{6}$$

Other sufficient conditions for (4) are

$$|\theta_{h_m}(\omega)| < \frac{1}{|H_o(j\omega)|}\ , \quad \forall\ \omega \in \mathbb{R}\ , \tag{7}$$

where $\theta_{h_m}(\omega) = h_m\omega$ for $\omega \in [0\ ,\ 2/h_m)$, $\theta_{h_m}(\omega) = 2$ for $\omega \geq 2/h_m$, and

$$h_m\omega < \frac{1}{|H_o(j\omega)|}\ , \quad \forall\ \omega \in \mathbb{R}. \tag{8}$$

Note that (8) implies (7), which in turn implies (5); but in many practical cases they all estimate the *DM* closely. For example, the *DM* is about 0.136 s for the feedback system defined by (3). The graphs of $1/|H_o(j\omega)|$, $\psi_h(j\omega)$, $\theta_h(\omega)$ and $h\omega$ shown in Fig. 2 illustrate that all of the sufficient conditions above are satisfied for $h = 0.11$ s., which is relatively close to the exact *DM*.

In what follows we use the *DM* lower bound $DM > \|sH_o\|^{-1}$ determined from (8), which can be computed easily from the $\mathcal{H}_\infty$-norm of a rational transfer function (the related Matlab command is `hinfnorm`). Note that for any rational minimum phase transfer function $w_h(s)$ satisfying (9), a lower bound of the *DM* is given by the largest $h > 0$ that satisfies (10):

$$|w_h(j\omega)| \geq \psi_h(\omega)\ , \quad \forall\ \omega\ , \quad \forall h > 0\ , \tag{9}$$

$$\|w_h H_o\| < 1\ . \tag{10}$$

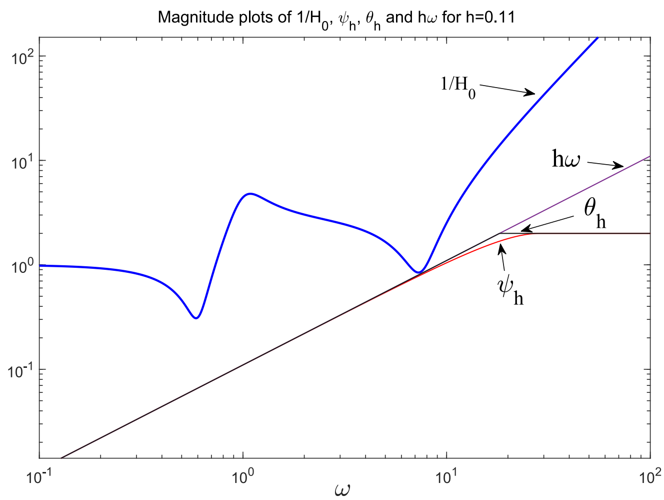

Fig. 2. Various functions involved in the computation of the lower bounds for the delay margin.

Obviously $w_h(s) = hs$ is a special case that satisfies (10). Various possible choices of $w_h(s)$ can be found in Özbay, Gümüşsoy, Kashima, and Yamamoto (2018) and Zhu et al. (2018). Thus, once the controller $\mathbf{C}_o$ is free in H_o, a lower bound of the largest achievable *DM* can be computed by solving a Nevanlinna–Pick interpolation problem resulting from (10) (see Theorem 4.4 of Zhu et al. (2018), and also Section 5.1.2 of Özbay et al. (2018)). However, the controller obtained from this design may have poor step response performance. As an example, consider a simple case, where $\mathbf{P}(s) = (s+a)/(s-p)$, with $a > 0$ and $p > 0$. This corresponds to a single interpolation condition, and a lower bound of the *DM* is the largest $h > 0$ satisfying $|w_h(p)| < 1$, with the corresponding optimal controller

$$\mathbf{C}_{opt}(s) = \frac{w_h(p)}{(s+a)} \left(\frac{(s-p)}{w_h(s) - w_h(p)} \right).$$

This controller, which is designed to maximize a lower bound of the *DM*, typically does not have high gain at low frequencies; hence it will lead to a large steady-state error e_{ss} for constant reference inputs, where $e_{ss} = \lim_{s\to 0} \left(1 - \frac{w_h(p)}{w_h(s)}\right)$. The steady-state error is nonzero whenever $w_h(0) \neq w_h(p)$. Typically, since w_h is chosen to have very small values at $s = 0$, this means that the steady-state error becomes large as p gets large.

3. Delay margin improvement

Before tackling the technical difficulties appearing in the case of general unstable plants, we revisit controller parametrization for stable plants in Section 3.1, and discuss how free parameters are designed to meet a desired lower bound for the *DM*. In Section 3.2 we consider unstable plants. We assume that an initial controller $\mathbf{C}_o$ was designed to stabilize the plant and to achieve certain steady-state performance objectives. The controller is then modified to *improve a lower bound on DM*, without modifying the steady-state performance.

3.1. DM from controller parametrization for stable plants

Suppose that $\mathbf{P}$ is stable. Let $\mathbf{C}_o$ be any controller such that the delay-free closed-loop system $\mathscr{S}(\mathbf{P}, C_o)$ is stable. Then the stabilized system has *delay robustness*; i.e., any controller $\mathbf{C}_o$ that stabilizes the delay-free plant $\mathbf{P}$ also stabilizes the delayed plant $e^{-sh}\mathbf{P}$ for all $h < h_m$, where h_m is the *DM*. For stable plants, it is possible to design controllers that achieve closed-loop stability for any given delay $h = \tau > 0$. Therefore, if the maximum expected delay τ is known, then the controller can be designed to stabilize $e^{-s\tau}\mathbf{P}$. Furthermore, the same controller stabilizes $e^{-sh}\mathbf{P}$ for all $h \leq \tau$. These results are formally stated in Proposition 3.1 for general stabilizing controllers, as well as integral-action controllers.

Proposition 3.1 (*Controller Design to Meet a Specified DM for Stable Plants*)**.** *Let $\mathbf{P} \in \mathcal{S}$ and $\mathbf{C}_o$ be a controller that stabilizes $\mathbf{P}$, i.e., for $\tilde{Q} \in \mathcal{S}$, let*

$$\mathbf{C}_o = \tilde{Q}\,(1 - \mathbf{P}\,\tilde{Q})^{-1}\,. \tag{11}$$

(a) *In (11), for any $Q \in \mathcal{S}$ and $a \in \mathbb{R}_+$, let $\tilde{Q} \in \mathcal{S}$ be such that $(\mathbf{PC}_o)(\infty) = 0$, i.e., let*

$$\tilde{Q} := \begin{cases} Q\,, & \text{if } \mathbf{P}(\infty) = 0 \\ \frac{1}{s+a} Q\,, & \text{if } \mathbf{P}(\infty) \neq 0 \end{cases} \tag{12}$$

(i) *Let the controller $\mathbf{C}_o$ in (11) be pre-specified, i.e., $\tilde{Q} \in \mathcal{S}$ is fixed. Then $\mathbf{C}_o$ in (11) stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_m)$, where τ_m is a lower bound on the DM:*

$$\tau_m = \|s\mathbf{P}\tilde{Q}\|^{-1}\,. \tag{13}$$

(ii) *For a given delay $h = \tau \in \mathbb{R}_+$, the controller $\mathbf{C}_o$ in (11) can be designed to stabilize $e^{-s\tau}\mathbf{P}$ by choosing $Q \in \mathcal{S}$ in (12) such that*

$$\|Q\| < \begin{cases} \tau^{-1}\,\|s\mathbf{P}\|^{-1}\,, & \text{if } \mathbf{P}(\infty) = 0 \\ \tau^{-1}\,\|\frac{s}{s+a}\mathbf{P}\|^{-1}\,, & \text{if } \mathbf{P}(\infty) \neq 0\,. \end{cases} \tag{14}$$

Furthermore, with Q chosen as in (14), the corresponding lower bound $\tau_m = \|s\mathbf{P}\tilde{Q}\|^{-1}$ defined in (13) can be found, where $\tau_m \geq \tau$. Therefore, the controller $\mathbf{C}_o$ in (11) also stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_m)$, where $\tau_m \geq \tau$. This means that arbitrarily large DM can be achieved by the controller choice determined via (14).

(b) *Integral-action controllers: Assume that $\mathbf{P}(0) \neq 0$. For any $Q_I \in \mathcal{S}$ and $a,\ b \in \mathbb{R}_+$ define*

$$\tilde{Q}_I := \frac{b}{s+b}\mathbf{P}(0)^{-1}(1 + \frac{s}{(s+a)}Q_I\,)\,. \tag{15}$$

With $\tilde{Q}_I$ as in (15), the controller $\mathbf{C}_I$ given by (16) is an integral-action controller that stabilizes $\mathbf{P}$:

$$\mathbf{C}_I = \tilde{Q}_I\,(1 - \mathbf{P}\,\tilde{Q}_I)^{-1}\,. \tag{16}$$

(i) *Let the controller $\mathbf{C}_I$ in (16) be pre-specified, i.e., $\tilde{Q}_I \in \mathcal{S}$ in (15) is fixed. The controller $\mathbf{C}_I$ in (16) stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_{mI})$, where τ_{mI} is a lower bound on the DM, defined as*

$$\tau_{mI} = \frac{1}{b}\,|\mathbf{P}(0)|\,\|\,\frac{s}{(s+b)}\mathbf{P}(1 + \frac{s}{(s+a)}Q_I)\|^{-1}\,. \tag{17}$$

(ii) *For a given delay $h = \tau \in \mathbb{R}_+$, the controller $\mathbf{C}_I$ in (16) can be designed to stabilize $e^{-s\tau}\mathbf{P}$ by choosing any $Q_I \in \mathcal{S}$, and $b > 0$ in (15) such that*

$$0 < b < \frac{1}{\tau}\,|\mathbf{P}(0)|\,\|\mathbf{P}(1 + \frac{s}{(s+a)}Q_I\,)\,\|^{-1}\,. \tag{18}$$

Furthermore, with b is chosen as in (18), the corresponding lower bound τ_{mI} can be found as in (17), and $\tau_{mI} \geq \tau$. Therefore, the controller $\mathbf{C}_I$ in (16) also stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_{mI})$, where $\tau_{mI} \geq \tau$.

Example 3.1. Consider the plant $\mathbf{P} \in \mathcal{S}$ in (19). Since $\mathbf{P}(0) = 1 \neq 0$, integral-action controllers can be designed as in (16). Choosing $Q_I = 0$ and $b = 2$, the integral-action controller in (16) and the corresponding delay-free closed-loop transfer function H_{yu} are as in (20)–(21):

$$P = \frac{(1 - 0.1\,s)}{(s^2 + 0.5\,s + 1)}\,, \tag{19}$$

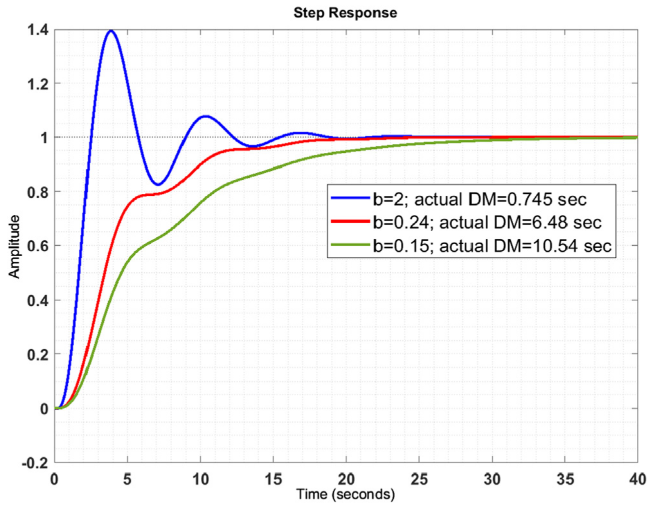

Fig. 3. Step responses of H_{yu}, for three different $\mathbf{C}_I$ with $b = 2$; $b = 0.24$; $b = 0.15$.

$$\mathbf{C}_I = \frac{2(s^2 + 0.5\,s + 1)}{s\,(s^2 + 2.5\,s + 2.2)} \tag{20}$$

$$H_{yu} = \frac{2\,(1 - 0.1\,s)}{(s + 2)(s^2 + 0.5\,s + 1)}\ . \tag{21}$$

(i) By (17), τ_{mI} is

$$\tau_{mI} = \frac{1}{b}\,\mathbf{P}(0)\,\|\frac{s}{(s+b)}\mathbf{P}\|^{-1} = 0.556\ \text{s}\ . \tag{22}$$

Then the controller $\mathbf{C}_I$ in (20) stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_{mI})$. The exact DM is 0.745 s.

(ii) For a fixed $\tau > 0$, $\|P\| = P(0)$ implies (18) is satisfied for $0 < b < \tau^{-1}$. For example, suppose that $\tau = 2$; then (18) is satisfied for $0 < b < 0.241$. The choice of b then determines τ_{mI}, and the controller $\mathbf{C}_I$ in (20) stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_{mI})$. For example, for $b = 0.24$, $\tau_{mI} = 2.073 > \tau$, and for $b = 0.15$, $\tau_{mI} = 3.25 > \tau$.

$$\mathbf{C}_I = \frac{0.24(s^2 + 0.5\,s + 1)}{s\,(s^2 + 0.74\,s + 1.144)} \quad \text{for} \quad b = 0.24$$

$$\mathbf{C}_I = \frac{0.15(s^2 + 0.5\,s + 1)}{s\,(s^2 + 0.65\,s + 1.09)} \quad \text{for} \quad b = 0.15\ .$$

Small values of b make the DM lower bound large, but this leads to a slower step response as seen in Fig. 3. This is a fundamental trade-off in controller design.

3.2. Delay margin improvement for unstable plants

Consider an unstable plant transfer function $\mathbf{P}$, factored into numerator and (monic) denominator polynomials:

$$\mathbf{P}(s) = \frac{n(s)}{d_s(s)\,d(s)}\ . \tag{23}$$

The roots of d and d_s are the $\mathbb{C}_+$-poles and $\mathbb{C}_-$-poles of $\mathbf{P}$, respectively. The degree of the polynomial d is $\deg(d) := \nu \geq 1$, where ν is the number of unstable poles of $\mathbf{P}$. Suppose that $p_i \in \mathbb{C}_+$, $i = 1, \ldots, \nu$ are the $\mathbb{C}_+$-poles of $\mathbf{P}$, ordered as follows: The first k poles are at zero, where $0 \leq k \leq \nu$, The next ℓ of the $\mathbb{C}_+$-poles are real, $0 \leq \ell \leq (\nu - k)$. The remaining $\mathbb{C}_+$-poles are m complex-conjugate pairs $p_{i,i+1} = \Re e(p_i) \pm j\Im m(p_i)$, where $2m = \nu - (k + \ell)$. Therefore, d can be expressed as

$$d(s) = s^k \prod_{i=k+1}^{k+\ell} (s - p_i) \prod_{i=k+\ell+1}^{k+\ell+m} (s^2 - 2\alpha_i s + \omega_i^2)\ , \tag{24}$$

where $\alpha_i := \Re e(p_i) > 0$ and $\omega_i := |\,p_i\,|$ for complex conjugate poles. For $\beta_i \geq 0$, $i = 1, \ldots, \nu$, define

$$\chi_\beta(s) := \prod_{i=1}^{\nu} (s + \beta_i + |p_i|\,)\ . \tag{25}$$

Lemma 3.1. *Suppose that $p_i \in \mathbb{C}_+$, $i = 1, \ldots, \nu$. Let $\beta_0 > 0$ and $\beta_i \geq 0$ be real constants satisfying $(\beta_i + |\,p_i\,|) > 0$, $i = 1, \ldots, \nu$. With $d(s)$ and $\chi_\beta(s)$ defined as in (24)–(25), the following norm equalities hold:*

$$\|\,s\left(1 - \frac{d(s)}{\chi_\beta(s)}\right)\| = \sum_{i=1}^{\nu} (\beta_i + p_i + |\,p_i\,|) =: \psi \tag{26}$$

$$\|\,s\left(1 - \frac{s\,d(s)}{(s + \beta_0)\,\chi_\beta(s)}\right)\| = \beta_0 + \psi\ . \tag{27}$$

A special case of Lemma 3.1 when all $p_i = 0$ can be stated as follows: For any set of positive constants $\beta_i > 0$, $i = 1, \ldots, k$, the following norm equality holds:

$$\|\,s\,(\,1 - \frac{s^k}{\prod_{i=1}^{k}(s + \beta_i)}\,)\,\| = \sum_{i=1}^{k} \beta_i\ . \tag{28}$$

The main result in Proposition 3.2 develops a controller design method to improve the *DM* over an existing stabilizing controller without changing its poles at $s = 0$.

Proposition 3.2 (*DM Improvement for Unstable* $\mathbf{P}$). *Suppose that $\mathbf{P} \notin \mathcal{S}$. Let $p_i \in \mathbb{C}_+$, $i = 1, \ldots, \nu$, be the $\mathbb{C}_+$-poles of $\mathbf{P}$, ordered as in (24).*

(a) *Let $\mathbf{C}_o$ be a stabilizing controller for $\mathbf{P}$ such that $H_o = \mathbf{P}\mathbf{C}_o\,(1 + \mathbf{P}\mathbf{C}_o)^{-1}$ is strictly proper. Then the controller $\mathbf{C}_o$ stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_m)$, where*

$$\tau_m := \|\,s\,H_o\,\|^{-1}\ . \tag{29}$$

(b) *Let $\mathbf{C}_o$ be as in part* **(a)**. *With $\beta_0 > 0$, define W as*

$$W(s) := \begin{cases} \dfrac{s}{s + \beta_0}\ , & \text{if } k = 0 \\ 1\ , & \text{if } k \neq 0\ . \end{cases} \tag{30}$$

For $i = 1, \ldots, k$, choose $\beta_i > 0$, and for $i = k+1, \ldots, \nu$, choose $\beta_i \geq 0$. Define $U := W\,d/\chi_\beta$ and let

$$\mathbf{C}_\beta = (1 - U)(1 + U\mathbf{C}_o\mathbf{P})^{-1}\mathbf{C}_o\ . \tag{31}$$

Then the controller $\mathbf{C}_\beta$ in (31) stabilizes $e^{-sh}\mathbf{P}$ for $h \in [0, \tau_{\beta m})$, where $\tau_{\beta m}$ in (32) is a lower bound on the DM:

$$\tau_{\beta m} = \|\,s\,H_\beta\,\|^{-1}\ . \tag{32}$$

Furthermore, the DM lower bound satisfies

$$\tau_{\beta m} \geq \left(\sum_{i=\varsigma}^{\nu} \beta_i + \sum_{i=1}^{\nu} (p_i + |\,p_i\,|)\right)^{-1} \|\,H_o\,\|^{-1} \tag{33}$$

where $\varsigma = 0$ if $k = 0$ and $\varsigma = 1$ if $k \neq 0$. A sufficient condition for the DM lower bound $\tau_{\beta m}$ to exceed the previous DM lower bound τ_m is

$$\left(\sum_{i=k+1}^{\nu} (p_i + |\,p_i\,|)\right) \frac{\|\,H_o\,\|}{\|\,s\,H_o\,\|} < 1\ . \tag{34}$$

If (34) holds, then choose β_i as follows: If $k = 0$, choose $\beta_0 > 0$; otherwise, choose $\beta_0 = 0$. For $i = 1, \ldots, k$, choose $\beta_i > 0$, and for $i = k+1, \ldots, \nu$ choose $\beta_i \geq 0$ such that

$$\sum_{i=\varsigma}^{\nu} \beta_i < \frac{\|\,s\,H_o\,\|}{\|\,H_o\,\|} - \left(\sum_{i=k+1}^{\nu} (p_i + |\,p_i\,|)\right). \tag{35}$$

Then we have $\tau_{\beta m} > \tau_m$.

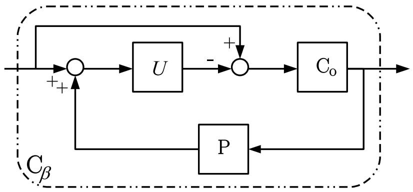

Fig. 4. Implementation of $\mathbf{C}_\beta$.

With $\mathbf{C}_\beta$ as (31), which can be implemented as in Fig. 4, the closed-loop input–output transfer function is

$$H_\beta := \mathbf{P}\mathbf{C}_\beta(1+\mathbf{P}\mathbf{C}_\beta)^{-1} = (1-U)H_o \,. \tag{36}$$

A special case of Proposition 3.2 is when the only unstable poles of $\mathbf{P}$ are at $s=0$ as stated in Corollary 3.1. These types of plants are of special interest in various applications, (Niculescu & Michiels, 2004).

Corollary 3.1 (*Plants with a Chain of Integrators*)**.** *Let $\mathbf{P} \notin \mathcal{S}$, where $d(s) = s^\nu$ in (24). Let $\mathbf{C}_o$ be a stabilizing controller for $\mathbf{P}$ such that $H_o = \mathbf{P}\mathbf{C}_o(1+\mathbf{P}\mathbf{C}_o)^{-1}$ is strictly proper. Then the controller $\mathbf{C}_o$ stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_m)$, where $\tau_m = \|\, s H_o \,\|^{-1}$. For $i = 1, \ldots, \nu$, choose $\beta_i > 0$. Let*

$$\mathbf{C}_\beta = \left(1 - \frac{s^\nu}{\prod_{i=1}^{\nu}(s+\beta_i)}\right)\left(1 + \frac{s^\nu}{\prod_{i=1}^{\nu}(s+\beta_i)}\mathbf{C}_o\mathbf{P}\right)^{-1}\mathbf{C}_o \,. \tag{37}$$

Then, the new complementary sensitivity is

$$H_\beta = \mathbf{P}\mathbf{C}_\beta(1+\mathbf{P}\mathbf{C}_\beta)^{-1} = \left(1 - \frac{s^\nu}{\prod_{i=1}^{\nu}(s+\beta_i)}\right)H_o \,. \tag{38}$$

(i) *The controller $\mathbf{C}_\beta$ in (37) stabilizes $e^{-sh}\mathbf{P}$ for $h \in [0, \tau_{\beta m})$, where $\tau_{\beta m} = \|\, s H_\beta \,\|^{-1}$. Furthermore,*

$$\tau_{\beta m} \geq \left(\sum_{i=1}^{\nu} \beta_i\right)^{-1} \|\, H_o \,\|^{-1} \,. \tag{39}$$

A sufficient condition for $\tau_{\beta m}$ to exceed τ_m is the choice of $\beta_i \in \mathbb{R}_+$, $i = 1, \ldots, \nu$ such that

$$\sum_{i=1}^{\nu} \beta_i < \frac{\|\, s H_o \,\|}{\|\, H_o \,\|} \,. \tag{40}$$

(ii) *For any given delay $h = \tau \in \mathbb{R}_+$, the controller $\mathbf{C}_\beta$ in (37) can be designed to stabilize $e^{-s\tau}\mathbf{P}$ by choosing $\beta_i \in \mathbb{R}_+$, $i = 1, \ldots, \nu$, to satisfy*

$$\sum_{i=1}^{\nu} \beta_i < \tau^{-1} \|\, H_o \,\|^{-1} \,. \tag{41}$$

Furthermore, once $\beta_i \in \mathbb{R}_+$, are chosen, the corresponding lower bound $\tau_{\beta m} = \|\, s H_\beta \,\|^{-1}$ can be found, where $\tau_{\beta m} \geq \tau$. Therefore, the controller $\mathbf{C}_\beta$ in (37) also stabilizes $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_{\beta m})$, where $\tau_{\beta m} \geq \tau$.

4. Examples with unstable plants

Example 4.1 (*Plant with Double Integrator*)**.** Consider

$$\mathbf{P}(s) = \frac{n(s)}{d(s)\,d_s(s)} = \frac{(s-4)^2}{s^2(s+4)} \,. \tag{42}$$

Since $\mathbf{P}$ is strictly proper, the transfer function H_o is strictly proper for every stabilizing controller $\mathbf{C}_o$. The first order controller $\mathbf{C}_o$ given in (43) stabilizes $\mathbf{P}$:

$$\mathbf{C}_o = \frac{1.05\,(s+0.2)}{(s+8)} \,. \tag{43}$$

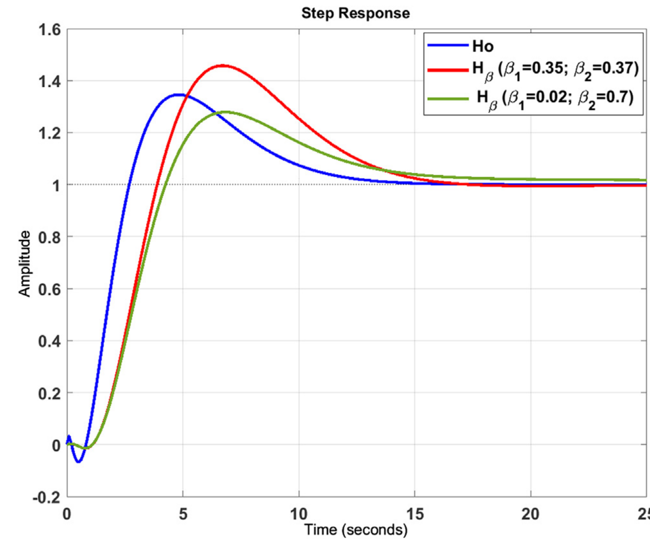

Fig. 5. Closed-loop step response of Example 4.1 for $h=0$, with $\mathbf{C}_o$ (H_o) and $\mathbf{C}_\beta$ (H_β) for two different sets of choices of β_1 and β_2.

(a) The controller $\mathbf{C}_o$ is guaranteed to stabilize $e^{-sh}\mathbf{P}$ for all $h \in [0, \tau_m)$, where $\tau_m = \|\, s H_o \,\|^{-1} = 0.95$ s (the actual DM is 1.32 s).
(b) Using $\mathbf{C}_o$ given in (43), for the delay-free closed-loop transfer function we have $\|\, H_o \,\| = 1.4513$ and $\|\, sH_o \,\| = 1.05$. By choosing $\chi_\beta = (s+\beta_1)(s+\beta_2)$ with $(\beta_1+\beta_2) < \frac{1.05}{1.4513} = 0.7235$, we can have $\tau_{\beta m} \geq \tau_m$. For example, with $\beta_1 = 0.35$, $\beta_2 = 0.37$, the controller (31) becomes

$$\mathbf{C}_\beta = \frac{0.756(s+4)(s+0.2)(s+0.1799)}{(s+11.17)(s+0.2273)(s^2+2.377s+2.956)} \,.$$

This leads to a new DM lower bound $\tau_{\beta m} = \|sH_\beta\|^{-1} = 1.3541$ s (with actual DM of 1.6227 s). An alternative choice, $\beta_1 = 0.02$, $\beta_2 = 0.7$, leads to the controller

$$\mathbf{C}_\beta = \frac{0.756(s+4)(s+0.2)(s+0.01944)}{(s+11.17)(s+0.1089)(s^2+2.493s+3.13)} \,,$$

which gives a new DM lower bound $\tau_{\beta m} = \|sH_\beta\|^{-1} = 1.5767$ s (with actual DM of 2.067 s).

Fig. 5 shows $y(t)$ for a unit-step input at $u(t)$ with $\mathbf{C}_o$ (closed-loop is H_o) and C_β (closed-loop is H_β) for the different set of choices of β_1 and β_2 above. The trade-off for *DM* improvement by using $\mathbf{C}_\beta$ is seen by comparing the step responses for H_o and H_β in Fig. 5: for H_o the settling time is 12.4 s, with the choice $\beta_1 = 0.35$, $\beta_2 = 0.37$ (respectively $\beta_1 = 0.02$, $\beta_2 = 0.7$) the settling time has increased to 15 s (respectively 17.6 s).

Suppose that a pre-specified delay of $\tau = 2.5$ s is given. The controller $\mathbf{C}_\beta$ in (37) stabilizes $e^{-2s}\mathbf{P}$ for $\beta_1 > 0$, $\beta_2 > 0$ satisfying the sufficient condition (41):

$$\beta_1 + \beta_2 < \tau^{-1}\|H_o\|^{-1} = 0.2756 \,.$$

For example, $\beta_1 = 0.15$, $\beta_2 = 0.12$ defines a new $\mathbf{C}_\beta$ that leads to $\tau_{\beta m} = 2.76$ s (the actual DM is 3.64 s); but in this case the settling time further increases to 37 s.

Example 4.2 (*Plants with Real RHP Poles*)**.** Consider a strictly proper $\mathbf{P}$ with two real $\mathbb{C}_+$-poles and no finite RHP zeros, with $p_1 = 0.2$, $p_2 = 1.1$,

$$\mathbf{P} = \frac{1}{(s-0.2)(s-1.1)} = \frac{1}{d(s)} \,. \tag{44}$$

For this plant a lower bound of the largest achievable DM is computed in Zhu et al. (2018) from the small gain condition (10): with a 5th order $w_h(s)$, this lower bound is found to be near 1.0 s. If we consider a simpler $w_h(s) = hs$, then the lower bound is 0.69 s. From the optimal solution of the associated Nevanlinna–Pick interpolation problem (see pp.15–17 of Özbay et al. (2018)) we compute the controller and the closed-loop map

$$\mathbf{C}_1(s) = \frac{132\ (s-0.16)}{(s\ +\ 100)}\ ,$$
$$H_1(s) = \frac{1.3377\ (s-0.16)}{(s/98.68+1)(s^2+0.02241s+0.008918)}.$$

This controller leads to $DM = 0.72$ s. However, this design does not consider other specifications. For example, in this case the step response is very slow, oscillatory, and the steady-state error is extremely large (final value of the step response is -24, settling time is around 500 s, and undershoot is more than 90%). It should also be noted that a lower bound of the achievable DM by a PD controller is near 0.5 s (see Theorem 4.1 and Fig. 2 of Ma, Chen, Liu, Chen, and Niculescu (2019)); the controller $\mathbf{C}_1$ is very close to a PD controller and gives a greater DM. On the other hand, this controller does not satisfy Assumption 2.1 of Ma et al. (2019), where controllers are restricted to a proper subset of all stabilizing PD controllers.

Keeping in mind the design above, we now move to another first order controller, which considers the step response related performance objectives: the controller $\mathbf{C}_o$ in (45) leads to a steady-state error less than 1%, and its settling time is less than 1 s, but in this case the DM is reduced to 0.1 s.

$$\mathbf{C}_o(s) = \frac{215\ (s+2.25)}{(s\ +\ 25)}\ . \tag{45}$$

From the discussion above, any first order controller obtained from the interpolation problem using the weight $w_h(s) = hs$, will not give a delay margin near 0.7 s. Also, note that it is not possible to stabilize this plant with a first order PI controller; so all first order controllers will lead to some nonzero steady-state error. We now apply the results of Proposition 3.2 part **(b)** to design a new controller $\mathbf{C}_\beta$, given by (31), without modifying the steady-state error. In this case, since $k = 0$, the term W in (30) is $W(s) = \frac{s}{(s+\beta_0)}$. In order to improve the DM, we make sure that (35) holds, i.e.,

$$\sum_{i=0}^{2} \beta_i < \frac{10.6950}{1.4826} - 2(0.2+1.1) = 4.6\ .$$

An admissible choice is $\beta_0 = 0.1$, $\beta_1 = \beta_2 = 0.05$, which leads to

$$\mathbf{C}_\beta(s) = \frac{602(s+2.25)(s^2+0.07411s+0.01027)}{(s+2.586)(s+0.001455)(s^2+23.91s+191)}.$$

An integral action approximation of $\mathbf{C}_\beta(s)$ above is given in (46), which improves the DM to 0.308 s, makes the steady-state error zero, but it increases the settling time to 50.5 s:

$$\mathbf{C}_{\beta I}(s) = \frac{600(s+2.25)(s^2+0.075s+0.01)}{s\ (s+2.5)(s^2+25s+200)}. \tag{46}$$

By increasing the values of β_0, β_1 and β_2 it is possible to decrease the settling time, but this will also reduce the delay margin. For example, if we choose $\beta_0 = \beta_1 = \beta_2 = 1$ the inequality (35) is still satisfied; the resulting controller $\mathbf{C}_\beta$ leads to $DM = 0.1458$ s and the step response settling time is 6.53 s, see Fig. 6.

Example 4.3. For $\mathbf{P}(s) = 1/(s-p)$ it is shown by Ma and Chen (2019) that the largest possible DM is achieved by a PD controller is $2/p$. Moreover, adding very small integral action gain leads to a very small reduction in the DM. So, one can obtain zero steady state error and DM nearly equal to $2/p$. Here we take an initial stabilizing controller $\mathbf{C}_o(s) = K > p$. Let K be designed such that the steady state error for a unit step input has magnitude less than 0.1, which means that $K \geq 11p$. The lower bound of the DM computed for this controller is $\tau_m = \|sH_o\|^{-1} = \frac{1}{11p}$. But this is a conservative lower bound, the actual DM is $\frac{0.4}{p}$. With a constant controller, largest DM achievable is $\frac{1}{p}$ and it is obtained when $K \searrow p$, but this gives a very poor step response.

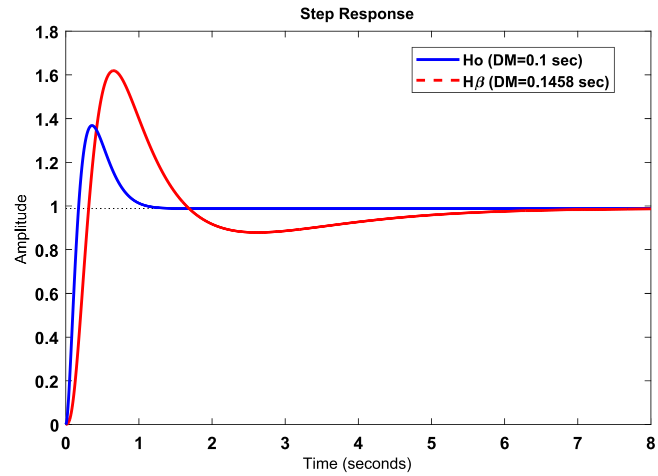

Fig. 6. Closed-loop step response of Example 4.2 for $h = 0$, with $\mathbf{C}_o$ (H_o) and $\mathbf{C}_\beta$ (H_β) for $\beta_0 = \beta_1 = \beta_2 = 1$.

For this plant, the controller $\mathbf{C}_\beta$ in (31) is in the form

$$\mathbf{C}_\beta(s) = \frac{K\ ((\beta_0+\beta_1+2p)s+\beta_0(\beta_1+p))}{(s+\beta_0)(s+\beta_1+p)+Ks}.$$

Note that $\mathbf{C}_\beta(0) = K$, so the steady state error for a unit step reference is the same as that of $\mathbf{C}_o = K$. Now let $\beta_1 \searrow 0$ and compute $\tau_{\beta m}$ from (32):

$$\tau_{\beta m} = \left\| \frac{Ks\ ((\beta_0+2p)s+\beta_0 p)}{(s+\beta_0)(s+p)(s+K-p)} \right\|^{-1}$$

If we let $\beta = \eta p$, then (35) implies that $\tau_{\beta m} > \tau_m$ for $0 < \eta < 8$. The actual DM and its lower bound $\tau_{\beta m}$ can be computed for different values of η. It can be shown that for values of $\eta < 0.52$, the lower bound of DM is greater than the original DM, which was $0.4/p$. For small values of η, e.g. $\eta = 0.01$, this controller gives a DM greater than $0.56/p$.

5. Conclusions

We proposed a method to modify an initially designed stabilizing controller to improve a lower bound of the delay margin. The initial controller is assumed to be designed so that steady-state tracking performance objectives are met. The modified controller does not change the steady state error for a unit step reference input, and it is obtained by introducing some parameters, $\beta_0, \ldots, \beta_\nu$, where ν is the number of unstable poles of the plant. It can be deduced from (36) that the order of the modified controller is $(\nu+1)$ higher than that of the initial controller.

In all of the examples we observed that choosing small values for $\beta_0, \ldots, \beta_\nu$ we obtain large DM, but the step response is slow. Increasing these values within the bounds of stability leads to faster step response, but smaller DM. These observations are justified by Eqs. (33) and (36). First, (33) shows that in order to have a large $\tau_{\beta m}$ we need to choose the sum of β_is small. On the other hand, by (36), the closed-loop system poles are the poles of H_0 and those of U, i.e. the roots of χ, which are $-\beta_0$ and $-(\beta_i + |p_i|)$, for $i = 1, \ldots, \nu$. So, in order to move the poles of U further to the left, to speed up the step response, we should increase the values of β_is. Hence additional optimization can be done by exploiting the freedom in the design parameters to tackle the competing objectives, namely, large DM and fast step response.

Appendix

Proof of Proposition 3.1. **(a-i)** The result is a simple application of the small gain theorem: as shown in Özbay et al. (2018), Qi et al. (2017) and Zhu et al. (2018), a controller $\mathbf{C}_o$ stabilizing $\mathbf{P}$ also stabilizes all plants $e^{-hs}\mathbf{P}(s)$ for all $h \in [0, \tau_m)$ where τ_m is the largest h satisfying the inequality (4); moreover, (7) is a sufficient condition for this. In the stable case, controller parametrization (11) leads to $H_o = \mathbf{P}\tilde{Q}$ from which (13) is obtained.
(a-ii) For any $\tau < \tau_m$, (14) is a direct consequence of part (a-i) when the definition (12) is used for $\tilde{Q}$ in (13).
(b) The parameter $\tilde{Q}_I$ defined as in (15) satisfies $\tilde{Q}_I(0) = \mathbf{P}(0)^{-1}$, which means $(1 - \mathbf{P}\tilde{Q}_I)(0) = 0$. Due to having at least one pole at $s = 0$, $\mathbf{C}_I$ in (16) is an integral-action controller. There may be more than one pole at $s = 0$ depending on the free parameter Q_I choice in (15).
(b-i) With $\tilde{Q}_I$ defined as in (15), the equality (17) is obtained directly from (13).
(b-ii) The inequality (18) is obtained by re-arranging the terms appearing in (17). $\square$

Proof of Lemma 3.1. **(a)** The following norms are used to prove that (27) holds: for any $\beta_i \geq 0$ such that $\beta_i + |p_i| > 0$, we have

$$\| \frac{s}{s + \beta_i + |\, p_i \,|} \| = 1,$$

$$1 \;\leq\; \| \prod_{i=1}^{\nu-1} \frac{(s - p_i)}{(s + \beta_i + |\, p_i \,|)} \| = 1.$$

For $\nu = 1$, (27) holds, in which case, $p_1 \in \mathbb{R}_+$:

$$\| \, s \, (\, 1 - \frac{(s - p_1)}{(s + \beta_1 + |\, p_1 \,|)} \,) \, \| = \beta_1 + 2p_1 \; . \tag{47}$$

For $\nu = 2$, there are two possible cases: If $p_1, p_2 \in \mathbb{R}_+$, then define $a_{12} = (\beta_1 + \beta_2 + 2(p_1 + p_2))$, $b_{12} = (\beta_1\beta_2 + \beta_1 p_2 + \beta_2 p_1)$, and check that

$$\begin{aligned}
&\| \, s \, (\, 1 - \frac{(s - p_1)(s - p_2)}{(s + \beta_1 + p_1)(s + \beta_2 + p_2)} \,) \, \| \\
&= \| \, \frac{s}{(s + \beta_2 + p_2)} \frac{[\, a_{12}s + b_{12} \,]}{(s + \beta_1 + p_1)} \, \| \\
&\leq \| \, \frac{s}{(s + \beta_2 + p_2)} \| \, \| \frac{a_{12}s + b_{12}}{(s + \beta_1 + p_1)} \, \| \\
&= \beta_1 + \beta_2 + 2(p_1 + p_2) \; .
\end{aligned}$$

The other case is $p_2 = \overline{p}_1$, with $(p_1 + \overline{p}_1) = 2\alpha_1$, and $|p_1| = \omega_1$; then it is a similar exercise to prove that

$$\begin{aligned}
&\| \, s \, (\, 1 - \frac{(s - p_1)(s - p_2)}{(s + \beta_1 + |\, p_1 \,|)(s + \beta_2 + |\, p_2 \,|)} \,) \, \| \\
&= \beta_1 + \beta_2 + 2\alpha_1 + 2\omega_1 \; .
\end{aligned}$$

Now for $\nu > 2$, assume that (27) holds for $(\nu - 1)$ and show that it also holds for ν: Let

$$\| \, s \, (\, 1 - \frac{\displaystyle\prod_{i=1}^{\nu-1}(s - p_i)}{\displaystyle\prod_{i=1}^{\nu-1}(s + \beta_i + |\, p_i \,|)} \,) \, \| = \sum_{i=1}^{\nu-1} \beta_i + \sum_{i=1}^{\nu-1}(p_i + |\, p_i \,|) \; .$$

Then it can be verified that

$$\begin{aligned}
&\| \; s(\, 1 - \frac{\prod_{i=1}^{\nu}(s - p_i)}{\prod_{i=1}^{\nu}(s + \beta_i + |\, p_i \,|)} \,) \; \| \\
&\leq \| \, \frac{s}{(s + \beta_\nu + |\, p_\nu \,|)} \| \; \| s(1 - \frac{\prod_{i=1}^{\nu-1}(s - p_i)}{\prod_{i=1}^{\nu-1}(s + \beta_i + |\, p_i \,|)}) \| \\
&\qquad + (\beta_\nu + |\, p_\nu \,|) + p_\nu \, \| \frac{\prod_{i=1}^{\nu-1}(s - p_i)}{\prod_{i=1}^{\nu-1}(s + \beta_i + |\, p_i \,|)} \| \\
&= \sum_{i=1}^{\nu-1} \beta_i + \sum_{i=1}^{\nu-1}(p_i + |\, p_i \,|) \; + \; \beta_\nu + (p_\nu \; + |\, p_\nu \,|) \; .
\end{aligned}$$

The norm in (27) is an equality since

$$\lim_{s \to \infty} s \, [\, 1 - \frac{\prod_{i=1}^{\nu}(s - p_i)}{\chi_\beta} \,] \; = \; \sum_{i=1}^{\nu} \beta_i + \sum_{i=1}^{\nu} p_i + \sum_{i=1}^{\nu} |\, p_i \,| \; .$$

(b) This case is an extension of (27), where an additional $p_0 = 0$ is included, and the constant $\beta_0 > 0$ in order to satisfy $(\beta_0 + |p_0|) > 0$. $\square$

Proof of Proposition 3.2. **(a)** The result expressed by (29) is the application of the small gain theorem as mentioned in the proof of Proposition 3.1(a-i).
(b) Let

$$Y := \frac{d}{\chi_\beta} = \frac{s^k \, \prod_{i=k+1}^{\nu}(s - p_i)}{\prod_{i=1}^{k}(s + \beta_i) \, \prod_{i=k+1}^{\nu}(s + \beta_i + |\, p_i \,|)}.$$

Then $X := Y\mathbf{P} \in \mathcal{S}$. With $N, \, D \in \mathcal{S}$, let $C_o = ND^{-1}$ be a coprime factorization. Since C_o stabilizes $\mathbf{P}$, $M_o = NX + DY \in \mathcal{S}$ is a unit in $\mathcal{S}$, where $M_o^{-1}NX = H_o$. The controller $\mathbf{C}_\beta$ in (31) can be written in factorized form as

$$\begin{aligned}
\mathbf{C}_\beta &= (1 - WY \,)(1 + WYND^{-1}\mathbf{P})^{-1}ND^{-1} \\
&= (N - NWY)(D + NWX)^{-1} \; .
\end{aligned}$$

Therefore, $\mathbf{C}_\beta$ stabilizes the delay-free $\mathbf{P}$ since $(N - NWY)X + (D + NWX)Y = M_o$. Now $\mathbf{C}_\beta$ stabilizes the plant with delay $e^{-sh}\mathbf{P}$ if $h < \tau_{\beta m}$ where $\tau_{\beta m}$ is defined in (32); this result is from the same small gain arguments used in the proof of Proposition 3.1(a-i). By Lemma 3.1-(b), with $\beta_0 = 0$ if $k \neq 0$,

$$\begin{aligned}
\tau_{\beta m}^{-1} &= \| \, sH_\beta \, \| \leq \| s(1 - WY) \| \, \| \, H_o \, \| \\
&\leq [\; \beta_0 + \sum_{i=1}^{\nu} \beta_i + \sum_{i=k+1}^{\nu} (p_i + |\, p_i \,|) \;] \| \, H_o \, \| \; . \quad \square
\end{aligned}$$

Proof of Corollary 3.1. This is a direct application of Proposition 3.2 with $\nu = k \geq 1$. $\square$

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