COPRIME FACTORIZATIONS IN STABLE LINEAR SYSTEMS

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ABSTRACT

We consider a block-diagonal linear (not necessarily time-invariant) map P with a right-coprime factorization ND^{-1} (or a left-coprime factorization $\tilde{D}^{-1}\tilde{N}$). We show that the individual blocks in P have right-coprime factorizations (left-coprime factorizations, respectively) *if and only if* the denominator map D ($\tilde{D}$) has a special block-triangular structure. We apply this condition to the stable linear feedback system $S(P_1, P_2)$.

I. INTRODUCTION

Consider the linear (not necessarily time-invariant) feedback system $S(P_1, P_2)$ shown in Figure 1. If the system is stable, then the map $P := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} : \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ has both a right-coprime fraction representation (N, D) and a left-coprime fraction representation $(\tilde{D}, \tilde{N})$, where N, D, $\tilde{D}$ and $\tilde{N}$ are linear causal stable maps. This result was proven in [Vid.1] for the case where P has elements in the quotient field of an entire ring. However, the conditions for existence of individual right-coprime fraction representations and left-coprime fraction representations of the subsystems P_1 and P_2, was left as an open question.

To show that the stability of the closed-loop does not imply that P_1 and P_2 individually have coprime factorizations, a special non-unique factorization domain was constructed in [Ana.1]; scalar p_1 and p_2 in the quotient field of this particular ring have no stable coprime factorizations although $\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ has a right-coprime factorization.

In this paper, we consider this problem from a general input-output approach, where the multiinput-multioutput subsystems P_1 and P_2 are represented by linear (not necessarily time-invariant) maps defined over extended spaces. Generalizing the concepts of factorizations and coprime factorizations, we obtain right- and left-coprime fraction representations of the map P when the system $S(P_1, P_2)$ is stable. The main result is Theorem 3.3, which states that: given coprime factorizations of P, individual coprime factorizations for P_1 and P_2 exist if and only if a right-coprime factorization of P has a lower block-triangular "denominator" D and a left-coprime factorization of P has an upper block-triangular "denominator" $\tilde{D}$. Note that Theorem 3.3 answers the question posed in [Vid.1]; the example constructed in [Ana.1] is only one case where the conditions of Theorem 3.3 fail. In the linear time-invariant case where P_1 and P_2 have rational function entries, the necessary and sufficient conditions in Theorem 3.3 are satisfied due to the existence of triangular (Hermite) forms [Vid.2].

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II. NOTATION AND DEFINITIONS

For similar notation see, for example, [Wil.1, Saf.1, Des.1].

Let $\mathcal{T} \subset \mathbb{R}$ and let $\mathcal{V}$ be a normed vector space. Let $\zeta := \{ F \mid F : \mathcal{T} \to \mathcal{V} \}$ be the vector space of $\mathcal{V}$–valued functions on $\mathcal{T}$. For any $T \in \mathcal{T}$, the projection map $\Pi_T : \zeta \to \zeta$ is defined by $\Pi_T F(t) := \begin{cases} F(t), & t \le T,\ t \in \mathcal{T} \\ \theta_\zeta, & t > T,\ t \in \mathcal{T} \end{cases}$, where θ_ζ is the zero element in ζ. Let $\Lambda \subset \zeta$ be a normed vector space which is closed under the family of projection maps $\{ \Pi_T \}_{T \in \mathcal{T}}$. For any $F \in \Lambda$, let the norm $\| \Pi_{(\cdot)} F \| : \mathcal{T} \to \mathbb{R}_+$ be a nondecreasing function. The *extended space* Λ_e is defined by $\Lambda_e := \{ F \in \zeta \mid \forall T \in \mathcal{T},\ \Pi_T F \in \Lambda \}$.

A map $F : \Lambda_e \to \Lambda_e$ is said to be *causal* iff for all $T \in \mathcal{T}$, Π_T commutes with $\Pi_T F$; equivalently, $\Pi_T F = \Pi_T F \Pi_T$.

We define two function spaces closely related to Λ_e (the superscripts i and o refer to "input" and "output", respectively): Let Λ_e^i and Λ_e^o be extended function spaces analogous to Λ_e except that their functions take values in the normed spaces $\mathcal{V}^i$ and $\mathcal{V}^o$, respectively; the associated projections Π_T are redefined accordingly.

Definition (Well-posed system): A *feedback system* is said to be *well-posed* iff for all possible inputs, all signals in the system are (uniquely) determined by causal maps.

Definition (Finite-gain stability): (1) A causal *map* $H : \Lambda_e^i \to \Lambda_e^o$ is called *finite-gain (f.g.) stable* iff there exists $m > 0$ such that $\| He \| \le m \| e \|$, for all $e \in \Lambda^i$.
(2) A well-posed *feedback system* is called *f.g. stable* iff for all possible inputs, all signals in the system are determined by causal *f.g. stable* maps.

Definition (Finite-gain unimodularity): A causal f.g. stable map $M : \Lambda_e \to \Lambda_e$ is said to be *f.g. unimodular* iff M is bijective and $M^{-1} : \Lambda_e \to \Lambda_e$ is causal f.g. stable.

Definition (Coprime factorizations): (e.g. [Fei.1, Man.1]) Let $N : \Lambda_e^i \to \Lambda_e^o$, $D : \Lambda_e^i \to \Lambda_e^i$, $\tilde{N} : \Lambda_e^i \to \Lambda_e^o$ and $\tilde{D} : \Lambda_e^o \to \Lambda_e^o$ be causal *linear* f.g. stable maps.
(1) The pair (N, D) [$(\tilde{D}, \tilde{N})$] is called *right-coprime (r.c.)* [*left-coprime (l.c.)*] iff there exist causal linear f.g. stable maps $U : \Lambda_e^o \to \Lambda_e^i$ and $V : \Lambda_e^i \to \Lambda_e^i$ [$\tilde{U} : \Lambda_e^o \to \Lambda_e^i$ and $\tilde{V} : \Lambda_e^o \to \Lambda_e^o$] such that

$$UN + VD = I_{\Lambda_e^i} \quad [\ \tilde{D}\tilde{V} + \tilde{N}\tilde{U} = I_{\Lambda_e^o}\ ], \qquad (1)$$

where $I_{\Lambda_e^i}$ [$I_{\Lambda_e^o}$] is the identity map on Λ_e^i [Λ_e^o].
(2) The pair (N, D) [$(\tilde{D}, \tilde{N})$] is called a *right fraction representation (r.f.r.)* [*left fraction representation (l.f.r.)*] of the causal linear map $P : \Lambda_e^i \to \Lambda_e^o$ iff (i) D [$\tilde{D}$] is bijective with a causal inverse $D^{-1} : \Lambda_e^i \to \Lambda_e^i$ [$\tilde{D}^{-1} : \Lambda_e^o \to \Lambda_e^o$], and (ii) $P = ND^{-1}$ [$P = \tilde{D}^{-1}\tilde{N}$].
(3) The pair (N, D) [$(\tilde{D}, \tilde{N})$] is called a *right-coprime fraction representation (r.c.f.r.)* [*left-coprime fraction representation (l.c.f.r.)*] of the causal linear map

$P : \Lambda_e^i \to \Lambda_e^o$ iff (i) (N, D) is r.c. [$(\tilde{D}, \tilde{N})$ is l.c.] , and (ii) (N, D) is an r.f.r. [$(\tilde{D}, \tilde{N})$ is an l.f.r.] of P.

III. MAIN RESULTS

Consider the system $S(P_1, P_2)$ shown in Figure 1: $P_1 : \Lambda_e^o \to \Lambda_e^i$ and $P_2 : \Lambda_e^i \to \Lambda_e^o$ are causal *linear* maps.

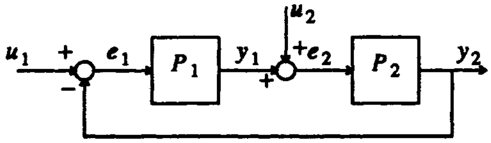

Figure 1: The feedback system $S(P_1, P_2)$.

Let $e := \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$, $u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Let the causal *linear* map $P : e \mapsto y$ be defined by

$$P : \Lambda_e^o \times \Lambda_e^i \to \Lambda_e^i \times \Lambda_e^o \ , \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} , \qquad (2)$$

where $Pe := \begin{bmatrix} P_1 e_1 + 0 e_2 \\ 0 e_2 + P_2 e_2 \end{bmatrix}$.

3.1. Fact: Let the well-posed *linear* system $S(P_1, P_2)$ be f.g. stable. Then the map P defined in equation (2) has an r.c.f.r. and an l.c.f.r.

3.2. Lemma: Let (N, D) be an r.c.f.r. and $(\tilde{D}, \tilde{N})$ be an l.c.f.r. of the linear map P ; then

(i) (A, B) is also an r.c.f.r. of P if and only if there exists an f.g. unimodular map $R : \Lambda_e^i \to \Lambda_e^i$ such that $A = NR$, $B = DR$;

(ii) $(\tilde{B}, \tilde{A})$ is also an l.c.f.r. of P if and only if there exists an f.g. unimodular map $L : \Lambda_e^o \to \Lambda_e^o$ such that $\tilde{B} = L\tilde{D}$, $\tilde{A} = L\tilde{N}$.

Comment: With suitable interpretations, conclusion 3.2.(i) above holds for *nonlinear* maps [see e.g. Ham.1, Des.2].

3.3. Theorem (i) Let (A, B) be an r.c.f.r. of $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$; then P_1 *and* P_2 have r.c.f.r.s (N_{11}, D_{11}) and (N_{22}, D_{22}) , respectively, if and only if there exists an f.g. unimodular map R such that

$$BR = \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix} , \qquad (3)$$

where $D_{11} : \Lambda_e^i \to \Lambda_e^i$ and $D_{22} : \Lambda_e^o \to \Lambda_e^o$, and $AR =: \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$.

(ii) Let $(\tilde{B}, \tilde{A})$ be an l.c.f.r. of $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$; then P_1 *and* P_2 have l.c.f.r.s $(\tilde{D}_{11}, \tilde{N}_{11})$ and $(\tilde{D}_{22}, \tilde{N}_{22})$, respectively, if and only if there exists an f.g. unimodular map L such that

$$L\tilde{B} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_{22} \end{bmatrix} , \qquad (4)$$

where $\tilde{D}_{11} : \Lambda_e^o \to \Lambda_e^o$ and $\tilde{D}_{22} : \Lambda_e^i \to \Lambda_e^i$,

and $L\tilde{A} =: \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}$.

Comments: (1) Equation (3) is a structure test on the "denominator" map: P *must* have an r.c.f.r. (N, D) where D is of the specific *lower block-triangular* form. In order to find the individual r.c.f.r.s of the subsystems from the given r.c.f.r. (A, B) of P , we only need to determine D_{11} and D_{22} ; D_{21} is not necessary for the calculation. Similar comments apply for the *upper block-triangular* form in equation (4). (2) Theorem 3.3 can be restated for n subsystems when $P = diag(P_1 \cdots P_n)$ has an r.c.f.r. or an l.c.f.r. (3) If condition (i) of Theorem 3.3 holds, then P has the structure in Figure 2.

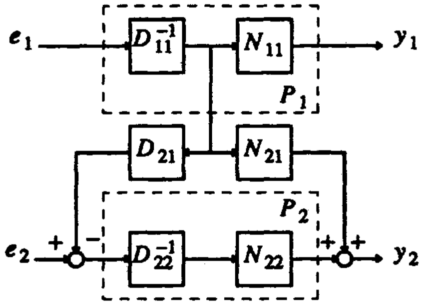

Figure 2

Since $N_{21} = N_{22} D_{22}^{-1} D_{21}$, P is in fact *decoupled* into two subsystems P_1 and P_2 . In other words, the blocks D_{21} and N_{21} can be removed for a simpler r.c.f.r. of P . (4) By Fact 3.1, the map P in any well-posed f.g. stable linear system has an r.c.f.r. (l.c.f.r.). The individual subsystems also have r.c.f.r.s (l.c.f.r.s) *if and only if* the condition stated in Theorem 3.3 is satisfied.

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