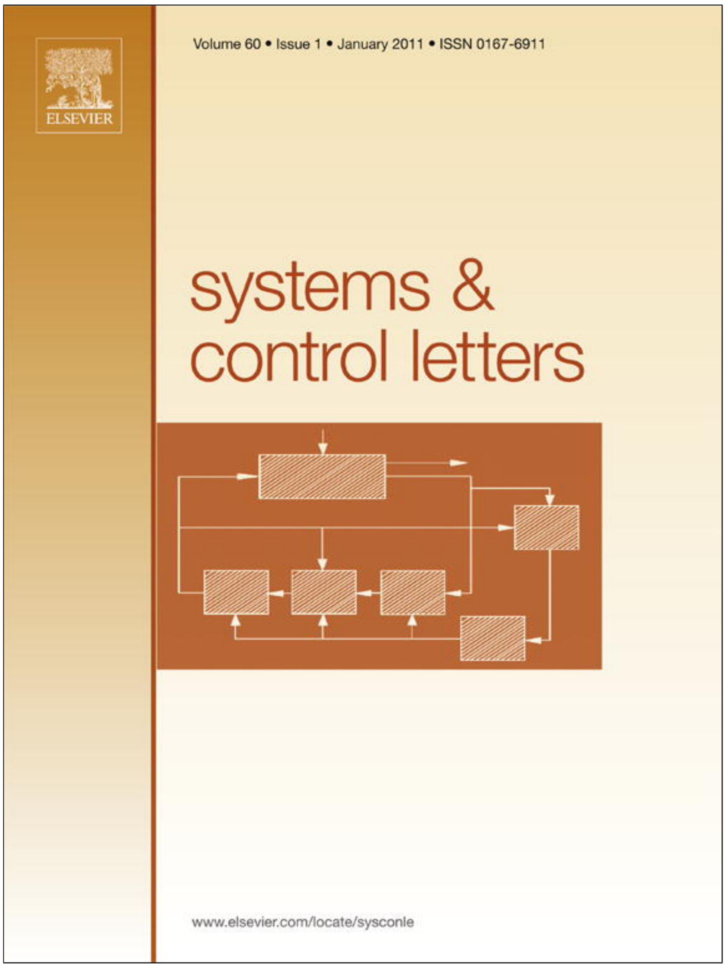
Volume 60 • Issue 1 • January 2011 • ISSN 0167-6911
ELSEVIER
systems &
control letters
www.elsevier.com/locate/sysconle

Systems & Control Letters 60 (2011) 885–891

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Low order integral-action controller synthesis

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ARTICLE INFO

Article history:
Received 20 September 2010
Received in revised form
17 May 2011
Accepted 14 July 2011

Keywords:
Stabilization
Tracking
Integral action
Controller synthesis

ABSTRACT

A simple controller synthesis method is developed for certain classes of linear, time-invariant, multi-input multi-output plants. The number of poles in each entry of these controllers depends on the number of right-half plane plant zeros, and is independent of the number of poles of the plant to be stabilized. Furthermore, these controllers have integral-action so that they achieve asymptotic tracking of step input references with zero steady-state error. The designed controller's poles and zeros are all in the stable region with the exception of one pole at the origin for the integral-action design requirement. The freedom available in the design parameters may be used for additional performance objectives, although the only goal here is stabilization and tracking of constant references.

1. Introduction

In this paper, we show that it is possible to design very simple controllers to stabilize a special class of linear time-invariant (LTI), multi-input multi-output (MIMO) plants that have restrictions on their (blocking and transmission) zeros that lie in the region of instability. The pole locations are not restricted, and the zeros that are in the stable region open left-half complex-plane (OLHP) are not restricted. An additional objective is to design these LTI controllers with integral-action so that the closed-loop system achieves asymptotic tracking of constant reference inputs with zero steady-state error.

Controllers stabilizing a complex plant and achieving a specified performance are usually at least as complex as the plant itself [1]. Low order controllers or controllers with the least number of poles are generally preferred for ease of implementation. In control system design, the issues of computation and implementation of high-order controllers are dealt with using reduction approaches such as (a) designing the high-order controller and then approximating it with a low-order one within an acceptable loss of performance; (b) reducing the order of the plant model with the prospect that a low-order model will lead to a low-order controller (see e.g., [2–8]). Model reduction is not the objective of this work. The synthesis approach developed in this paper directly gives simple controller design that stabilizes the original plant without the need to reduce the plant model. Since the resulting controllers are simple, they need not be approximated with lower order ones for implementation purposes.

Robust asymptotic tracking of reference inputs is achieved with poles duplicating the dynamic structure of the exogenous signals that the regulator has to process. Due to this internal model principle, integral-action controllers have poles at the origin of the complex plane [9]. The standard method of designing controllers with integral-action starts by augmenting the plant dynamics with extra states corresponding to the integral of the output error, i.e., the plant's transfer-matrix is replaced by P/s. In the MIMO case with m inputs and outputs, the integrator augmented to the plant introduces m additional states. Using a full-order observer to estimate the n states of the original plant and state feedback on the $(n+m)$ states, the resulting $(m \times m)$ controller transfer-matrix is always strictly-proper, has m of its eigenvalues at the origin, and the remaining eigenvalues may be anywhere in the complex plane. The entries of the controller's transfer-matrix C would have up to $(n+1)$ poles, one of which is at the origin. Although this standard method may not result in a simple controller, it applies to any LTI plant. On the other hand, for the special classes of plants we consider here, a much simpler integral-action controller design can be achieved. The special class of plants here has no restrictions as far as the location of the poles is concerned (stable or unstable) and the zeros in the OLHP or infinity are also not restricted. However, we assume that the zeros in the region of instability are on the positive real axis and have "large" magnitude (including infinity).

Based on the restrictions imposed on the zeros in the unstable region, we consider three special classes of (square) MIMO plants in Section 3. All results apply to single-input single-output (SISO)

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doi:10.1016/j.sysconle.2011.07.003

plants as a special case. In all cases, there may be any number of (transmission or blocking) zeros in the OLHP. Section 3.1: The class of plants in this subsection allows no (transmission or blocking) zeros in the unstable region or at infinity. This is a very simple case to treat and is included in the discussion only for completeness. As shown in Proposition 1, integral-action controllers can be constructed with first-order terms in every nonzero entry of the controller's transfer-matrix C, with exactly one pole at the origin, and one OLHP zero in each nonzero term of C. In other words, C is a proportional + integral (PI) controller, with constant matrix proportional and integral terms in the MIMO case. Section 3.2: The class of plants in this subsection allows only blocking zeros on the real-axis of the unstable region, including any number of blocking zeros at infinity. Asymptotically tracking controller design for a more restricted sub-class of plants that have only exactly one blocking zero at infinity has also been considered in the context of funnel control (see e.g., [10] and the references therein). Proposition 2 shows that plants with r blocking zeros (with large magnitudes) on the positive real-axis can be stabilized using integral-action controllers that have exactly r poles in every entry of the controller's $(m \times m)$ transfer-matrix, where one of these poles is at $s = 0$. The case where the unstable zeros are all at infinity is particularly interesting: The remaining $(r-1)$ controller poles are all in the region of stability (OLHP). Furthermore, the controllers are bi-proper and they have stable inverse. For SISO plants $(m = 1)$ that have n poles and r positive large zeros (or zeros at infinity), the proposed design gives an r-th order integral-action controller, which is bi-proper, and has one pole at $s = 0$, and $(r-1)$ poles in the OLHP. On the other hand, a design based on augmenting the SISO plant as P/s would result in a strictly-proper controller of order $(n + 1)$, with one pole at $s = 0$ and some of the n poles possibly in the closed right-half plane. Although this augmentation method creates a more complex controller, it is available for any plant, whereas the proposed simple design applies to the described plant classes only. Section 3.3: The class of plants in this subsection allows any number of transmission zeros at infinity in addition to blocking zeros. Proposition 3 gives a straightforward method of obtaining simple integral-action controllers. Illustrative SISO and MIMO examples are also given, and a comparison of the number of poles of the controller is provided with the standard integral-action design method based on full-order observer and state-feedback applied to an augmented plant model.

Although we discuss continuous-time systems here, all results apply also to discrete-time systems with appropriate modifications. The following fairly standard notation is used:

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ denote real, positive real, and complex numbers, respectively. The extended closed right-half plane is $\mathcal{U} = \{s \in \mathbb{C} \mid \mathcal{Re}(s) \geq 0\} \cup \{\infty\}$; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I is the identity matrix (of appropriate dimension). A transfer-matrix $M \in \mathcal{M}(\mathbf{S})$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathbf{S})$. The H_∞-norm of $M \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M\|$ (i.e., the norm $\|\cdot\|$ is the usual operator norm $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). For simplicity, we drop (s) in transfer-matrices such as $P(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$: For $P \in \mathbf{R_p}^{m\times m}$, $C \in \mathbf{R_p}^{m\times m}$, $P = D^{-1}N$ denotes a left-coprime-factorization (LCF), and $C = N_c D_c^{-1}$ denotes a right-coprime-factorization (RCF), where $N, D, N_c, D_c \in \mathbf{S}^{m\times m}$, $\det D(\infty) \neq 0$, $\det D_c(\infty) \neq 0$. For full-rank P, we say that $z \in \mathcal{U}$ is a $\mathcal{U}$-zero of P if rank $N(z) < m$; these zeros include both transmission zeros and blocking zeros in $\mathcal{U}$. If $z \in \mathcal{U}$ is a blocking zero of P, then $P(z) = 0$ and equivalently $N(z) = 0$. We use diag $[x_1, \ldots, x_m]$ to denote the $(m \times m)$ diagonal matrix whose diagonal entries are x_j, $j = 1, \ldots, m$. We use δn to denote the polynomial degree of n.

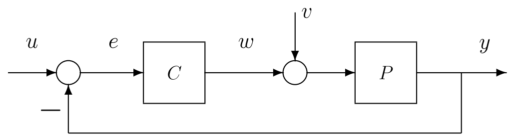

Fig. 1. Unity-feedback system $Sys(P, C)$.

2. Problem description

Consider the standard LTI, MIMO unity-feedback system $Sys(P, C)$ shown in Fig. 1, where $P \in \mathbf{R_p}^{m\times m}$ and $C \in \mathbf{R_p}^{m\times m}$ denote the plant's and the controller's transfer-matrices, respectively. It is assumed that the feedback system is well-posed, P and C have no hidden-modes in the unstable region, and the plant $P \in \mathbf{R_p}^{m\times m}$ is full normal rank m. The objective is to design a low-order stabilizing controller C with integral-action, so that the closed-loop system achieves asymptotic tracking of step-input references with zero steady-state error.

Let $P = D^{-1}N$ be an LCF of the plant and $C = N_c D_c^{-1}$ be an RCF of the controller. Let the (input-error) transfer-function from u to e be denoted by H_{eu} and let the (input–output) transfer-function from u to y be denoted by H_{yu}; then

$$H_{eu} = (I + PC)^{-1} = I - PC(I + PC)^{-1} = I - H_{yu}. \tag{1}$$

Definition 1. (i) The system $Sys(P, C)$ is stable if the closed-loop transfer-function from (u, v) to (y, w) is stable. (ii) The controller C is said to stabilize P if C is proper and the system $Sys(P, C)$ is stable. (iii) The stable system $Sys(P, C)$ has integral-action if H_{eu} has blocking zeros at $s = 0$. (iv) The controller C is an integral-action controller if C stabilizes P and the denominator D_c of any RCF $C = N_c D_c^{-1}$ has blocking zeros at $s = 0$, i.e., $D_c(0) = 0$. □

The controller C stabilizes $P \in \mathcal{M}(\mathbf{R_p})$ if and only if

$$M := DD_c + NN_c \tag{2}$$

is unimodular [11,12]. Suppose that the system $Sys(P, C)$ is stable and that step input references are applied to the system. Then the steady-state error $e(t)$ due to all step input vectors at $u(t)$ goes to zero as $t \to \infty$ if and only if $H_{eu}(0) = 0$. Therefore, by Definition 1, the stable system $Sys(P, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. Write $H_{eu} = (I + PC)^{-1} = D_c\ M^{-1}D$. Then by Definition 1, $Sys(P, C)$ has integral-action if $C = N_c D_c^{-1}$ is an integral-action controller since $D_c(0) = 0$ implies $H_{eu}(0) = (D_c M^{-1} D)(0) = 0$.

Lemma 1 states the necessary condition on P, for existence of integral-action controllers.

Lemma 1 (*Necessary Condition for Integral-Action*)**.** *Let $P \in \mathbf{R_p}^{m\times m}$. Let rank$P(s) = m$. If the system $Sys(P, C)$ has integral-action, then P has no transmission zeros at $s = 0$.* □

In order to design controllers with integral-action, we assume from now on that the plants under consideration have no zeros at $s = 0$, i.e., rank$P(0) = m$.

3. Low order controller synthesis

The plants under consideration here for low-order stabilizing controller synthesis have no restrictions on their poles; there are no restrictions on the zeros in the OLHP $\mathbb{C} \setminus \mathcal{U}$, and at infinity. However, the finite $\mathcal{U}$-zeros are restricted. In order to design controllers with integral-action, based on the necessary condition of Lemma 1, we assume everywhere that the plant has no zeros at $s = 0$, i.e., rank$P(0) = m$.

In Section 3.1, we consider plants with no zeros in the right-half plane $\mathcal{U}$ including infinity; integral-action controllers (with only one pole, which is at $s = 0$) can be designed for these plants. In Section 3.2, we consider the case where the $\mathcal{U}$-zeros are all blocking zeros and are positive real, or at infinity. In Section 3.3, the $\mathcal{U}$-zeros are only at infinity but instead of appearing in every entry of P with the same multiplicity, there may be any number of transmission zeros in addition to the blocking zeros. In all of these cases, the plants may have any number of transmission and blocking zeros anywhere in the OLHP.

3.1. Plants with no right-half plane zeros

The plants in this section have no restrictions on their poles, and also no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$. However, there are no $\mathcal{U}$-zeros, i.e, the blocking and transmission zeros of P are all in OLHP. Therefore, P^{-1} is stable. Proposition 1 gives a systematic controller synthesis method for such plants.

Proposition 1 (*Controller Synthesis for Plants with no Zeros in $\mathcal{U}$ Including Infinity*). *Let $P \in \mathbf{R_p}^{m\times m}$, where $\operatorname{rank}P(0) = m$. Choose any nonsingular $K \in \mathbb{R}^{m\times m}$ and any $g \in \mathbb{R}_+$. Choose $\alpha \in \mathbb{R}_+$ such that*

$$\alpha > \left\| \frac{s}{s+g} P^{-1} K^{-1} \right\| . \tag{3}$$

Then the integral-action controller in (4) stabilizes P:

$$C = \alpha \frac{(s+g)}{s} K . \quad \square \tag{4}$$

The integral-action controller in (4) with a first-order term in every (non-zero) entry is a proportional + integral (PI) controller, with a proportional-constant matrix term of αK and an integral-constant matrix term of $\alpha g K$.

3.2. Plants with large blocking zeros on the positive real-axis

The plants in this section have no restrictions on their poles, and also no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$. The $\mathcal{U}$-zeros of the plant P are positive real, and they are blocking zeros (appearing in every entry of P). Therefore, P can be written as

$$P = D^{-1}N = \left(\prod_{i=1}^{r} \frac{(1-s/z_i)}{(s+a)} P^{-1} \right)^{-1} \prod_{i=1}^{r} \frac{(1-s/z_i)}{(s+a)} I , \tag{5}$$

for any $a \in \mathbb{R}_+$, where $z_i \in \mathbb{R}_+ \cup \{\infty\}$, $i = 1, \ldots, r$, are the $\mathcal{U}$-blocking zeros of P, and P has no other transmission zeros in $\mathcal{U}$, i.e.,

$$D = \frac{\prod_{i=1}^{r}(1-s/z_i)}{(s+a)^r} P^{-1} \in \mathcal{M}(\mathbf{S}).$$

The zeros z_i need not be distinct. Any number of these $\mathcal{U}$-blocking zeros, or even all r of them, may be at infinity. If none of the $\mathcal{U}$-zeros is finite, then (5) becomes

$$P = D^{-1}N = \left(\frac{1}{(s+a)^r} P^{-1} \right)^{-1} \frac{1}{(s+a)^r} I . \tag{6}$$

Proposition 2 gives a systematic controller synthesis method for plants in the form of (5).

Proposition 2 (*Controller Synthesis for Plants with Blocking Zeros in $\mathcal{U}$*). *Let $P \in \mathbf{R_p}^{m\times m}$ be as in (5), with $\operatorname{rank}P(0) = m$. Let $D(\infty)^{-1} = \left(\frac{(s+a)^r}{\prod_{i=1}^{r}(1-s/z_i)} P(s) \right) |_{s\to\infty}$. Choose any monic r-th order strictly-Hurwitz polynomial $\rho(s)$. Define Φ as*

$$\Phi(s) := s \left[\frac{\prod_{i=1}^{r}(1-s/z_i)}{\rho(s)} P^{-1}(s) D(\infty)^{-1} - I \right] . \tag{7}$$

If $\| \Phi(s) \|^{-1} > (\frac{r}{z_i} + \sum_{i=1}^{r} \frac{1}{z_i})$ for each finite $\mathcal{U}$- zero $z_i \in \mathbb{R}_+$ of P, then choose $\alpha \in \mathbb{R}_+$ such that $\alpha < z_i$ for $i = 1, \ldots, r$ and

$$\alpha > \frac{r}{\|\Phi\|^{-1} - \sum_{i=1}^{r} \frac{1}{z_i}} . \tag{8}$$

Then the bi-proper integral-action controller in (9) stabilizes P:

$$C = \frac{\alpha^r \rho(s)}{(s+\alpha)^r - \alpha^r \prod_{i=1}^{r}(1-s/z_i)} D(\infty) . \quad \square \tag{9}$$

Remarks. (1) For $r = 1$, the controller in (9) is a PI controller, and for $r = 2$, it is a proportional + integral + derivative (PID) controller [13].

(2) Since P^{-1} has poles at the plant's blocking zeros $z_i \in \mathcal{U}$, the expression for Φ in (7) does not contain z_i because the terms $(1 - s/z_i)$ cancel with the corresponding factors in P^{-1}.

(3) If all r of the $\mathcal{U}$-zeros of the plant (5) are at infinity as in (6), then Φ in (7) becomes

$$\Phi(s) := s \left[\frac{1}{\rho(s)} P^{-1} D(\infty)^{-1} - I \right] , \tag{10}$$

and the condition $\|\Phi(s)\|^{-1} > (\frac{r}{z_i} + \sum_{i=1}^{r} \frac{1}{z_i})$ obviously holds since there are no finite $\mathcal{U}$-zeros. In this case, $\alpha \in \mathbb{R}_+$ is chosen to satisfy (8) as

$$\alpha > r \, \| \Phi(s) \| , \tag{11}$$

and the r-th order integral-action controller in (9) becomes

$$C = \frac{\alpha^r \rho(s)}{(s+\alpha)^r - \alpha^r} D(\infty) . \tag{12}$$

The MIMO controller C in (12) is bi-proper. Every entry has the r OLHP zeros of the strictly-Hurwitz polynomial $\rho(s)$; every entry has r poles, which are the roots of the polynomial $d(s)$ defined by

$$d(s) := (s+\alpha)^r - \alpha^r . \tag{13}$$

One of the r poles is at $s = 0$ and the remaining $(r-1)$ poles are in the stable region $\mathbb{C} \setminus \mathcal{U}$.

(4) In the SISO case, suppose that P has r zeros on the positive real-axis and a total of n poles. The order of the proposed controller C in (9) is exactly equal to r, which cannot exceed the order n of the plant.

(5) By Definition 1-(iv), the denominator-matrix of any integral-action controller $C = N_c D_c^{-1}$ is of the form $D_c = \frac{s}{(s+a)} \hat{D}_c$ for any $a \in \mathbb{R}_+$ and some $\hat{D}_c \in \mathcal{M}(\mathbf{S})$. If C is to have no $\mathcal{U}$-poles other than the one at $s = 0$, then without loss of generality, $\hat{D}_c = I$. By (2), $C = N_c(\frac{s}{(s+a)}\hat{D}_c)^{-1} = N_c(\frac{s}{(s+a)} I)^{-1}$ stabilizes P if and only if

$$M(s) = \frac{s}{(s+a)} D(s) + N(s) N_c(s) \tag{14}$$

is unimodular for some $N_c \in \mathcal{M}(\mathbf{S})$. For SISO plants that have distinct $\mathcal{U}$-zeros, one method to find $N_c \in \mathbf{S}$ satisfying (14) is to find a stable solution for $N_c(s) = N(s)^{-1}(M(s) - \frac{s}{(s+a)} D(s))$ using interpolation at the plant's $\mathcal{U}$-zeros, i.e., $M(z_i) = \frac{z_i}{(z_i+a)} D(z_i)$. A stable $G(s)$ can then be constructed such that $M(s) = I + G(s)$, which satisfies $\|G(s)\| < 1$ (see for example [14–16]). The simple method in Proposition 2 does not require interpolation and applies to MIMO plants with multiple zeros. $\square$

3.3. *Plants with blocking or transmission zeros at infinity*

As in Sections 3.1 and 3.2, the plants in this section have no restrictions on their poles, and also no restrictions on the zeros in the open left-half complex plane $\mathbb{C} \setminus \mathcal{U}$. The $\mathcal{U}$-zeros of the plant P are at infinity, but are not necessarily just limited to blocking zeros; every entry in the transfer-matrix of P may have different relative degree and some entries may not even be strictly proper. Therefore, the numerator matrix N in any LCF $P = D^{-1}N$ has an improper inverse, which we write as

$$N^{-1}(s) = \left[\frac{n_{ij}(s)}{d_{ij}(s)}\right]_{i,j\in\{1,\ldots,m\}}. \tag{15}$$

Since P has no transmission zeros in $\mathcal{U}$ except at infinity, the polynomials $d_{ij}(s)$ are strictly-Hurwitz. For $i, j = 1, \ldots, m$, define the integers r_{ij} as in (16) and define r_j as in (17):

$$r_{ij} := \begin{cases} \delta n_{ij} - \delta d_{ij}, & \delta n_{ij} > \delta d_{ij} \\ 0, & \delta n_{ij} \le \delta d_{ij}, \end{cases} \tag{16}$$

$$r_j := \max_j \{r_{ij}\}. \tag{17}$$

Let $a \in \mathbb{R}_+$; then

$$\frac{n_{ij}}{d_{ij}(s+a)^{r_j}} \in \mathbf{S}, \quad i = 1, \ldots, m. \tag{18}$$

Define $\Lambda \in \mathbf{S}^{m\times m}$ as

$$\begin{aligned} \Lambda(s) &:= \mathrm{diag}\,[\lambda_1(s), \ldots, \lambda_m(s)] \\ &= \mathrm{diag}\left[\frac{1}{(s+a)^{r_1}}, \ldots, \frac{1}{(s+a)^{r_m}}\right], \end{aligned} \tag{19}$$

where $\lambda_j(s) = 1$ if $r_j = 0$. The following are some examples of plants with transmission zeros at infinity: (1) The transmission zero at infinity of $P = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s^2-1)} \\ 1 & \frac{1}{s-2} \end{bmatrix}$ is not a blocking zero. An LCF is

$$P = D^{-1}N = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ \frac{s-2}{s+1} & \frac{1}{s+1} \end{bmatrix}.$$

Then

$$N^{-1} = \begin{bmatrix} \frac{1}{3}(s+1)^2 & -\frac{1}{3}(s+1) \\ -\frac{1}{3}(s-2)(s+1)^2 & \frac{1}{3}(s+1)^2 \end{bmatrix}, \qquad r_1 = 3, r_2 = 2$$

and (19) becomes

$$\Lambda(s) = \mathrm{diag}\left[\frac{1}{(s+a)^3}, \frac{1}{(s+a)^2}\right], \quad a \in \mathbb{R}_+.$$

(2) Some entries of

$$P = \begin{bmatrix} \frac{s+1}{s-1} & \frac{-(s+1)}{s-1} \\ \frac{1}{s-2} & \frac{1}{(s+1)(s-2)} \end{bmatrix} = D^{-1}N$$

$$= \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ \frac{1}{s+1} & \frac{1}{(s+1)^2} \end{bmatrix}$$

are not strictly-proper and $r_1 = 0$. Then

$$N^{-1} = \begin{bmatrix} \frac{1}{s+2} & \frac{(s+1)^2}{s+2} \\ \frac{-(s+1)}{s+2} & \frac{(s+1)^2}{s+2} \end{bmatrix}$$

and

$$\Lambda(s) = \mathrm{diag}\left[1, \frac{1}{s+a}\right], \quad a \in \mathbb{R}_+ \text{ in (19)}.$$

(3) There may be a blocking zero at infinity in addition to transmission zeros at infinity that do not appear in every entry of P; e.g.,

$$P = \begin{bmatrix} \frac{1}{s-1} & \frac{-1}{s-1} \\ \frac{1}{(s+1)(s-2)} & \frac{1}{(s+1)^2(s-2)} \end{bmatrix}$$

$$= D^{-1}N = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix}^{-1} \frac{1}{(s+1)} \begin{bmatrix} 1 & -1 \\ \frac{1}{s+1} & \frac{1}{(s+1)^2} \end{bmatrix}.$$

Then

$$N^{-1} = (s+1) \begin{bmatrix} \frac{1}{s+2} & \frac{(s+1)^2}{s+2} \\ \frac{-(s+1)}{s+2} & \frac{(s+1)^2}{s+2} \end{bmatrix},$$

and with $r_1 = 1, r_2 = 2$, (19) becomes

$$\Lambda(s) = \mathrm{diag}\left[\frac{1}{s+a}, \frac{1}{(s+a)^2}\right], a \in \mathbb{R}_+.$$

Proposition 3 gives a systematic controller synthesis method for MIMO plants with blocking or transmission zeros at infinity.

Proposition 3 (*Controller Synthesis for Plants with Transmission Zeros at Infinity*). *Let $P \in \mathbf{R_p}^{m\times m}$ have no finite transmission zeros in $\mathcal{U}$. Let $P = D^{-1}N$ be any LCF of P. Define Λ as in (19). For $j = 1, \ldots, m$, choose any monic r_j-th order strictly-Hurwitz polynomial $\rho_j(s)$. Define Ψ as*

$$\Psi(s) := s\left[D(s)D(\infty)^{-1}\mathrm{diag}\left[\frac{(s+a)^{r_1}}{\rho_1(s)}, \ldots, \frac{(s+a)^{r_m}}{\rho_m(s)}\right] - I\right]. \tag{20}$$

Choose $\alpha \in \mathbb{R}_+$ such that

$$\alpha > \max_j r_j \|\Psi\|. \tag{21}$$

Then the integral-action controller in (22) stabilizes P:

$$\begin{aligned} C = N^{-1}\Lambda\,\mathrm{diag}\Bigg[&\frac{\alpha^{r_1}\rho_1(s)}{(s+\alpha)^{r_1} - \alpha^{r_1}}, \\ &\frac{\alpha^{r_2}\rho_2(s)}{(s+\alpha)^{r_2} - \alpha^{r_2}}, \ldots, \frac{\alpha^{r_m}\rho_m(s)}{(s+\alpha)^{r_m} - \alpha^{r_m}}\Bigg] D(\infty). \quad \square \end{aligned} \tag{22}$$

Remarks. The poles of the MIMO integral-action controller in (22) are the poles of the stable matrix $N^{-1}\Lambda$ and the roots of the Hurwitz polynomials $d_j(s)$ defined as

$$d_j(s) := (s+\alpha)^{r_j} - \alpha^{r_j}, \tag{23}$$

which has one root at $s = 0$ and the remaining $(r_j - 1)$ roots in $\mathbb{C} \setminus \mathcal{U}$. $\square$

3.4. Comparison with design based on augmented plant model

The standard integral-action controller design method is based on augmentation, where the integral of error is included in the state-space representation of the plant. Let $[A_p, B_p, C_p, D_p]$ be a minimal state-space representation of the plant $P \in \mathbb{R}_p^{m\times m}$, where $A_p \in \mathbb{R}^{n\times n}$. A full n-th order observer is designed by choosing $L \in \mathbb{R}^{n\times m}$ such that the eigenvalues of $(A_p - LC_p)$ are in $\mathbb{C} \setminus \mathcal{U}$. The estimator state is used for state-feedback $K_a = [K_n \ K_m] \in \mathbb{R}^{m\times(n+m)}$ such that the eigenvalues of $(A_a - B_a K_a)$ are in $\mathbb{C} \setminus \mathcal{U}$. An $(n+m)$-th dimensional state-space representation of the augmented plant $\frac{1}{s}P \in \mathbb{R}_p^{m\times m}$ gives the augmented state-space matrices (A_a, B_a) as

$$A_a = \begin{bmatrix} A_p & 0 \\ -C_p & 0 \end{bmatrix}, \qquad B_a = \begin{bmatrix} B_p \\ -D_p \end{bmatrix}. \tag{24}$$

Let C_a denote the integral-action controller designed using this augmentation. Then a state-space representation (A_c, B_c, C_c, D_c) for the controller C_a is:

$$\begin{aligned} A_c &= \left[A_a - B_a K_a + L_a\left([C_p\ 0] - D_p K_a\right)\right] \\ &= \begin{bmatrix} A_p - (B_p - LD_p)K_n + LC_p & -(B_p - LD_p)K_m \\ 0 & 0 \end{bmatrix}, \\ B_c &= L_a = \begin{bmatrix} -L \\ I \end{bmatrix}, \qquad C_c = -K_a, \qquad D_c = 0. \end{aligned} \tag{25}$$

The transfer function of this controller is given by

$$\begin{aligned} C_a &= C_c(sI_{(n+m)} - A_c)^{-1}B_c + D_c \\ &= -K_a\left[sI - A_a + B_a K_a - L_a([\,C\ 0\,] - DK_a)\right]^{-1} L_a. \end{aligned} \tag{26}$$

The controller C_a in (26) has m of its $(n+m)$ eigenvalues at $s = 0$. The other n eigenvalues of C_a may be anywhere in the complex plane, so C_a may be unstable. Integral-action controllers designed using full-order observer and state feedback based on an augmented plant model always give a strictly proper controller transfer function. In the SISO case with an n-th order plant P, the design based on the augmented plant P/s gives an integral-action controller C_a as in (26), which is $(n+1)$-th order.

The order of the bi-proper controllers designed for the plant classes in Sections 3.1–3.3 do not depend on the number of eigenvalues of the plant $P \in \mathbf{R_p}^{m\times m}$. For the class of plants with no $\mathcal{U}$-zeros discussed in Proposition 1, PI controller $C \in \mathbf{R_p}^{m\times m}$ in (4) can be realized with m states, with all m eigenvalues at zero. For the class of plants with r blocking zeros (large, possibly at infinity) discussed in Proposition 2, the controller $C \in \mathbf{R_p}^{m\times m}$ in (9) (and the special case of (12) when all blocking zeros of P are at infinity) can be realized with rm states, with m of the eigenvalues at zero; the remaining $(r-1)m$ eigenvalues are in the OLHP. In the SISO case with an n-th order plant P, these controllers are r-th order, where $r \le n$, while C_a in (26) is of order $(n+1)$.

3.5. Examples

We now explore an SISO and an MIMO example to compare the plant-augmentation based controller C_a to the designs in (9) and (12). The number of poles in each entry of the controller in the proposed integral-action synthesis method is lower than the number of poles in the entries of the controller designed using a full order observer approach based on the augmented plant. For the SISO case, the proposed controller order equals the number of $\mathcal{U}$-zeros for the plant class under consideration.

Example 1. Consider the fourth-order SISO unstable plant given in (27), which has one finite $\mathcal{U}$-zero $z_1 \in \mathbb{C}_+$, and two zeros at infinity, i.e., $r = 3$:

$$P = \frac{(s+1)(s-z_1)}{(s-0.2)(s-1)(s^2+31)}. \tag{27}$$

For this plant, $D(\infty)^{-1} = -z_1$. By Proposition 2, if we choose the monic 3rd order strictly-Hurwitz polynomial $\rho(s) = (s+1.4)^3$, then by (7), $\|\Phi\| = 6.4$. We can apply the controller in (9) for z_1 large enough to satisfy $\|\Phi\| > (\frac{r}{z_i} + \sum_{i=1}^{r} \frac{1}{z_i}) = \frac{4}{z_1}$, i.e., $z_1 > 4\|\,\Phi\,\| = 25.60$. Suppose that the finite $\mathcal{U}$-zero is at $z_1 = 27$. By (8), choose any α satisfying $25.1650 < \alpha < 27$; for example, let $\alpha = 26$. The third-order integral-action controller as in (9) is

$$C = \frac{\alpha^3\,\rho(s)D(\infty)}{(s+\alpha)^3 - \alpha^3(1-s/z_1)} = \frac{-650.9630\,(s+1.4)^3}{s\,(s^2+78s+2679)}, \tag{28}$$

which is bi-proper and has a stable inverse. The r zeros of the controller are chosen by design as $\rho(s)$ can be any monic strictly-Hurwitz r-th order polynomial. One pole is at $s = 0$ due to the integral-action requirement in the design, and the remaining two poles are in the OLHP.

We now design an integral-action controller as in (26) based on the augmented plant P/s. A minimal state-space representation of the augmented version P/s of this SISO fourth-order plant has $n+1 = 5$ states. We choose $L = \begin{bmatrix}10.6039 & -1.2194 & -0.5827 & -0.0470\end{bmatrix}^T \in \mathbb{R}^{4\times 1}$ to place the eigenvalues of $(A_p - LC_p)$ at $\{-2, -3, -4, -5\}$. Then we choose $K_a = \begin{bmatrix}7.1000 & -17.5900 & 52.4897 & 0.3857 & 0.0607\end{bmatrix} \in \mathbb{R}^{1\times 5}$ to place the eigenvalues of $(A_a - B_a K_a)$ at $\{-0.5, -1.2, -1.3, -1.4, -1.5\}$. The integral-action controller C_a as in (26) is

$$C_a = \frac{66.0744s^4 - 122.3214s^3 - 246.1680s^2 - 5.5818s - 7.2800}{s\,(s^4 + 21.1000s^3 + 161.3300s^2 + 285.0706s + 143.8425)}, \tag{29}$$

which is fifth-order. Comparing the third-order integral-action controller in (28) with the fifth-order controller in (29), the design based on Proposition 2 gives a lower order controller, whose order equals the number of $\mathcal{U}$-zeros of the plant. □

Example 2. In this example we consider a chemical reactor plant obtained by linearizing the model given in [17], where the concentration of the inlet reactant and the rate of heat input are manipulated to regulate the outlet reactant concentration and the reactor temperature. The linearization around one of the operating points gives the unstable plant transfer-matrix in (30), where P has poles at $s = 0.0614 \in \mathcal{U}$ and $s = -0.0167$, and a blocking zero at infinity:

$$P = \frac{1}{100y}\begin{bmatrix} 1.67s - 0.1232 & -0.00189 \\ 4.143 & 4.184s + 0.1218 \end{bmatrix}, \quad y = (s-0.0614)(s+0.167). \tag{30}$$

With $r = 1$, the plant in (30) can be written as in (6), where $N = \frac{1}{(s+a)} I_2$ and

$$D = \frac{100(s+0.167)}{6.9873(s+a)} \begin{bmatrix} \dfrac{4.184s+0.1218}{(s+0.0167)} & \dfrac{0.00189}{(s+0.0167)} \\ \dfrac{-4.143}{(s+0.0167)} & \dfrac{1.67s-0.1232}{(s+0.0167)} \end{bmatrix},$$

where $a \in \mathbb{R}_+$. By Proposition 2, we take a simple first-order $\rho(s) = (s+1)$. Then the norm in (10) is $\|\Phi\| = 1.5$. If we choose

$\alpha = 3 > r\|\Phi\|$ satisfying (11), then the controller with first-order terms as in (12) becomes

$$C = \frac{(s+1)}{s}\,\mathrm{diag}\left[179.64 \qquad 71.7\right]. \tag{31}$$

For different choices of $\rho(s)$ and α, we would obtain different first-order controllers. A minimal state-space realization of the controller in (31) has 2 states, with eigenvalues both at $s = 0$.

We now design an integral-action controller as in (26) based on the augmented plant $\frac{1}{s}P$. A minimal state-space representation of the (2×2) plant in (30) has $n = 4$ states, and the augmented description has $n + m = 6$ states. We choose $L \in \mathbb{R}^{4\times 2}$ to place the observer poles (eigenvalues of $(A_p - LC_p)$) at $\{-50, -50, -40, -40\}$. Then we choose $K_a \in \mathbb{R}^{2\times 6}$ to place the eigenvalues of $(A_a - B_a K_a)$ at $\{-1.5444 \pm j0.7764, -1.5835 \pm j0.7018, -2, -2\}$. The controller C_a as in (26) then has 6 eigenvalues, at $\{-120575, 120482, -48.498 \pm j10385, 0, 0\}$. In this case, C_a has one eigenvalue in $\mathcal{U}$ in addition to the two at $s = 0$. The transfer-function of C_a is strictly-proper, with fifth-order denominator terms. □

4. Conclusions

For plants whose zeros in the unstable region are 'large' and particularly at infinity, we developed a systematic synthesis methodology that results in a simple integral-action controller, whose poles other than the one integrator providing the integral-action all are in the stable region. We investigated both blocking zeros and transmission zeros at infinity. The plant classes under consideration do not put any constraints on where the poles are and also do not restrict the OLHP zeros. Since the controller has only one integrator but is otherwise stable, the plants here are in fact strongly stabilizable [18].

The proposed controllers for each plant class we considered here have flexibility in the choice of the design parameters (e.g., the numerator polynomial for the controller is chosen arbitrarily). The effect of the parameter choices on the system performance can be explored in future extensions, although the scope of this current work is limited to the challenging goal of stabilization using simple controllers while achieving asymptotic tracking of step-input references with zero steady-state error.

Appendix

Proof of Proposition 1. Let $N_c = I$ and $D_c = \frac{s}{\alpha(s+g)}K^{-1} = C^{-1}$. By (2), $C = N_c D_c^{-1}$ stabilizes $P = (P^{-1})^{-1}I$ if and only if $M = NN_c + DD_c$ is unimodular, where

$$M = I + \frac{1}{\alpha}\frac{s}{(s+g)}P^{-1}(s)K^{-1}.$$

A sufficient condition for M to be unimodular is that $\|\frac{1}{\alpha}\frac{s}{(s+g)}P^{-1}(s)K^{-1}\| < 1$, which holds for α satisfying (3) (this condition follows from the small-gain theorem, see e.g., [11]). Hence, C in (4) stabilizes P, which has no zeros in $\mathcal{U}$. □

Proof of Proposition 2. Let $d(s) := (s+\alpha)^r - \alpha^r \prod_{i=1}^{r}(1 - s/z_i)$, which becomes (13) when all zeros are at infinity. Let $N_c = \alpha^r I$ and $D_c = \alpha^r C^{-1} = \frac{d}{\rho}D(\infty)^{-1}$; by choice of $\rho(s)$, C^{-1} is stable. By (2), $C = N_c D_c^{-1}$ stabilizes $P = D^{-1}N$ given by (5) if and only if $M = NN_c + DD_c$ is unimodular, where

$$M = \frac{\alpha^r \prod_{i=1}^{r}(1 - s/z_i)}{(s+a)^r}I + D(s)D(\infty)^{-1}\frac{d}{\rho}.$$

Since $a, \alpha \in \mathbb{R}_+$, M is unimodular if and only if $\hat{M} := M\frac{(s+a)^r}{(s+\alpha)^r}$ is unimodular, where

$$\begin{aligned}
\hat{M} &= \frac{\alpha^r \prod_{i=1}^{r}(1 - s/z_i)}{(s+\alpha)^r}I + DD(\infty)^{-1}\frac{(s+a)^r}{\rho}\frac{d(s)}{(s+\alpha)^r} \\
&= I + \left[DD(\infty)^{-1}\frac{(s+a)^r}{\rho} - I\right]\frac{d(s)}{(s+\alpha)^r} \\
&= I + s\left[\frac{\prod_{i=1}^{r}(1 - s/z_i)}{\rho}P^{-1}(s)D(\infty)^{-1} - I\right]\frac{d(s)}{s\,(s+\alpha)^r} \\
&= I + \Phi(s)\frac{d(s)}{s\,(s+\alpha)^r}.
\end{aligned}$$

Since $(D(s)D(\infty)^{-1}\frac{(s+a)^r}{\rho} - I)(\infty) = 0$, Φ is proper. A sufficient condition for $\hat{M}$ to be unimodular is that $\|\Phi(s)\frac{d(s)}{s\,(s+\alpha)^r}\| < 1$. For $z_i > \alpha$, the norm $\|\frac{d(s)}{s\,(s+\alpha)^r}\|$ is:

$$\begin{aligned}
\left\|\frac{d(s)}{s\,(s+\alpha)^r}\right\| &= \left\|\frac{(s+\alpha)^r - \alpha^r\prod_{i=1}^{r}(1 - s/z_i)}{s\,(s+\alpha)^r}\right\| \\
&= \left\|\frac{(s+\alpha)^r - \alpha^r}{s\,(s+\alpha)^r} + \frac{\alpha^r - \alpha^r\prod_{i=1}^{r}(1 - s/z_i)}{s\,(s+\alpha)^r}\right\| \\
&\le \left\|\frac{(s+\alpha)^r - \alpha^r}{s\,(s+\alpha)^r}\right\| + \left\|\frac{\alpha^r - \alpha^r\prod_{i=1}^{r}(1 - s/z_i)}{s\,(s+\alpha)^r}\right\| \\
&\le \frac{r}{\alpha} + \sum_{i=1}^{r}\frac{1}{z_i}.
\end{aligned}$$

Therefore, if $\left[\frac{r}{z_i} + \sum_{i=1}^{r}\frac{1}{z_i}\right] < \|\Phi(s)\|^{-1}$, then for $\alpha < z_i$ satisfying (8),

$$\begin{aligned}
\left\|\Phi(s)\frac{d(s)}{s(s+\alpha)^r}\right\| &\le \|\Phi(s)\|\left\|\frac{d(s)}{s\,(s+\alpha)^r}\right\| \\
&\le \|\Phi(s)\|\left[\frac{r}{\alpha} + \sum_{i=1}^{r}\frac{1}{z_i}\right] < 1,
\end{aligned}$$

and hence, $\hat{M}$ is unimodular; equivalently, the controller C in (9) stabilizes P. By Definition 1-(iv), C is an integral-action controller since $d(0) = 0$ implies $D_c(0) = \frac{d}{\rho}D(\infty)^{-1}|_{s=0} = 0$. Since ρ and d are both r-th order polynomials, $C = \frac{\alpha^r\rho(s)}{d(s)}D(\infty)$ and C^{-1} are both proper. □

Proof of Proposition 3. Let $d_j(s)$ be defined as in (23). Let $N_c = N^{-1}\Lambda\,\mathrm{diag}\left[\alpha^{r_1}, \ldots, \alpha^{r_1}\right]$ and $D_c = \alpha^r C^{-1} = D(\infty)^{-1}\mathrm{diag}\left[\frac{d_1(s)}{\rho_1(s)}, \ldots, \frac{d_m(s)}{\rho_m(s)}\right]$; note that C^{-1} is stable by choice of $\rho(s)$. By (2), $C = N_c D_c^{-1}$ stabilizes $P = D^{-1}N$ if and only if $M = NN_c + DD_c$ is unimodular, where

$$\begin{aligned}
M &= N(s)N^{-1}\Lambda\,\mathrm{diag}\left[\alpha^{r_1}, \ldots, \alpha^{r_1}\right] + D(s)D(\infty)^{-1} \\
&\quad \times \mathrm{diag}\left[\frac{d_1(s)}{\rho_1(s)}, \ldots, \frac{d_m(s)}{\rho_m(s)}\right].
\end{aligned}$$

Now since $a, \alpha \in \mathbb{R}_+$, M is unimodular if and only if $\hat{M} := M\mathrm{diag}\left[\frac{(s+a)^{r_1}}{(s+\alpha)^{r_1}}, \ldots, \frac{(s+a)^{r_m}}{(s+\alpha)^{r_m}}\right]$ is unimodular, where

$$\begin{aligned}\hat{M} &= I + s\left[D(s)D(\infty)^{-1}\mathrm{diag}\left[\frac{(s+a)^{r_1}}{\rho_1},\ldots,\frac{(s+a)^{r_m}}{\rho_m}\right]\right.\\ &\quad \left.- I\right]\mathrm{diag}\left[\frac{d_1(s)}{s(s+\alpha)^{r_1}},\ldots,\frac{d_m(s)}{s(s+\alpha)^{r_m}}\right]\\ &= I + \Psi(s)\,\mathrm{diag}\left[\frac{d_1(s)}{s(s+\alpha)^{r_1}},\ldots,\frac{d_m(s)}{s(s+\alpha)^{r_m}}\right].\end{aligned}$$

Since $(D(s)D(\infty)^{-1}\mathrm{diag}\left[\frac{(s+a)^{r_1}}{\rho_1},\ldots,\frac{(s+a)^{r_m}}{\rho_m}\right] - I)(\infty) = 0$, Ψ is proper. A sufficient condition for $\hat{M}$ to be unimodular is that $\left\|\Psi(s)\,\mathrm{diag}\left[\frac{d_1}{s(s+\alpha)^{r_1}},\ldots,\frac{d_m}{s(s+\alpha)^{r_m}}\right]\right\| < 1$. The norm

$$\begin{aligned}&\left\|\mathrm{diag}\left[\frac{d_1}{s(s+\alpha)^{r_1}},\ldots,\frac{d_m}{s(s+\alpha)^{r_m}}\right]\right\|\\ &\quad= \max_j\left\|\frac{(s+\alpha)^{r_j}-\alpha^{r_j}}{s\,(s+\alpha)^{r_j}}\right\| \le \max_j\frac{r_j}{\alpha}.\end{aligned}$$

Therefore, for α satisfying (21),

$$\begin{aligned}&\left\|\Psi(s)\mathrm{diag}\left[\frac{d_1(s)}{s(s+\alpha)^{r_1}},\ldots,\frac{d_m(s)}{s(s+\alpha)^{r_m}}\right]\right\| \le \|\Psi(s)\|\\ &\quad\times\left\|\mathrm{diag}\left[\frac{d_1(s)}{s(s+\alpha)^{r_1}},\ldots,\frac{d_m(s)}{s(s+\alpha)^{r_m}}\right]\right\|\\ &\le \|\Psi(s)\|\max_j\frac{r_j}{\alpha} < 1.\end{aligned}$$

Hence, $\hat{M}$ is unimodular; equivalently, C in (22) stabilizes P. Since $d_j(0) = 0$ implies $D_c(0) = D(\infty)^{-1}\mathrm{diag}\left[\frac{d_1(0)}{\rho_1(0)},\ldots,\frac{d_m(0)}{\rho_m(0)}\right] = 0$, by Definition 1-(iv), C is an integral-action controller. $\square$

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