Proposition 1 has an immediate consequence.

Corollary 1: Consider an interval matrix $\mathcal{A}$, its center $\boldsymbol{A}^0$ and its right end $\boldsymbol{A}^+$, defined in accordance with equality (1). If $\boldsymbol{A}^0$ is an essentially nonnegative matrix, then $\lambda_{\max}(\boldsymbol{A}^+)$ is the right end-point of the eigenvalue range of $\mathcal{A}$, i.e., the following equality holds:

$$I^+(\mathcal{A}) = \lambda_{\max}(\boldsymbol{A}^+). \tag{22}$$

Proof: We have the matrix equality $\boldsymbol{\Theta} = \boldsymbol{A}^+$ and condition (a) in the hypothesis of Proposition 1 (ii) is fulfilled. □

The values $I^+(\mathcal{A})$ provided by Proposition 1 (ii) (or, equivalently, Corollary 1) for the interval matrices $\mathcal{A}_1$ (4), $\mathcal{A}_3$ (6) are given in the fifth column of Table I, in Section IV-D. Note that Proposition 1 (ii) cannot be applied to $\mathcal{A}_2$ (5), but Proposition 1 (i) yields the right outer bound $I_e^+(\mathcal{A}_2)$ given in the fifth column of Table I.

D. Brief Comparative Analysis

Table I summarizes the key points of a comparative analysis on the use of Theorem 1 [1] versus the three methods discussed in Section IV, by referring to the interval matrices $\mathcal{A}_1$ (4), $\mathcal{A}_2$ (5), $\mathcal{A}_3$ (6). We have extended this analysis to numerous other examples that were not reproduced here for brevity reasons. The considered examples do not intend to prove that methods in Sections IV-A–IV-C ensure high accuracy; a thorough testing of these methods is obviously beyond the objective of our note. The note focuses on the power/significance of Theorem 1 [1], and the mentioned methods serve only as comparison instruments.

As an overall point of view, Theorem 1 [1] presents a theoretical interest, but it is not suitable for applications. Its hypothesis and Assumption 1 [1] are quite restrictive conditions, and even when these conditions are fulfilled, the right outer bound calculated by Theorem 1 [1] may be less accurate than the values given by the methods in Sections IV-A–IV-C.

V. Conclusion

The commented paper proposes numerical tools for assessing the stability of interval systems, which provide a right outer bound of the eigenvalue range. The approach focuses on the theoretical support, and pays less attention to the practical use of these tools in applications. This type of approach generates some debatable problems, for which the reader does not find direct answers in the original text or in the cited references. Therefore, our note can be regarded as a natural continuation of the commented paper, bringing the following contributions. Section II shows that Example 1 is erroneous and the true value of the right outer bound, calculable by Theorem 1, is less accurate than claimed by the commented paper. Section III analyzes several drawbacks encountered in the exploitation of Theorem 1. Section IV presents three methods for assessing the stability margin of interval systems, which are founded on different bases than Theorem 1. The whole note stimulates a broad understanding of the research progress in the considered area and constructs meaningful comparisons between works relying on different instruments, but targeting similar objectives.

References

[1] L. Kolev and S. Petrakieva, "Assessing the stability of linear time-invariant continuous interval dynamic systems," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 393–397, Mar. 2005.

[2] J. Rohn, "Bounds on eigenvalues of interval matrices," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 78, no. 3, pp. S1049–S1050, 1998.

[3] L. Kolev and I. Nenov, "Cheap and tight bounds on the solution set of perturbed systems of nonlinear equations," *Rel. Comput.*, vol. 7, no. 5, pp. 399–408, 2001.

[4] L. Kolev, "Eigenvalue range determination for interval and parametric matrices," *Int. J. Circuits Theor. Appl.*, vol. 38, no. 10, pp. 1027–1061, 2010.

[5] O. Pastravanu and M. Voicu, "Necessary and sufficient conditions for componentwise stability of interval matrix systems," *IEEE Trans. Autom. Control*, vol. 49, no. 6, pp. 1016–1021, Jun. 2004.

[6] S. S. Wang and W. G. Lin, "A new approach to the stability analysis of interval systems," *Control-Theory Adv. Technol.*, vol. 7, no. 2, pp. 271–284, 1991.

[7] *Global Optimization Toolbox 3. User's Guide*. Natick, MA: The MathWorks, Inc., 2010.

[8] D. Hertz, "Interval analysis: Eigenvalue bounds of interval matrices," in *Encyclopedia of Optimization*, C. A. Floudas and P. M. Pardalos, Eds. New York: Kluwer Academic Publishers, 2001, vol. 6, pp. 11–17.

[9] Q. Yuan, Z. He, and H. Leng, "An evolution strategy method for computing eigenvalue bounds of interval matrices," *Appl. Math. Comp.*, vol. 196, no. 1, pp. 257–265, 2008.

[10] L. X. Xin, "Necessary and sufficient conditions for stability of a class of interval matrices," *Int. J. Control*, vol. 45, no. 1, pp. 211–214, 1987.

[11] M. E. Sezer and D. D. Šiljak, "On stability of interval matrices," *IEEE Trans. Autom. Control*, vol. 39, no. 2, pp. 368–371, Feb. 1994.

[12] D. Liu and A. Molchanov, "Criteria for robust absolute stability of time-varying nonlinear continuous-time systems," *Automatica*, vol. 38, no. 4, pp. 627–637, 2002.

Strong Stabilization of a Class of MIMO Systems

A. N. Gündeş and H. Özbay

***Abstract*—Stabilization of finite dimensional linear, time-invariant, multi-input multi-output plants by stable feedback controllers, known as the strong stabilization problem, is considered for a class of plants with restrictions on the zeros in the right-half complex plane. The plant class under consideration has no restrictions on the poles, or on the zeros in the open left-half complex plane, or on the zeros at the origin or at infinity; but only one finite positive real zero is allowed. A systematic strongly stabilizing controller design procedure is proposed. The freedom available in the design parameters may be used for additional performance objectives although the only goal here is strong stabilization. In the special case of single-input single-output plants within the class considered, the proposed stable controllers have order one less than the order of the plant.**

***Index Terms*—Linear time-invariant (LTI), multi-input multi-output (MIMO), parity interlacing property (PIP).**

I. Introduction

This note discusses the strong stabilization problem for a class of linear time-invariant (LTI), multi-input multi-output (MIMO) plants that have restrictions on their zeros in the region of instability. Strong stabilization refers to output feedback stabilization of a given plant by a stable controller. Interest in the strong stabilization problem is due to important practical considerations as well as due to the equivalence of simultaneous stabilization of two plants to the strong stabilization

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of one related system [15]. Although stable stabilizing controller design is important, not all plants are strongly stabilizable. A given plant is strongly stabilizable if and only if it satisfies the parity interlacing property (PIP); a plant is said to satisfy the PIP if the number of poles (counted according to their McMillan degrees) between any pair of blocking-zeros on the extended positive real-axis is even [15], [16].

For single-input multi-output plants, and single-input single-output (SISO) plants as a special case, several procedures are available for obtaining strongly stabilizing controllers involving interpolation constraints to construct a unit in stable rational functions and usually resulting in very high order controllers (e.g. [5], [15], [16]). A parametrization of all strongly stabilizing controllers can be obtained for SISO plants using interpolation with infinite dimensional transfer functions [15]. Extensions of these interpolation techniques to MIMO plants are also available (e.g., [13]), and strong stabilization of MIMO plants has been studied extensively in the literature, some using numerical approaches and some under H_∞ or H_2 performance criteria (e.g., [2]–[4], [8], [9], [11], [12], [17], [18]). Analytical synthesis methods to design stable stabilizing controllers were explored for MIMO plants that have at most two blocking-zeros on the extended non-negative real axis in [10], where connections to the sufficient conditions in [18] were also established. These results excluded plants that have transmission-zeros (instead of blocking-zeros) and plants that have more than a total of two zeros at the origin and infinity. In the special case of SISO plants, this implied that the results were not applicable for plants with relative degree larger than two. In this work, we obtain a stable stabilizing controller design procedure that applies to a large class of MIMO strongly stabilizable plant with any number of (transmission and blocking) zeros at the origin and at infinity, and at most one finite positive real zero. The constraints of [10] on the number of zeros at the origin and at infinity are removed here and the results are generalized to include transmission-zeros as well as blocking-zeros. The plant class under consideration has no restrictions on the poles; the zeros in the open left-half complex plane are also completely unrestricted. However, these plants have no unstable zeros except on the extended non-negative real axis. We assume two "positive eigenvalue" conditions for certain matrices, and show that these conditions are sufficient for existence of strongly stabilizing controllers. These eigenvalue conditions are in fact equivalent to PIP in the case of single-output (SISO or fat) plants, and hence they are necessary for strong stabilizability. Analogous eigenvalue conditions assumed for the case of tall plants are similarly necessary for the case of single-input plants.

Various design methods are available for MIMO plants without restrictions on the unstable zeros but assuming other sufficient conditions in addition to PIP to obtain strongly stabilizing controllers, e.g., [1], [2], [4], [12]. When the plant has two complex conjugate zeros located in such a way that the PIP is about to be violated (as the imaginary part goes to zero), many of the existing finite dimensional controller design techniques fail because the minimum order of the strongly stabilizing controllers can be very large (grows as the imaginary part gets smaller) [14]. The method proposed here is simple, and allows freedom in the design parameters, which may be used for additional performance objectives that are not considered here. Using standard robustness arguments, the designed controllers provide robust closed-loop stability if the plant is subject to stable additive or pre-multiplicative perturbations. In the special case of SISO plants, the proposed design method gives a stable stabilizing controller whose order is one less than the order of the given plant.

The technical note is organized as follows: Section II gives the problem formulation, and defines the class of plants considered for strong stabilization. The main result in Section III, Theorem I, provides a systematic procedure of constructing strongly stabilizing controllers for the class of MIMO plants considered. Illustrative examples are given in Section IV, where SISO plant examples are also provided to demonstrate that the proposed method gives a low-order controller (order one less than that of the plant). Concluding remarks are in Section V.

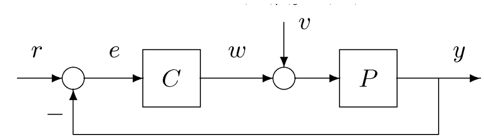

Fig. 1. Unity-Feedback System $Sys(P, C)$.

Although we discuss continuous-time systems, all results apply also to discrete-time systems with appropriate modifications. The following standard notation is used:

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ denote real, positive real, and complex numbers, respectively. The extended closed right-half plane is $\mathcal{U} = \{s \in \mathbb{C} | \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; I is the identity matrix (of appropriate dimension). A transfer-matrix $M \in \mathcal{M}(\mathbf{S})$ is called unimodular iff $M^{-1} \in \mathcal{M}(\mathbf{S})$. The H_∞-norm of $M \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M\|$ (i.e., the norm $\|\cdot\|$ is the usual operator norm $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). For simplicity, we drop (s) in transfer-matrices such as $G(s)$ where this causes no confusion. When $m \leq \mu$, we use coprime factorizations over $\mathbf{S}$; i.e., for $P \in \mathbf{R_p}^{m\times\mu}$, $P = D^{-1}N$ denotes a left-coprime-factorization (LCF), where $N \in \mathbf{S}^{m\times\mu}$, $D \in \mathbf{S}^{m\times m}$, $\det D(\infty) \neq 0$. When $m > \mu$, the results are stated in terms of a right-coprime-factorization (RCF) $P = \tilde{N}\tilde{D}^{-1}$, where $\tilde{N} \in \mathbf{S}^{m\times\mu}$, $\tilde{D} \in \mathbf{S}^{\mu\times\mu}$, $\det\tilde{D}(\infty) \neq 0$. For full normal rank P (i.e., $\mathrm{rank}P(s) = \min\{m, \mu\}$), we say that $z \in \mathcal{U}$ is a $\mathcal{U}$-zero of P if $\mathrm{rank}N(z) < \min\{m, \mu\}$ (equivalently, $\mathrm{rank}\tilde{N}(z) < \min\{m, \mu\}$); these zeros include both transmission-zeros and blocking-zeros in $\mathcal{U}$. In the product notation used throughout,it is assumed that $\prod_{j=\nu}^{\eta} g_j = 1$ if $\nu > \eta$.

II. Problem Description and Plant Classes

Consider the standard LTI, MIMO unity-feedback system $Sys(P, C)$ shown in Fig. 1, where $P \in \mathbf{R_p}^{m\times\mu}$ and $C \in \mathbf{R_p}^{\mu\times m}$ denote the plant's and the controller's transfer-matrices, respectively. The objective is to design a stabilizing controller C, which is stable itself. It is assumed that the feedback system is well-posed, P and C have no unstable hidden-modes, and the plant $P \in \mathbf{R_p}^{m\times\mu}$ is full normal rank equal to $\min\{m, \mu\}$. We discuss the "square or fat" plant case $(m \leq \mu)$ in detail; the "tall" plant case $(m > \mu)$ is similar and can be obtained using simple modifications as explained briefly in Remark 1. Let $P = D^{-1}N$ be an LCF of the plant and $C = N_c D_c^{-1}$ be an RCF of the controller, where N, D, N_c, $D_c \in \mathcal{M}(\mathbf{S})$ are matrices with appropriate sizes,$\det D(\infty) \neq 0$, $\det D_c(\infty) \neq 0$. The system $Sys(P, C)$ is said to be stable iff the closed-loop transfer-function from (r, v) to (y, w) is stable. The controller C is said to stabilize P iff C is proper and the system $Sys(P, C)$ is stable. The controller C stabilizes $P \in \mathcal{M}(\mathbf{R_p})$ if and only if

$$M := DD_c + NN_c \tag{1}$$

is unimodular. The stabilizing controller C is stable if and only if M in (1) is unimodular with a unimodular D_c; in this case C is said to strongly stabilize P. There exist strongly stabilizing controllers for a plant P if and only if P satisfies the PIP. Let $z_1, \ldots, z_\ell \in \mathbb{R} \cap \mathcal{U}$ be the non-negative real-axis blocking-zeros of P in the extended closed right-half-plane, i.e., $N(z_k)=0$ for $1\le k\le \ell$. Then P satisfies the PIP if and only if $\det D(z_k)$ is sign invariant for $1\le k\le \ell$ (e.g., [15]).

The plants under consideration for strongly stabilizing controller synthesis have no restrictions on their poles; there are no restrictions on the zeros in the open left-half complex plane $\mathbb{C}\setminus\mathcal{U}$, at the origin $s=0$, and at infinity. The finite non-zero $\mathcal{U}$-zeros are restricted. We only consider the case where the plant P has at most one non-zero finite transmission-zero in the region of instability $\mathcal{U}$ and it does not have a $\mathcal{U}$-pole at that same point. It may have any number of transmission-zeros at the origin and at infinity; if the plant has zeros at the origin, then we assume that it does not also have a pole at $s=0$.

At the $\mathcal{U}$-zeros of $P\in\mathbf{R}_{\mathbf{p}}^{m\times\mu}$, where $m\le\mu$, the numerator N in any LCF $P=D^{-1}N$ drops rank; i.e., $z\in\mathcal{U}$ is a $\mathcal{U}$-zero if $\mathrm{rank}N(z)<m$. Let N^R denote an $\mu\times m$ right-inverse of $N\in\mathbf{S}^{m\times\mu}$; if P has any $\mathcal{U}$-zeros, then N^R is not stable. For square plants, $N^R=N^{-1}$. Write N^R as in (2), where $x_{ij},y_{ij}\in\mathbf{S}$, $i=1,\ldots,\mu$, $j=1,\ldots,m$

$$N^R=\left[\frac{x_{ij}}{y_{ij}}\right]_{i=1,\ldots,\mu;j=1,\ldots,m}. \tag{2}$$

Then the largest numerator invariant-factor $\lambda_z\in\mathbf{S}$ is a least-common-multiple of all y_{ij}, and hence, $(\lambda_z N^R)\in\mathcal{M}(\mathbf{S})$. There are four possibilities for λ_z depending on whether P has a finite $\mathcal{U}$-zero or zeros at infinity or at the origin:

Case (i) If P has no transmission-zeros in the unstable region $\mathcal{U}$, then $P\in\mathbf{R}_{\mathbf{p}}^{m\times\mu}$ has a stable right-inverse $P^R\in\mathbf{R}_{\mathbf{p}}^{\mu\times m}$; an LCF is given by $P=D^{-1}N$, where N has a stable right-inverse, and $\lambda_z=1$.

Case (ii) If P has a finite non-zero {\cal U}b zero, then the general expression for the largest invariant-factor λ_z of N is

$$\lambda_z=\frac{(1-s/z)}{(s+a)}\left(\prod_{i=1}^{n_\infty}\frac{1}{(s+a_i)}\right)\left(s^{n_o}\prod_{j=1}^{n_o}\frac{1}{(s+b_j)}\right) \tag{3}$$

where $a\in\mathbb{R}_+$, $a_i\in\mathbb{R}_+$ for $1\le i\le n_\infty$, $b_j\in\mathbb{R}_+$ for $1\le j\le n_o$. The total number of $\mathcal{U}$-zeros of λ_z is $n=n_\infty+n_o+1$, where n_o is the number of zeros at the origin $s=0$, and n_∞ is the number of zeros at infinity. If P has no zeros at infinity or at the origin, and has one finite positive zero, then the expression (3) is still valid with $n_\infty=0$ or with $n_o=0$. If $n_\infty\ne 0$, we assume that all eigenvalues of W defined in (4) have positive real parts. If $n_o\ne 0$, we assume that all eigenvalues of Y defined in (5) have positive real parts

$$W:=D(z)^{-1}D(\infty) \tag{4}$$
$$Y:=D(0)^{-1}D(z). \tag{5}$$

In the single-output case (including SISO), the eigenvalue conditions become $W>0$ and $Y>0$, which is equivalent to the PIP: If $P(z)=P(\infty)=P(0)=0$, then P satisfies the PIP if and only if $D(z),D(\infty),D(0)$ all have the same sign.

Case (iii) If P does not have a finite non-zero $\mathcal{U}$-zero but it has zeros at infinity and at the origin, then take $z=\infty$ in (3) and the expression for the largest invariant-factor λ_z of N is

$$\lambda_z=\frac{1}{(s+a)}\left(\prod_{i=1}^{n_\infty}\frac{1}{(s+a_i)}\right)\left(s^{n_o}\prod_{j=1}^{n_o}\frac{1}{(s+b_j)}\right) \tag{6}$$

where $a\in\mathbb{R}_+$, $a_i\in\mathbb{R}_+$ for $1\le i\le n_\infty$, $b_j\in\mathbb{R}_+$ for $1\le j\le n_o$. The total number of $\mathcal{U}$-zeros of λ_z is $n=n_\infty+n_o+1$; the number of zeros at infinity now becomes $n_\infty+1$, and n_o is the number of zeros at the origin $s=0$. In this case, $W=D(\infty)^{-1}D(\infty)=I$. If $n_o\ne 0$, we assume that all eigenvalues of $Y:=D(0)^{-1}D(\infty)$ have positive real parts, which is again equivalent to PIP for single-output plants.

Case (iv) If the $\mathcal{U}$-zeros of P are only at the origin, and P has no finite positive zeros and no zeros at infinity, then $z=0$ and $n_\infty=0$; the term $(1-s/z)/(s+a)$ in (3) is replaced with $(s-z)/(s+a)=s/(s+a)$ and hence, the expression for λ_z in (3) becomes

$$\lambda_z=\frac{s-z}{(s+a)}s^{n_o}\prod_{j=1}^{n_o}\frac{1}{(s+b_j)} \tag{7}$$

where, with $z=0$ and $n_\infty=0$, the number of zeros of λ_z at $s=0$ is $n_o+1=n$. In this case, $Y=D(0)^{-1}D(0)=I$. $\square$

Some examples of plants in the classes being considered are as follows: For the non-square plant $P_1=\begin{bmatrix}(1/(s-p) & 0 & (s+1)/(s-p)\\ 0 & (s-z)/(s+1)^2 & (s-z)/(s-1)\end{bmatrix}$, let $p,z\in\mathbb{R}_+$; P_1 has zeros at $z\in\mathbb{R}_+$ and at $s=\infty$, with $n_\infty=1$. The square plant, as shown in the equation at the bottom of the page, has $\mathcal{U}$-zeros at $z=2$ and at $s=\infty$, with $n_\infty=1$ and $n_o=0$, i.e., $n=2$. The plant $P_3=\begin{bmatrix}(s+2)(s-3)/(s+5)(s-4) & (s+2)(4s-1)/(s+5)(s-4)\\ 3(s+2)/(s+1)(s-3) & (s+2)/(s+1)(s-3)\end{bmatrix}$ has $\mathcal{U}$-zeros at $s=0$ and at $s=\infty$; since it has no finite non-zero zero, we would consider $z=\infty$ as in (6) and hence, $n_\infty=0$, $n_o=1$, i.e., $n=2$. Let $P_4=\begin{bmatrix}P_2 & G\\ 0 & P_3\end{bmatrix}$, where G can be any stable 2×2 matrix; P_4 has $\mathcal{U}$-zeros at $z=2$, $s=\infty$ and $s=0$, with $n_\infty=1$, $n_o=1$, i.e., $n=3$. Let $P_5=(s/y(s))P_2$, where $y(s)$ is any polynomial and r is the relative degree of $s/y(s)$; P_5 has $\mathcal{U}$-zeros at $z=2$, $s=\infty$ and $s=0$, with $n_\infty=r+1$, $n_o=1$, $n=r+3$. On the other hand, $P_6=(s/y(s))P_3$ has $\mathcal{U}$-zeros at $s=\infty$ and $s=0$, with $n_\infty=r$, $n_o=2$, $n=r+3$. The plants P_5 and P_6 have blocking-zeros at $s=\infty$ and $s=0$, whereas all $\mathcal{U}$-zeros in P_1,P_2,P_3,P_4 are transmission-zeros.

In Section III we propose strongly stabilizing controllers for the plant class described with λ_z as in (3) when the $\mathcal{U}$-zeros are at $s=z,0,\infty$, or as in (6) and (7) when the $\mathcal{U}$-zeros are all at $s=\infty$ and $s=0$.

III. Strongly Stabilizing Controllers

Case (i): It is obvious that plants that have no transmission-zeros in the unstable region (the extended closed right-half plane $\mathcal{U}$) as in Case (i) are strongly stabilizable. Let P^R denote a stable right-inverse of $P=D^{-1}N\in\mathbf{R}_{\mathbf{p}}^{m\times\mu}$, with $m\le\mu$, and let N^R denote a stable right-inverse of N; P^R and N^R are stable because P has no $\mathcal{U}$-zeros. All stable controllers C that stabilize such plants are trivially obtained

$$C=P^R(D^{-1}E-I)=N^R(E-D) \tag{8}$$

for any unimodular $E\in\mathbf{S}^{m\times m}$ (an RCF of C in (8) is $N_c=C=N^R(E-D)$, $D_c=I$). If the plant is square, then an LCF is given by

$$P_2=\begin{bmatrix}(2s^2+26s+100)/(s-5)(s+6) & (-s^3+s^2+83s+12)/(s+3)(s-5)(s+6)\\ (s-2)(s+3)/(s+4)(s^2+9) & (s-2)/(s^2+9)\end{bmatrix}$$

$P = D^{-1}N = (P^{-1})^{-1}I$, and (8) becomes $C = E - P^{-1}$. If $m > \mu$, let P^L, $\tilde{N}^L$ denote a stable left-inverse of $P = \tilde{N}\tilde{D}^{-1} \in \mathbf{R}_{\mathbf{p}}^{m\times\mu}$, and a stable left-inverse of $\tilde{N}$. All stable controllers that stabilize such plants are $C = (\tilde{E}\tilde{D}^{-1} - I)P^L = (\tilde{E} - \tilde{D})\tilde{N}^L$, for any unimodular $\tilde{E} \in \mathbf{S}^{\mu\times\mu}$.

Since the challenge is to find stable controllers when the plant has zeros in $\mathcal{U}$, we focus on cases (ii)-(iii)-(iv).

Cases (ii)-(iii)-(iv): Theorem 1 gives a systematic strongly stabilizing controller design method for the plant class described in Section II. The plants $P \in \mathbf{R}_{\mathbf{p}}^{m\times\mu}$ under consideration are strongly stabilizable, and have at most one finite non-zero $\mathcal{U}$-zero $z \in \mathcal{U}$, described with λ_z as in (3), (6) or (7). It is also assumed that P has no poles coinciding with its transmission-zeros in $\mathcal{U}$. Hence, if P has a zero at $z \in \mathcal{U}$, then $D(z)$ is non-singular; if it has a zero at $s = 0$, then $D(0)$ is non-singular. Theorem 1 shows that these plants are strongly stabilizable under the sufficient condition that W and Y in (4), (5) have eigenvalues with positive real parts. These conditions are equivalent to PIP for single-output plants as shown in Section II and hence, they are *necessary* and sufficient for strong stabilization of this class of single-output plants that have a finite non-zero zero and zeros at $s = \infty$ or $s = 0$.

Theorem 1 (Strongly stabilizing controller synthesis): $P = D^{-1}N \in \mathbf{R}_{\mathbf{p}}^{m\times\mu}$, $m \le \mu$, be described with λ_z as in (3), (6) or (7). If $n_\infty \ne 0$, then assume that all eigenvalues of W have positive real parts. Define

$$\Gamma_\ell := \prod_{i=\ell}^{n_\infty} \frac{\alpha_i}{s+\alpha_i} \tag{9}$$

for $\ell = 2,\ldots,n_\infty$; if $\ell > n_\infty$, then $\Gamma_\ell = 1$. Choose $\alpha_1 \in \mathbb{R}_+$, $\alpha_i \in \mathbb{R}_+$ for $i = 2,\ldots,n_\infty$ satisfying (10), (11), where $\Phi_i \in \mathcal{M}(\mathbf{S})$ is defined as in (12)

$$\alpha_1 > \left\| s\left(D(s)D(\infty)^{-1} - I\right)\right\| \tag{10}$$
$$\alpha_i > \left\| s\left(D(s)D(z)^{-1} - I\right)\Phi_i\right\| \tag{11}$$

$$\Phi_i := \Bigg[I - D(s)D(z)^{-1} + D(s)(sI + \alpha_1 W)D(\infty)^{-1}\frac{1}{\alpha_1} \times \prod_{\nu=2}^{i-1}\frac{s+\alpha_\nu}{\alpha_\nu}\Bigg]^{-1}. \tag{12}$$

Define F_∞ and U_{n_∞} as

$$F_\infty := \begin{cases} I, & \text{if } n_\infty = 0 \\ \alpha_1 W(sI + \alpha_1 W)^{-1}\Gamma_2(s), & \text{if } n_\infty \ne 0, \end{cases} \tag{13}$$
$$U_{n_\infty} := D(s) + (D(z) - D(s))\,F_\infty(s). \tag{14}$$

If $n_o \ne 0$, then assume that all eigenvalues of $Y := D(0)^{-1}D(z)$ have positive real parts. Define

$$\hat{\Gamma}_\ell := \prod_{j=\ell}^{n_o}\frac{1}{s+\beta_j} \tag{15}$$

for $\ell = 2,\ldots,n_o$; if $\ell > n_o$, then $\hat{\Gamma}_\ell = 1$. Choose $\beta_1 \in \mathbb{R}_+$, $\beta_j \in \mathbb{R}_+$ for $j = 2,\ldots,n_o$ satisfying (16), (17), where $\Psi_j \in \mathcal{M}(\mathbf{S})$ is defined as in (18)

$$\beta_1 < \left\| s^{-1}\left(D(s)YU_{n_\infty}^{-1}(s) - I\right)\right\|^{-1} \tag{16}$$
$$\beta_j < \left\| s^{-1}\left(D(s)D(z)^{-1} - I\right)\Psi_j(s)\right\|^{-1} \tag{17}$$

$$\Psi_j(s) := \Bigg[I - D(s)D(z)^{-1} + D(s)(sI + \beta_1 Y) \times F_\infty^{-1}(s)D^{-1}(z)\frac{1}{s^{j-1}}\prod_{\nu=2}^{j-1}(s+\beta_\nu)\Bigg]^{-1}. \tag{18}$$

Define F_o as

$$F_o(s) = \begin{cases} I, & \text{if } n_o = 0 \\ s^{n_o}(sI + \beta_1 Y)^{-1}\hat{\Gamma}_2(s), & \text{if } n_o \ne 0. \end{cases} \tag{19}$$

Then the stable controller C given by

$$C(s) = N^R(s)\,(D(z) - D(s))\,F_\infty(s)F_o(s) \tag{20}$$

strongly stabilizes P. Furthermore, with $C \in \mathcal{M}(\mathbf{S})$ as in (20), the controller in (21) also strongly stabilizes P for all $Q \in \mathbf{S}^{\mu\times m}$ satisfying (22)

$$C_q = C + Q \tag{21}$$
$$\|Q\| < \left\|(I + PC)^{-1}P\right\|^{-1}. \tag{22}$$

□

Remark 1 (Strongly stabilizing controllers for tall plants): Theorem 1 can be modified for plants with more outputs than inputs. Let $P = \tilde{N}\tilde{D}^{-1} \in \mathbf{R}_{\mathbf{p}}^{m\times\mu}$, $m > \mu$. Let the $\mu \times m$ matrix $\tilde{N}^L$ denote a left-inverse of $\tilde{N} \in \mathbf{S}^{m\times\mu}$, and write the fat matrix $\tilde{N}^L$ as in (2). Then $(\lambda_z\tilde{N}^L) \in \mathcal{M}(\mathbf{S})$, where λ_z is given as in (3), (6) or (7). If $n_\infty \ne 0$, then assume that all eigenvalues of $\tilde{W}$ have positive real parts; if $n_o \ne 0$, then assume that all eigenvalues of $\tilde{Y}$ have positive real parts

$$\tilde{W} := \tilde{D}(\infty)\tilde{D}(z)^{-1}, \quad \tilde{Y} := \tilde{D}(z)\tilde{D}(0)^{-1}. \tag{23}$$

In the special case of single-inputs plants ($\mu = 1 < m$), the eigenvalue conditions become $\tilde{W} > 0$ and $\tilde{Y} > 0$; therefore they are equivalent to the PIP and hence, these conditions are necessary and sufficient for existence of strongly stabilizing controllers for this class of plants. For $\ell = 2,\ldots,n_\infty$, define $\Gamma_\ell := \prod_{i=\ell}^{n_\infty}\tilde{\alpha}_i/(s+\tilde{\alpha}_i)$; if $\ell > n_\infty$, then $\Gamma_\ell = 1$. Choose $\tilde{\alpha}_1 \in \mathbb{R}_+$, $\tilde{\alpha}_i \in \mathbb{R}_+$ for $i = 2,\ldots,n_\infty$ satisfying (24), where $\tilde{\Phi}_i \in \mathcal{M}(\mathbf{S})$ is defined as in (25)

$$\begin{aligned}\tilde{\alpha}_1 &> \left\| s\left(\tilde{D}(\infty)^{-1}\tilde{D}(s) - I\right)\right\|, \\ \tilde{\alpha}_i &> \left\| s\tilde{\Phi}_i\left(\tilde{D}(z)^{-1}\tilde{D}(s) - I\right)\right\|\end{aligned} \tag{24}$$

$$\tilde{\Phi}_i := \Bigg[I - \tilde{D}(z)^{-1}\tilde{D}(s) + \tilde{D}(\infty)^{-1}(sI + \tilde{\alpha}_1\tilde{W}) \times \tilde{D}(s)\frac{1}{\tilde{\alpha}_1}\prod_{\nu=2}^{i-1}\frac{s+\tilde{\alpha}_\nu}{\tilde{\alpha}_\nu}\Bigg]^{-1}. \tag{25}$$

Define $\tilde{F}_\infty := I$ if $n_\infty = 0$, $\tilde{F}_\infty := (sI + \tilde{\alpha}_1\tilde{W})^{-1}\tilde{\alpha}_1\tilde{W}\Gamma_2(s)$ if $n_\infty \ne 0$; define $\tilde{U}_{n_\infty} := \tilde{D}(s) + \tilde{F}_\infty(\tilde{D}(z) - \tilde{D}(s))$. For $\ell = 2,\ldots,n_o$, define $\hat{\Gamma}_\ell := \prod_{j=\ell}^{n_o}1/(s+\tilde{\beta}_j)$; if $\ell > n_o$, then $\hat{\Gamma}_\ell = 1$. Choose $\tilde{\beta}_1 \in \mathbb{R}_+$, $\tilde{\beta}_j \in \mathbb{R}_+$ for $j = 2,\ldots,n_o$ satisfying (26), where $\tilde{\Psi}_j \in \mathcal{M}(\mathbf{S})$ is defined as in (27)

$$\begin{aligned}\tilde{\beta}_1 &< \left\| s^{-1}\left(\tilde{U}_{n_\infty}^{-1}(s)\tilde{Y}\tilde{D}(s) - I\right)\right\|^{-1}, \\ \tilde{\beta}_j &< \left\| s^{-1}\tilde{\Psi}_j(s)\left(\tilde{D}(z)^{-1}\tilde{D}(s) - I\right)\right\|^{-1}\end{aligned} \tag{26}$$

$$\tilde{\Psi}_j(s) := \Bigg[I - \tilde{D}(z)^{-1}\tilde{D}(s) + \tilde{D}(z)^{-1}\tilde{F}_\infty^{-1} \times (sI + \tilde{\beta}_1\tilde{Y})\tilde{D}(s)\frac{1}{s^{j-1}}\prod_{\nu=2}^{j-1}(s+\tilde{\beta}_\nu)\Bigg]^{-1}. \tag{27}$$

Define $\tilde{F}_o := I$ if $n_o = 0$, $\tilde{F}_o := s^{n_o}(sI + \tilde{\beta}_1 \tilde{Y})^{-1}\hat{\Gamma}_2(s)$ if $n_o \neq 0$. Then

$$C(s) = \tilde{F}_o \tilde{F}_\infty(s)\left(\tilde{D}(z) - \tilde{D}(s)\right)\tilde{N}^L(s) \tag{28}$$

is a stable controller that strongly stabilizes P. □

Remark 2 (The order of the proposed controllers): In the case of SISO plants, the order of the controller C in (20) is one less than the plant's order. Although coprime factorizations are unique only up to a unit in **S**, it can be assumed that the chosen numerator in the factorization $P = D^{-1}N$ is in the form of the largest invariant-factor λ_z in (3), (6) or (7). For discussing the order, we write the numerator and denominator factors of the plant in polynomial form as

$$P = \frac{(1 - s/z)s^{n_o}\eta}{d} \tag{29}$$

where η is an $\tilde{n}$-th order polynomial whose roots are the zeros of the plant in the stable region $\mathbb{C} \setminus \mathcal{U}$, and d is a polynomial of degree $\delta = n_\infty + n_o + \tilde{n} + 1$. Then a coprime factorization $P = D^{-1}N$ over **S** is given by

$$\begin{aligned} D &= \frac{d}{(1 - s/z)s^{n_o}\eta}\lambda_z \\ &= \frac{d}{\eta(s+a)\prod_{i=1}^{n_\infty}(s+a_i)\prod_{j=1}^{n_o}(s+b_j)} \end{aligned} \tag{30}$$

$$N = \lambda_z = \frac{(1 - s/z)s^{n_o}}{(s+a)\prod_{i=1}^{n_\infty}(s+a_i)\prod_{j=1}^{n_o}(s+b_j)}. \tag{31}$$

Using D, N in (30), (31), the controller C in (20) becomes

$$\begin{aligned} &C = \lambda_z^{-1}(D(z) - D)F_\infty F_o = \\ &[D(z)\eta(s+a)\prod_{i=1}^{n_\infty}(s+a_i)\prod_{j=1}^{n_o}(s+b_j) - d]\alpha_1 W s^{n_o}/ \\ &(1 - s/z)s^{n_o}\eta(s + \alpha_1 W)(s + \beta_1 Y)\Gamma_2^{-1}\hat{\Gamma}_2^{-1} \end{aligned}$$

where the numerator of $(D(z) - D)$ has a zero at $s = z$ and hence, cancels the term $(1 - s/z)$ from the denominator of C. The polynomial terms $\eta(s + \alpha_1 W)(s + \beta_1 Y)\Gamma_2^{-1}\hat{\Gamma}_2^{-1}$ that remain in the denominator after cancelations have order $\tilde{n} + n_\infty + n_o$, where the degree of Γ_2^{-1} is $n_\infty - 1$ and the degree of $\hat{\Gamma}_2^{-1} = n_o - 1$. Therefore, the order of the controller C is $\tilde{n} + n_\infty + n_o = \delta - 1$, where δ is the order of the plant.

We showed that the controller order is one less than the plant order for the case where the plant has at least one non-zero zero on the extended non-negative real-axis so that λ_z is as in (3). Using entirely similar steps, it can be concluded that the controller order is again one less than the plant order when λ_z is given by (6) or (7). □

Remark 3 (Robustness of the proposed strongly stabilizing controllers): Under the assumptions of Theorem 1, let the stable controller C be given by (20), and $C_q = C + Q$ for any $Q \in \mathbf{S}^{\mu \times m}$ satisfying (22). Standard robustness arguments lead to the following conclusions [15], [19]: **a)** (*Additive perturbations*): The controller C_q strongly stabilizes $P + \Delta$ for all $\Delta \in \mathbf{S}^{m\times\mu}$ satisfying $\|\Delta\| < \|C_q M_q^{-1} D\|^{-1}$. **b)** (*Coprime factor perturbations*): Let $\Delta_N \in \mathbf{S}^{m\times\mu}$, $\Delta_D \in \mathbf{S}^{m\times m}$ be such that $\|[\Delta_D\ \Delta_N]\| < \|\left[\begin{smallmatrix} C_q \\ I \end{smallmatrix}\right] M_q^{-1}\|^{-1}$,where $M_q = D + NC_q$ is unimodular by design for all $C_q = C + Q$. Then the controller C_q strongly stabilizes all plants in the form $(D + \Delta_D)^{-1}(N + \Delta_N)$. Once C is fixed, Q can be optimized to maximize the allowable perturbation magnitude. □

IV. Examples

We consider two MIMO examples to illustrate the strongly stabilizing controller design approach using the synthesis procedure of Theorem 1. The plant models in Examples 1–2 are obtained from process control applications. The only objective considered in these designs is strong stabilization. Other performance objectives (e.g. robustness optimization as shown above) may be possible to achieve by choosing Q and other free parameters in Theorem 1 accordingly. An SISO example demonstrates that the controller order is one less than that of the plant following the procedure of Theorem 1 with a coprime factorization $P = D^{-1}N = D^{-1}\lambda_z$.

Example 1: The unstable plant in this example is obtained from a linearized model of a sugar mill process [7]. Let the plant's transfer-matrix be $P = \begin{bmatrix} -5/(25s+1) & (s^2 - 0.005(s+1)/s(s+1)) \\ 1/(25s+1) & -0.0023/s \end{bmatrix}$, where the $\mathcal{U}$-zeros of P are at $z = 0.137$ and infinity; P has also a zero $s = -0.1205 \notin \mathcal{U}$. For any $e \in \mathbb{R}_+$, an LCF is $P = D^{-1}N = \begin{bmatrix} 1 & -50/23 \\ 0 & s/(s+e) \end{bmatrix}^{-1} \begin{bmatrix} -165/23(25s+1) & s/(s+1) \\ s/(25s+1)(s+e) & -0.0023/(s+e) \end{bmatrix}$. The choice of $e \in \mathbb{R}_+$ effects the closed-loop pole locations. With $z = 0.137 \in \mathcal{U}$, and any $a, a_1 \in \mathbb{R}_+$, the largest invariant factor $\lambda_z = (1 - s/0.137)/(s+a)(s+a_1)$ of N is in the form of (3), with $n_\infty = 1$, $n_o = 0$. Since $n_\infty = 1 \neq 0$, we check that the eigenvalues of $W = D(z)^{-1}D(\infty) = \begin{bmatrix} 1 & 50e/23z \\ 0 & 1 + e/z \end{bmatrix}$ are both positive (for any $e \in \mathbb{R}_+$), and hence, the assumptions of Theorem 1 hold for this plant. Choose α_1 satisfying (10) as $\alpha_1 > \|s(DD(\infty)^{-1} - I)\| = \|\begin{bmatrix} 0 & 0 \\ 0 & -es/(s+e) \end{bmatrix}\| = e$. With $F_\infty = \alpha_1 W(sI + \alpha_1 W)^{-1}$ and $F_o = I$, for $\alpha_1 > e$, the controller in (20) is $C = N^{-1}(D(z) - D)F_\infty F_o = \begin{bmatrix} 0 & -e\alpha_1 s(25s+1)/(s+0.1205)(zs + e\alpha_1 + z\alpha_1) \\ 0 & -165e\alpha_1(s+1)/23(s+0.1205)(zs + e\alpha_1 + z\alpha_1) \end{bmatrix}$. The stable controller $C_q = C + Q$ also stabilizes the given plant for any $Q \in \mathbf{S}^{2\times 2}$ satisfying $\|Q\| < 0.1394$ as in (22). For example, we can choose $Q = \begin{bmatrix} (25s+1)/250(s+10) & 0 \\ 0 & 0 \end{bmatrix}$, where $\|Q\| = 0.1$, and we can choose $e = 10$, $\alpha_1 = 20 > e$. Then with the controller, as shown at the bottom of the page, the closed-loop poles are at $\{-8.9034, -10.5670 \pm j10.6591\}$. □

Example 2: In this example we consider a chemical reactor plant obtained by linearizing the model given in [6], where the concentration of the inlet reactant and the rate of heat input are manipulated to regulate the outlet reactant concentration and the reactor temperature. The linearization around one of the operating points gives the unstable plant transfer-matrix $P = 1/100d \begin{bmatrix} (1.67s - 0.1232) & -0.00189 \\ 4.143 & (4.184s + 0.1218) \end{bmatrix}$, with $d = (s - 0.0614)(s + 0.167)$, where P has poles at $s = 0.0614 \in \mathcal{U}$ and $s = -0.0167$, and a zero at infinity. For any second-order monic Hurwitz polynomial w, an LCF is given as $P = D^{-1}N = \begin{bmatrix} 1 & 0.005 \\ 0 & (d/w) \end{bmatrix}^{-1} \begin{bmatrix} 1.67/(100(s+0.167)) & 0.02092/(100(s+0.167)) \\ 4.143/100w & (4.184s + 0.1218)/100w \end{bmatrix}$. Since P has no finite $\mathcal{U}$-zeros, we set $z = \infty$; for any $a, a_1 \in \mathbb{R}_+$, the largest invariant factor $\lambda_z = (s+a)^{-1}(s+a_1)^{-1}$ of N is in the form of (3), with $n_\infty = 1$, $n_o = 0$; the number of zeros of λ_z at infinity is $n = 1 + n_\infty = 2$. In this case, $W = D(\infty)D(\infty)^{-1} = I$ obviously satisfies the positive eigenvalue assumption of Theorem 1.

$$C_q = C + Q = \begin{bmatrix} (25s+1)/250(s+10) & -200s(25s+1)/(s+0.1205)(0.137s+202.74) \\ 0 & -33000(s+1)/23(s+0.1205)(0.137s+202.74) \end{bmatrix}$$

With $F_\infty = \alpha_1(s+\alpha_1)^{-1}I$ and $F_o = I$, the controller in (20) is $C = N^{-1}(D(z)-D)F_\infty F_o = \alpha_1(s+\alpha_1)^{-1}N^{-1}(D(\infty)-D) = ((\alpha_1(w-d))/(6.9873(s+0.0167)(s+\alpha_1))\begin{bmatrix}0 & -2.092\\ 0 & 167\end{bmatrix}$, where α_1 satisfies (10). Suppose we choose $w = (s+2)(s+4)$; then (10) is satisfied for $\alpha_1 > 5.8944$. If we pick $\alpha_1 = 8$, then $C_q = C + Q$ is also a strongly stabilizing controller for any $Q \in \mathbf{S}^{2\times 2}$ satisfying $\|Q\| < 0.9975$. For example, choose $Q = \begin{bmatrix}(0.9(s+0.167))/(s+6) & 0\\ 0 & 0\end{bmatrix}$, where $\|Q\| = 0.9$. Then with the controller, shown as the equation at the bottom of the page, the closed-loop poles are $\{-1.7321, -6.0150, -3.1868 \pm j5.1764\}$. $\square$

A. Example 3

Consider the unstable plant, shown as the second equation at the bottom of the page, which has zeros at the origin with $n_o = 2$, at infinity with $n_\infty = \ell + 1$ and at $z = 1.6$. We consider two different cases for $\ell = 1$ and $\ell = 2$; in both cases the PIP is satisfied with two poles, or four poles, between 0 and 1.6 and no poles between 1.6 and $+\infty$. A coprime factorization is $P = D^{-1}N$ where, as shown in the third equation at the bottom of the page, and as shown in the last equation at the bottom of the page, $a > 0$. The eigenvalues of $W = D(z)^{-1}D(\infty)$ are positive for both $\ell = 1, \ell = 2$. The eigenvalues of Y are negative when ℓ is odd, positive when ℓ is even. For even values of ℓ, the procedure of Theorem 1 can be applied to find a strongly stabilizing controller. $\square$

Example 4: Consider $P = s(s-16)/(s-16+\epsilon)(s-1)(s+4)$, $\epsilon > 0$; this SISO unstable plant has a finite non-zero zero $z = 16 \in \mathcal{U}$, one zero at infinity, one at $s = 0$, i.e., $n_\infty = 1$, $n_o = 1$. The order of the given plant is $\delta = 3$. As ϵ approaches zero, the plant pole at $p_1 = (16-\epsilon)$ approaches the zero at $z = 16$ and the plant gets closer to violating the PIP; if the pole and zero cancel ($\epsilon = 0$), then there would be a single plant pole at $p_2 = 1$ between the zeros at $s = 0$ and infinity, Choosing $a = 4$, $a_1 = b_1 = 30$ in (30), (31), a coprime factorization is $P = D^{-1}N = (-(s-1)(s-16+\epsilon)/16(s+30)^2)^{-1}(s(1-s/16)/(s+4)(s+30)^2)$.Let $\epsilon = 1$. Following Theorem 1, verify $W = 141.0667 > 0, Y = 0.4253 > 0$. With $\Gamma_2 = 1$ since $n_\infty = 1$, (10) is satisfied for $\alpha_1 > 76$; we choose $\alpha_1 = 80$ and find U_1 using (14). With $\hat{\Gamma}_2 = 1$, (16) is satisfied for $\beta_1 < 0.5755$; we choose $\beta_1 = 0.5$. With $F_\infty = \alpha_1 W/s + \alpha_1 W$ and $F_o = s/s + \beta_1 Y$, the strongly stabilizing controller in (20) becomes $C = -11205(s-0.5426)(s+4)/(s^2+11286s+2400)$. The order of this stable controller is $\delta - 1 = 2$. The closed-loop poles are $\{-4, -1.6403, -3.1725, -29.7629 \pm j77.660\}$. Now repeat the design for $\epsilon = 0.1$. With $W = 1410.7 > 0$, $Y = 0.0401 > 0$, choose $\alpha_1 = 80 > 76.9$ satisfying (10), $\beta_1 = 0.58 < 0.5989$ satisfying (16). The strongly stabilizing controller in (20) is $C = -112770(s-0.9546)(s+4)/(s^2+112850s+2630)$. The order of this stable controller is again $\delta - 1 = 2$. The closed-loop poles are $\{-4, -1.8769 \pm j1.4986, -29.1834 \pm j79.9238\}$. As ϵ gets smaller, the positive real-axis zero of the controller gets closer to the plant pole at $p_2 = 1$. $\square$

V. Conclusion

We proposed a simple strongly stabilizing controller synthesis method for a class of unstable MIMO plants satisfying the PIP, with at most one positive real zero and any number of zeros at $s = 0$, at infinity, and in the open left-half complex plane. No restrictions were imposed on the number and locations of the poles. We explicitly constructed robust strongly stabilizing controllers for all plants in this class. The design offers freedom in the design parameters that may be used for other performance criteria. In the special case of SISO plants, the order of the (nominal) strongly stabilizing controller obtained using the proposed design procedure here is one less than the order of the plant.

Appendix
Proof of Theorem 1

Define $H_\alpha := (sI + \alpha_1 W)$ and $H_\beta := (sI + \beta_1 Y)$; by assumption, $H_\alpha^{-1} \in \mathcal{M}(\mathbf{S})$ and $H_\beta^{-1} \in \mathcal{M}(\mathbf{S})$. We first show that the controller proposed in (20) is stable. The largest invariant-factor $\lambda_z \in \mathbf{S}$ is as in either (3), (6) or (7). Let $\lambda_z N^R =: N_s$; then $N_s \in \mathcal{M}(\mathbf{S})$ and $N^R = \lambda_z^{-1}N_s$. We first show that $C = N^R(D(z)-D)F_\infty F_o = N_s\lambda_z^{-1}(D(z)-D)F_\infty F_o = N_s\lambda_z^{-1}(D(z)-D)\alpha_1 W H_\alpha^{-1}\Gamma_2 s^{n_o}(sI+\beta_1 Y)^{-1}\hat{\Gamma}_2 \in \mathcal{M}(\mathbf{S})$. Since $(D(z)-D(s))|_{s=z} = 0$, the term $((s+a)/(1-s/z))(D(z)-D(s))$ is stable. If $n_\infty \neq 0$, then λ_z^{-1} contains the term $\prod_{i=1}^{n_\infty}(s+a_i) \notin \mathbf{S}$, but $(\prod_{i=1}^{n_\infty}(s+a_i))(\alpha_1 W H_\alpha^{-1}\Gamma_2) \in \mathbf{S}$. If $n_o \neq 0$, then λ_z^{-1} contains the term $s^{-n_o}\prod_{j=1}^{n_o}(s+b_j) \notin \mathbf{S}$, but $(s^{-n_o}\prod_{j=1}^{n_o}(s+b_j))(s^{n_o}H_\beta^{-1}\hat{\Gamma}_2) \in \mathbf{S}$. If P has no finite positive zero, but it has transmission-zeros at infinity, we take $z = \infty$ as in (6); hence, $W = I$. If all $\mathcal{U}$-zeros of P are at $s = 0$, then $(1-s/z)/(s+a)$ in λ_z is replaced with $s/(s+a)$ as in (7). In this case, $F_\infty = I, Y = I, ((s+a)/s)(D(0)-D(s)) \in \mathcal{M}(\mathbf{S})$, $(s^{-n_o}\prod_{j=1}^{n_o}(s+b_j))(s^{n_o}(sI+\beta_1 Y)^{-1}\hat{\Gamma}_2) \in \mathbf{S}$. Therefore, the controller in (20) is stable in all cases. It remains to show that C stabilizes P: *Step 1*: Let $N_c = C$ and $D_c = I$; by (1), $C = N_c D_c^{-1}$ stabilizes $P = D^{-1}N$ if and only if $M = D + NC$ is unimodular. With $U_o := D(z)$, write $M = D + NN^R(D(z)-D)F_\infty F_o = U_o F_\infty F_o +$

$$C_q = C + Q = \begin{bmatrix}(0.9(s+0.167))/(s+6) & (-14.1183(s+1.3590)))/((s+0.0167)(s+8))\\ 0 & (1127(s+1.3590))/((s+0.0167)(s+8))\end{bmatrix}$$

$$P = \begin{bmatrix}s/(s^2-3s+2.5)(s-1.5)^\ell & s(s-1.6)/(s+1)(s^2-3s+2.5)(s-1.5)^\ell\\ s/(s+1)(s^2-3s+2.5)(s-1)^\ell & s(s-1.6)/(s^2-3s+2.5)(s-1)^\ell\end{bmatrix}$$

$$D = \begin{bmatrix}(s^2-3s+2.5)(s-1.5)^\ell/(s+a)^{\ell+2} & 0\\ 0 & (s^2-3s+2.5)(s-1)^\ell/(s+a)^{\ell+2}\end{bmatrix}$$

$$N = \begin{bmatrix}s/(s+a)^{\ell+2} & s(s-1.6)/(s+1)(s+a)^{\ell+2}\\ s/(s+1)(s+a)^{\ell+2} & s(s-1.6)/(s+a)^{\ell+2}\end{bmatrix}$$

$D(I - F_\infty F_o)$. If $n_\infty = 0$, then $F_\infty = I$; go to step 2. If $n_\infty > 0$, define $U_1 := ((\alpha_1/(s+\alpha_1))D(\infty) + (s/(s+\alpha_1))D)(s+\alpha_1)H_\alpha^{-1} = [I + (1/(s+\alpha_1))s(DD(\infty)^{-1} - I)]D(\infty)(s+\alpha_1)H_\alpha^{-1}$, where $D(\infty)(s+\alpha_1)H_\alpha^{-1} \in \mathcal{M}(\mathbf{S})$ is unimodular since $\alpha_1 > 0$, and $s(DD(\infty)^{-1} - I) \in \mathcal{M}(\mathbf{S})$ since $(DD(\infty)^{-1} - I)$ is strictly-proper. For α_1 satisfying (10), $\|(1/(s+\alpha_1))s(DD(\infty)^{-1}-I)\| \le \|(1/(s+\alpha_1))\|\,\|s(DD(\infty)^{-1}-I)\| = (1/\alpha_1)\|s(DD(\infty)^{-1}-I)\| < 1$ implies that U_1 is unimodular (see, e.g., [15]). Write $M = D + (D(z) - D)\alpha_1 W H_\alpha^{-1}\Gamma_2 F_o = (\alpha_1 D(z) W + sD)H_\alpha^{-1}\Gamma_2 F_o + D(I - \Gamma_2 F_o) = U_1\Gamma_2 F_o + D(I - \Gamma_2 F_o)$. If $n_\infty = 1$, then $\Gamma_2 = 1$; go to step 2. If $n_\infty > 1$, write $M = U_1(\alpha_2/(s+\alpha_2))\Gamma_3 F_o + D(I - (\alpha_2/(s+\alpha_2))\Gamma_3 F_o) = U_2\Gamma_3 F_o + D(I - \Gamma_3 F_o)$, where $U_2 := (\alpha_2/(s+\alpha_2))U_1 + (s/(s+\alpha_2))D = [I + (1/(s+\alpha_2))s(DU_1^{-1} - I)]U_1 = D + (D(\infty) - DW) H_\alpha^{-1}\alpha_1(\alpha_2/(s+\alpha_2))$. For $i = 2$, $\prod_{\nu=2}^{i-1}((s+\alpha_\nu)/\alpha_\nu) = 1$ in (12). Then $\Phi_2 = (I - DD(z)^{-1} + D(1/\alpha_1)H_\alpha D(\infty)^{-1})^{-1} = (I + (s/\alpha_1)DD(\infty)^{-1})^{-1} = \alpha_1 D(\infty)H_\alpha^{-1}U_1^{-1} \in \mathcal{M}(\mathbf{S})$ since U_1 is unimodular, and (11) becomes $\alpha_2 > \|s(DD(z)^{-1} - I)\Phi_2\| = \|s(DD(z)^{-1} - I)\,(I + (s/\alpha_1)DD(\infty)^{-1})^{-1}\| = \|s(DU_1^{-1} - I)\|$, where $U_1(\infty) = D(\infty)$ implies $s(DU_1^{-1} - I) \in \mathcal{M}(\mathbf{S})$. Therefore, U_2 is unimodular for α_2 satisfying (11). If $n_\infty = 2$, then $\Gamma_3 = 1$; go to step 2. If $n_\infty > 2$, write $M = U_2(\alpha_3/(s+\alpha_3))\Gamma_4 F_o + D(I - (\alpha_3/(s+\alpha_3))\Gamma_4 F_o) = U_3\Gamma_4 F_o + D(I - \Gamma_4 F_o)$, where $U_3 := (\alpha_3/(s+\alpha_3))U_2 + (s/(s+\alpha_3))D = [I + (1/(s+\alpha_3))s(DU_2^{-1} - I)]U_2 = D + (D(\infty) - DW) H_\alpha^{-1}\alpha_1\prod_{i=2}^{3}(\alpha_i/(s+\alpha_i))$. For $i = 3$, by (12), $\Phi_3 = (I - DD(z)^{-1} + D(1/\alpha_1)H_\alpha D(\infty)^{-1}((s+\alpha_2)/\alpha_2))^{-1} = (\alpha_1\alpha_2/(s+\alpha_2))D(\infty)H_\alpha^{-1}U_2^{-1} \in \mathcal{M}(\mathbf{S})$ since U_2 is unimodular, and (11) becomes $\alpha_3 > \|s(DD(z)^{-1} - I)\Phi_3\| = \|s(DU_2^{-1} - I)\|$. Therefore, U_3 is unimodular for α_3 satisfying (11). If $n_\infty = 3$, then $\Gamma_4 = 1$; go to step 2. If $n_\infty > 3$, then continue similarly with $U_k = D + (D(\infty) - DW)H_\alpha^{-1}\alpha_1\prod_{i=2}^{k}(\alpha_i/(s+\alpha_i))$. Write $M = U_k((\alpha_{k+1})/(s+\alpha_{k+1}))\Gamma_{k+2}F_o + D(I - ((\alpha_{k+1})/(s+\alpha_{k+1}))\Gamma_{k+2}F_o) = U_{k+1}\Gamma_{k+2}F_o + D(I - \Gamma_{k+2}F_o)$, where $U_{k+1} := ((\alpha_{k+1})/(s+\alpha_{k+1}))U_k + (s/(s+\alpha_{k+1}))D = [I + (1/(s+\alpha_{k+1}))s(DU_k^{-1} - I)]U_k$. For $i = k+1$, by (12), $\Phi_{k+1} = (I - DD(z)^{-1} + DH_\alpha D(\infty)^{-1}(1/\alpha_1)\prod_{\nu=2}^{k}((s+\alpha_\nu)/\alpha_\nu))^{-1} = \alpha_1 D(\infty)H_\alpha^{-1}U_k^{-1}\prod_{i=2}^{k}(\alpha_i/(s+\alpha_i)) \in \mathcal{M}(\mathbf{S})$ since U_k is unimodular, and (11) becomes $\alpha_{k+1} > \|s(DD(z)^{-1} - I)\Phi_{k+1}\| = \|s(DU_k^{-1} - I)\|$. For α_{k+1} satisfying (11), $\|(1/(s+\alpha_{k+1}))s(DU_k^{-1} - I)\| \le \|(1/(s+\alpha_{k+1}))\|\,\|s(DU_k^{-1} - I)\| = (1/(\alpha_{k+1}))\|s(DU_k^{-1} - I)\| < 1$ implies U_{k+1} is unimodular. If $n_\infty = k+1$, then $\Gamma_{k+2} = 1$ and $M = U_{n_\infty}F_o + D(I - F_o)$, where U_{n_∞} is unimodular; go to step 2. *Step 2*: If $n_o = 0$, then $F_o = I$; go to step 3. If $n_o > 0$, define $V_1 := ((s/(s+\beta_1))U_{n_\infty} + (\beta_1/(s+\beta_1))DY)\,(s+\beta_1)H_\beta^{-1} = [I + (\beta_1\, s/(s+\beta_1))s^{-1}(DYU_{n_\infty}^{-1} - I)]U_{n_\infty}(s+\beta_1)H_\beta^{-1} = D + s(D(z) - D)F_\infty H_\beta^{-1}$, where the unimodular matrix $U_{n_\infty} = D(z)$ if $n_\infty = 0$ and U_{n_∞} is given by (14) if $n_\infty \ne 0$; $U_{n_\infty}(s+\beta_1)H_\beta^{-1}$ is unimodular since $\beta_1 > 0$, and $s^{-1}(DYU_{n_\infty}^{-1} - I) \in \mathcal{M}(\mathbf{S})$ since $[DYU_{n_\infty}^{-1} - I]$ is zero at $s = 0$. For β_1 satisfying (16), $\|(\beta_1\, s/(s+\beta_1))s^{-1}(DYU_{n_\infty}^{-1} - I)\| \le \|(\beta_1\, s/(s+\beta_1))\|\|s^{-1}(DYU_{n_\infty}^{-1} - I)\| = \beta_1\,\|s^{-1}(DYU_{n_\infty}^{-1} - I)\| < 1$ implies that V_1 is unimodular. Write $M = U_{n_\infty}s^{n_o}H_\beta^{-1}\hat\Gamma_2 + D(I - s^{n_o}H_\beta^{-1}\hat\Gamma_2) = s^{n_o-1}V_1\hat\Gamma_2 + D(1 - s^{n_o-1}\hat\Gamma_2)$. If $n_o = 1$ then $\hat\Gamma_2 = 1$, $M = V_1$; go to step 3. If $n_o > 1$, write $M = s^{n_o-2}V_1(s/(s+\beta_2))\hat\Gamma_3 + D\,(1 - s^{n_o-2}(s/(s+\beta_2))\hat\Gamma_3) = s^{n_o-2}V_2\hat\Gamma_3 + D\,(1 - s^{n_o-2}\hat\Gamma_3)$, where $V_2 = (s/(s+\beta_2))V_1 + (\beta_2/(s+\beta_2))D = [I + (\beta_2\, s/(s+\beta_2))(s^{-1}[DV_1^{-1} - I])]V_1 = D + (U_{n_\infty} - D)sH_\beta^{-1}(s/(s+\beta_2)) = D + (D(z) - D)F_\infty sH_\beta^{-1}(s/(s+\beta_2))$. For $j = 2$, $\prod_{\nu=2}^{j-1}(s+\beta_\nu) = 1$ in (18). Then $\Psi_2 = (I - DD(z)^{-1} + D(1/s)H_\beta F_\infty^{-1}(s)D(z)^{-1})^{-1} = sD(z)F_\infty H_\beta^{-1}V_1^{-1} \in \mathcal{M}(\mathbf{S})$ since V_1 is unimodular, and (17) becomes $\beta_2 < \|s^{-1}(DD(z)^{-1} - I)\Psi_2\|^{-1} = \|s^{-1}(DV_1^{-1} - I)\|^{-1}$. Therefore, V_2 is unimodular for β_2 satisfying (17). If $n_o = 2$, then $\hat\Gamma_3 = 1$, $M = V_2$; go to step 3. If $n_o > 2$, then continue similarly with $V_k = D + (U_{n_\infty} - D)sH_\beta^{-1}\prod_{j=2}^{k}(s/(s+\beta_j)) = D + (D(z) - D)F_\infty sH_\beta^{-1}\prod_{j=2}^{k}(s/(s+\beta_j))$. Write $M = s^{n_o-k-1}V_k(s/(s+\beta_{k+1}))\hat\Gamma_{k+2} + D(1 - s^{n_o-k-1}(s/(s+\beta_{k+1}))\hat\Gamma_{k+2}) = s^{n_o-k-1}V_{k+1}\hat\Gamma_{k+2} + D(1 - s^{n_o-k-1}\hat\Gamma_{k+2})$, where $V_{k+1} = (s/(s+\beta_{k+1}))V_k + ((\beta_{k+1})/(s+\beta_{k+1}))D = [I + ((\beta_{k+1}s)/(s+\beta_{k+1}))(s^{-1}[DV_k^{-1} - I])]V_k$. For $j = k+1$, by (18), $\Psi_{k+1} = (I - DD(z)^{-1} + D(1/s^k)H_\beta F_\infty^{-1}(s)D^{-1}(z)\prod_{\nu=2}^{k}(s+\beta_\nu))^{-1} = sD(z)F_\infty H_\beta^{-1}V_k^{-1}\prod_{j=2}^{k}(s/(s+\beta_j)) \in \mathcal{M}(\mathbf{S})$ since V_k is unimodular, and (17) becomes $\beta_{k+1} < \|s^{-1}(DD(z)^{-1} - I)\Psi_{k+1}\|^{-1} = \|s^{-1}(DV_k^{-1} - I)\|^{-1}$. For β_{k+1} satisfying (17), $\|((\beta_{k+1}s)/(s+\beta_{k+1}))(s^{-1}[DV_k^{-1} - I])\| \le \|((\beta_{k+1}s)/(s+\beta_{k+1}))\|\|(s^{-1}[DV_k^{-1} - I])\| = \beta_{k+1}\|(s^{-1}[DV_k^{-1} - I])\| < 1$ implies that V_{k+1} is unimodular. If $n_o = k+1$, then $\hat\Gamma_{k+2} = 1$ and $M = V_{n_o}$ is unimodular; go to step 3. *Step 3*: If $n_o = 0$, then $M = U_{n_\infty}$ is unimodular; $U_{n_\infty} = U_o = D(z)$ if $n_\infty = 0$ and U_{n_∞} is as in (14) if $n_\infty \ne 0$. If $n_o > 0$, then $M = V_{n_o}$ is also unimodular. Since $M = D + NC$ is unimodular, the controller C in (20) stabilizes $P = D^{-1}N$. For $Q \in \mathcal{M}(\mathbf{S})$ satisfying (22), by standard "small-gain" argument, $\|M^{-1}NQ\| = \|(D + NC)^{-1}NQ\| = \|(I + PC)^{-1}PQ\| < 1$ implies $I + M^{-1}NQ$ is unimodular. Therefore, $M_q := D + NC_q = (D + NC) + NQ = M + NQ = M(I + M^{-1}NQ)$ is also unimodular, and hence, $C_q \in \mathcal{M}(\mathbf{S})$ also stabilizes P.

REFERENCES

[1] A. E. Barabanov, "Design of H_∞ optimal stable controller," in *Proc. IEEE Conf. Decision Control*, 1996, pp. 734–738.

[2] D. U. Campos-Delgado and K. Zhou, "A parametric optimization approach to $\mathcal{H}^\infty$ and $\mathcal{H}^2$ strong stabilization," *Automatica*, vol. 39, no. 7, pp. 1205–1211, 2003.

[3] P. Cheng, Y.-Y. Cao, and Y. Sun, "On strong $\gamma_k - \gamma_{cl}$ H_∞ stabilization and simultaneous $\gamma_k - \gamma_{cl}$ H_∞ control," in *Proc. 46th IEEE Conf. Decision Control*, 2007, pp. 5417–5422.

[4] Y. S. Chou, J. L. Leu, and Y. C. Chu, "Stable controller design for MIMO systems: An LMI approach," *IET Control Theory Appl.*, vol. 1, pp. 817–829, 2007.

[5] P. Dorato, H. Park, and Y. Li, "An algorithm with interpolation in units in H^∞, with applications to feedback stabilization," *Automatica*, vol. 25, pp. 427–430, 1989.

[6] N. H. El-Farra, P. Mhaskar, and P. D. Christofides, "Hybrid predictive control of nonlinear systems: Method and applications to chemical processes," *Int. J. Robust Nonlin. Control*, vol. 14, pp. 199–225, 2004.

[7] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*. Englewood Cliffs, NJ: Prentice Hall, 2001.

[8] S. Gümüşsoy and H. Özbay, "Remarks on strong stabilization and stable $\mathcal{H}^\infty$ controller design," *IEEE Trans. Autom. Control*, vol. 50, no. 12, pp. 2083–2087, Dec. 2005.

[9] Y. Halevi, "Stable LQG controllers," *IEEE Trans. Autom. Control*, vol. 39, no. 10, pp. 2104–2106, Oct. 1994.

[10] H. Özbay and A. N. Gündeş, "Strongly stabilizing controller synthesis for a class of MIMO plants," in *Proc. 17th IFAC World Congress*, 2008, pp. 359–363.

[11] H. Özbay, "On strongly stabilizing controller synthesis for time delay systems," in *Proc. 17th IFAC World Congress*, 2008, pp. 6342–6346.

[12] I. Petersen, "Robust $\mathcal{H}^\infty$ control of an uncertain system via a strict bounded real output feedback controller," in *Proc. 45th IEEE Conf. Decision Control*, 2006, pp. 571–577.

[13] A.-W. A. Saif, D.-W. Gu, and I. Postlethwaite, "Strong stabilization of MIMO systems via unimodular interpolation in H_∞," *Int. J. Control*, vol. 69, no. 6, pp. 797–818, 1989.

[14] M. C. Smith and K. P. Sondergeld, "On the order of stable compensators," *Automatica*, vol. 22, pp. 127–129, 1986.

[15] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.

[16] D. C. Youla, J. J. Bongiorno, and C. N. Lu, "Single-loop feedback stabilization of linear multivariable dynamical plants," *Automatica*, vol. 10, pp. 159–173, 1974.
[17] M. Zeren and H. Özbay, "On the synthesis of stable H_∞ controllers," *IEEE Trans. Autom. Control*, vol. 44, no. 2, pp. 431–435, Feb. 1999.
[18] M. Zeren and H. Özbay, "On the strong stabilization and stable H_∞-controller design problems for MIMO systems," *Automatica*, vol. 36, pp. 1675–1684, 2000.
[19] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.

Design of Observer-Based H_∞ Robust Repetitive-Control System

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***Abstract*—This technical note deals with the problem of designing a robust observer-based repetitive-control system that provides a given H_∞ disturbance attenuation performance for a class of plants with time-varying structured uncertainties. A continuous-discrete two-dimensional model is built that accurately describes the features of repetitive control, thereby enabling the control and learning actions to be preferentially adjusted. A sufficient condition for the repetitive-control system to have a disturbance-attenuation bound in the H_∞ setting is given in terms of a linear matrix inequality (LMI). It yields the parameters of the repetitive controller and the state observer. Finally, a numerical example demonstrates the effectiveness of the method, whose main advantage is the easy, preferential adjustment of control and learning through the tuning of two parameters in the LMI-based condition.**

***Index Terms*—Disturbance attenuation, linear matrix inequality (LMI), repetitive control, robust control, state observer, two-dimensional (2-D) system.**

I. Introduction

Repetitive control has a learning capability. For a given periodic reference input, a repetitive controller gradually reduces the tracking error through repeated learning actions, resulting in the tracking of the reference input without steady-state error.

The key feature of a repetitive controller is that it contains an internal model of a periodic signal, which theoretically guarantees asymptotic tracking [1]. It contains a pure-time-delay positive-feedback loop, which adds the tracking error of the previous period to the present error to produce a control signal. This action simulates human learning. From the standpoint of system theory, a repetitive-control system is a neutral-type delay system. A repetitive controller contains an infinite number of poles on the imaginary axis. [2] pointed out that this type of system can be stabilized only when the relative degree of the plant is zero. When the relative degree is larger than that, the repetitive controller has to be modified by the insertion of a low-pass filter into the time-delay feedback line. This modification means that the controller now contains only an approximate internal model of a periodic signal. As a result, tracking performance is not guaranteed for periodic signals in the high-frequency band. Since the best tracking performance is obtainable only when the plant has a relative degree of zero, the design of a repetitive-control system for this limiting case is theoretically significant.

Analysis of a repetitive-control system reveals two types of actions: continuous control within each repetition period and discrete learning between periods. Due to the difficulty of guaranteeing stability, almost all methods of designing repetitive-control systems consider only the overall results in the time domain. Consequently, they are incapable of making fundamental improvements in control performance. For example, [3] discussed the stability and robustness provided by a structured-singular-value method; but they used a trial-and-error technique to find approximate upper and lower bounds on a structured singular value. [4] presented a sufficient stabilization condition in the form of a linear matrix inequality (LMI); but the tracking performance depends on the iterative adjustment of the parameters of a low-pass filter and the repetitive controller.

[5] presented a method of designing a robust, static, output-feedback repetitive-control system that is based on two-dimensional (2-D) system theory [6], [7]; but it only considers the robust stability of the system. To enable that method to handle a larger class of systems, this technical note extends the static output feedback to dynamic output feedback and presents the configuration of an observer-based repetitive-control system. It focuses especially on the problem of designing a robust repetitive-control system with a prescribed bound on disturbance attenuation for a class of linear systems with a relative degree of zero and time-varying, structured, periodic uncertainties. First, we build a continuous-discrete 2-D model to describe the system. Next, to obtain satisfactory disturbance-attenuation performance, we formulate the design problem as an H_∞ robust-stabilization problem for a continuous-discrete 2-D system. Then, we derive a sufficient robust-stability condition in the form of an LMI by using 2-D system stability theory and the singular-value decomposition (SVD) of the output matrix. The advantage of this method over others, including the one in [5], is that it allows control and learning to be preferentially adjusted by means of two parameters in the LMI. Finally, a numerical example demonstrates the validity of the method.

Throughout this technical note, $\mathbb{R}_+$ is the set of non-negative real numbers; $\mathbb{C}^p$ is the n-dimensional vector space over complex numbers; $\mathbb{Z}_+$ is the set of non-negative integers; $\aleph$ is the linear space of all the functions from $[0, T]$ to $\mathbb{C}^p$. $L_2(\mathbb{R}_+, \mathbb{C}^p)$, or just L_2, is the linear space of square integrable functions from $\mathbb{R}_+$ to $\mathbb{C}^p$; and $\ell_2(\mathbb{Z}_+, \aleph)$, or just ℓ_2, is the linear space of all the functions from $\mathbb{Z}_+$ to $\aleph$ (discrete-time signal).

II. Problem Description

Consider the repetitive-control system in Fig. 1. $r(t)$ is a given periodic reference signal with a period of T. The compensated single-input, single-output (SISO) plant has a relative degree of zero and time-varying structured uncertainties

$$\begin{cases} \dot{x}_p(t) = [A + \delta A(t)]\, x_p(t) + [B + \delta B(t)]\, u(t) + B_w w(t) \\ y(t) = C x_p(t) + D u(t) \end{cases} \tag{1}$$

where $x_p(t) \in \mathbb{R}^n$ is the state of the plant; $u(t), y(t) \in \mathbb{R}$ are the control input and output, respectively; and $w(t) \in L_2[0, +\infty)$ is the disturbance input. Setting $B_w \neq 0$ adds the disturbance to the system,

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