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Simultaneous Stabilization and Step Tracking for MIMO Systems with LTI Controllers

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Abstract— **It is shown that any number of MIMO plants that have no zeros in the region of instability can be simultaneously stabilized using low order linear, time-invariant integral-action controllers. These plants may be stable or unstable and may have poles anywhere in the complex plane. The common controller achieves asymptotic tracking of step-input references with zero steady-state error and has a low order transfer-function. Systematic synthesis methods are presented, and a parametrization of all simultaneously stabilizing controllers with integral-action is also provided.**

I. INTRODUCTION

Simultaneous stabilization of a finite family of (three or more) plants using linear, time-invariant (LTI) controllers is a challenging and important control problem. The issue of simultaneous stabilization of a set of models to be controlled arises in linearization of nonlinear process models at various operating points. Maintaining stability under sensor or actuator failures for reliable operation also leads to dynamic models corresponding to failure modes to be all controlled using a common controller [10]. The robust control problem deals with controller design in the face of an infinite number of plant models all within a neighborhood of a nominal model, which represent perturbations of the nominal plant.

Simultaneous stabilization is a hard open problem in linear systems theory. Conditions for existence of simultaneously stabilizing controllers have been explored extensively [11], [4]. The well established result that the simultaneous stabilization of n plants is equivalent to strong stabilization of $n-1$ plants leads to explicit conditions for existence of simultaneously stabilizing controllers for $n = 2$: Two plants are simultaneously stabilizable if and only if a related system is strongly stabilizable, i.e., can be stabilized using a stable controller. Strong stabilizability of this single system can be checked via the parity interlacing property of the positive real poles and (blocking) zeros [12], [11], [4]. For simultaneous stabilizability of *three or more* plants, there are no *necessary and sufficient* conditions available [2], [3], [4] in the general case. Alternative strategies such as time-varying or sampled-data controllers have also been developed to overcome the limitations of LTI controllers (e.g., [9]). The problem considered in this work is the simultaneous stabilization of a finite set of LTI, multi-input multi-output (MIMO) plants using linear, time-invariant output-feedback controllers. Single-input single-output (SISO) plants are also included. The results here deal with the problem using only time-invariant controllers. The synthesis methods result in low order controllers to avoid complexity issues for computation and implementation. An added design goal is asymptotic tracking of constant reference inputs, achieved with poles duplicating the dynamic structure of the exogenous signals that the regulator has to process; these integral-action controllers obey the well-known internal model principle [5].

Exploring sufficient conditions for simultaneous stabilizability of three or more plants is important since explicit existence conditions for the completely general case of three or more arbitrary plants are not possible to obtain (see [4]). Sufficient conditions lead to identifying classes of practically relevant plants for which simultaneous stabilization is achievable. For SISO plants, an especially interesting class was considered in [1], where it was shown that scalar plants that are all minimum-phase, strictly proper, and have the same high-frequency gain sign can be simultaneously stabilized by stable and strictly proper controllers. Also for SISO plants, an algorithm for simultaneous stabilization of up to four plants in groups was given in [8]. For MIMO plants, a simultaneous controller synthesis with integral-action that applies only to stable plants was presented in [6]. The goal here is to extend simultaneous integral-action controller synthesis that applies to unstable as well as stable MIMO or SISO plant classes. Such simultaneous stabilizers would therefore achieve asymptotic tracking (and equivalently output disturbance rejection) of constant reference inputs with zero steady-state error in addition to closed-loop stability.

The plant classes considered in Section III-A have blocking-zeros at infinity but otherwise have no right-half plane zeros. Theorem 1 gives a sufficient condition for simultanous stabilizability of such plants based on their high frequency gain matrices. Proposition 1 develops a synthesis method for simultaneously stabilizing integral-action controllers whose transfer-functions are the same order as the number of blocking zeros at infinity. Section III-B extends the synthesis to plants that may have other transmission zeros at infinity in addition to the blocking zeros that factor out of every entry. In all cases, the plants may be stable or unstable, with any number of poles anywhere in the complex plane; they may have zeros in the left half plane and infinity. The synthesis approaches developed here are illustrated with numerical examples. Although we discuss continuous-time systems here, all results apply also to discrete-time systems with appropriate modifications.

The following notation is used: $\mathcal{U}$ denotes the extended closed right-half plane, i.e., $\mathcal{U} = \mathbb{C}_+ \cup \{\infty\} = \{ s \in \mathbb{C} \mid \mathcal{R}e(s) \geq 0 \} \cup \{\infty\}$; $\mathbb{R}$, $\mathbb{R}_+$ denote real and positive

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real numbers; $\mathbf{R_p}$ denotes real proper rational functions of s; $\mathbf{S} \subset \mathbf{R_p}$ is the stable subset with no poles in $\mathcal{U}$; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in $\mathbf{S}$; $M \in \mathcal{M}(\mathbf{S})$ is called unimodular if $M^{-1} \in \mathcal{M}(\mathbf{S})$; I_m is the $m \times m$ identity matrix; we use I when the dimension is unambiguous. The H_∞-norm of $M(s) \in \mathcal{M}(\mathbf{S})$ is denoted by $\|M(s)\|$ (i.e., the norm $\|\cdot\|$ is defined as $\|M\| := \sup_{s \in \partial\mathcal{U}} \bar{\sigma}(M(s))$, where $\bar{\sigma}$ is the maximum singular value and $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$). We use δn to denote the degree of the polynomial n. For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$; i.e., for $G \in \mathbf{R_p}^{m\times m}$, $C \in \mathbf{R_p}^{m\times m}$, $G = Y^{-1}X$ denotes a left-coprime-factorization (LCF), $C = ND^{-1}$ denotes a right-coprime-factorization (RCF), where $X, Y, N, D \in \mathbf{S}^{m\times m}$, $\det Y(\infty) \neq 0$, $\det D(\infty) \neq 0$. Let $\text{rank} G(s) = r \leq m$; then $z \in \mathcal{U}$ is a tansmission-zero of G if $\text{rank} X(z) < r$ and it is a blocking-zero of G if $X(z) = 0$. We refer to poles and zeros in the region of instability $\mathcal{U}$ as $\mathcal{U}$-poles and $\mathcal{U}$-zeros.

II. Problem Description

Consider the standard LTI, MIMO unity-feedback system $Sys(G, C)$ shown in Fig. 1, where $G \in \mathbf{R_p}^{m\times m}$, and $C \in \mathbf{R_p}^{m\times m}$ denote the plant's and the controller's transfer-functions, and $\text{rank}\, G = m$. The objective is to design a simple simultaneously stabilizing controller C that achieves asymptotic tracking of step-input references with zero steady-state error for a finite set of plants.

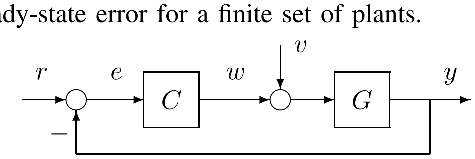

Fig. 1. Unity-Feedback System $Sys(G, C)$.

Let $G = Y^{-1}X$ be an LCF and $C = ND^{-1}$ be an RCF, where $Y, X, D, N \in \mathbf{S}^{m\times m}$, $\det Y(\infty) \neq 0$, $\det D(\infty) \neq 0$. Then C stabilizes $G \in \mathcal{M}(\mathbf{R_p})$ if and only if

$$M := Y\,D + X\,N \tag{1}$$

is unimodular [11]. Let the (input-error) transfer-function from r to e be denoted by H_{er} and let the (input-output) transfer-function from r to y be denoted by H_{yr}; then

$$H_{er} = (I+GC)^{-1} = I - GC(I+GC)^{-1} = I - H_{yr}\ . \tag{2}$$

Definition 1: i) The system $Sys(G, C)$ is stable and has integral-action if the closed-loop transfer-function from (r, v) to (y, w) is stable, and the (input-error) transfer-function H_{er} has blocking-zeros at $s = 0$. *ii)* The controller C is said to be an integral-action controller if C stabilizes G and $D(0) = 0$ for any RCF $C = ND^{-1}$. □

Suppose that the system $Sys(G, C)$ is stable and that step input references are applied at $r(t)$. The steady-state error $e(t)$ due to step inputs at $r(t)$ goes to zero as $t \to \infty$ if and only if $H_{er}(0) = 0$. By Definition 1, the stable system $Sys(G, C)$ achieves asymptotic tracking of constant reference inputs with zero steady-state error if and only if it has integral-action. By (1), write $H_{er} = (I + GC)^{-1} = DM^{-1}Y$. Then by Definition 1, $Sys(G, C)$ has integral-action if $C = ND^{-1}$ is an integral-action controller since $D(0) = 0$ implies $H_{er}(0) = (DM^{-1}Y)(0) = 0$.

III. Simultaneous Controller Synthesis

It is assumed throughout that plants to be simultaneously stabilized with integral-action controllers have no transmission-zeros at $s = 0$ since this condition is necessary for existence of integral-action controllers.

A. Plants with blocking zeros at infinity

Define finite sets $\mathcal{G}$, $\widehat{\mathcal{G}}$, and $\mathcal{P}$ of MIMO plants that all have *exactly* r, 1, and 0 blocking-zeros at infinity, respectively. There may be any number of plants in these sets; some plants may be stable and some unstable. These plants have no other (transmission and blocking) $\mathcal{U}$-zeros; they may have (transmission and blocking) zeros anywhere in the stable region $\mathbb{C} \setminus \mathcal{U}$. There are no restrictions on the poles; they may be anywhere in $\mathbb{C}$. In the SISO case, the relative degree of these plants is r, 1 and 0, respectively. What is meant by r blocking-zeros at infinity for the MIMO case is that for any $a \in \mathbb{R}_+$, the plants $G_j \in \mathcal{G}$ can be expressed as

$$G_j = Y_j^{-1}X = [\ \frac{1}{(s+a)^r}\, G_j^{-1}\ ]^{-1}\ [\ \frac{1}{(s+a)^r} I\ ]\ ; \tag{3}$$

$Y_j = \frac{1}{(s+a)^r}\, G_j^{-1} \in \mathcal{M}(\mathbf{S})$, and $X = \frac{1}{(s+a)^r} I$ for each G_j. Similarly, the plants $\hat{G}_j \in \widehat{\mathcal{G}}$ can be expressed as

$$\hat{G}_j = \hat{Y}_j^{-1}\hat{X} = [\ \frac{1}{s+a}\, \hat{G}_j^{-1}\ ]^{-1}\ [\ \frac{1}{s+a} I\ ]\ ; \tag{4}$$

for any $a \in \mathbb{R}_+$, $\hat{Y}_j = \frac{1}{s+a}\, \hat{G}_j^{-1} \in \mathcal{M}(\mathbf{S})$, and $\hat{X} = \frac{1}{s+a} I$ for each $\hat{G}_j$. We denote the plants in $\mathcal{P}$ as $P_j \in \mathcal{P}$; since these plants have no zeros in $\mathcal{U}$, including infinity, we have $P_j^{-1} \in \mathcal{M}(\mathbf{S})$. The largest set under consideration for simultaneous stabilization is $\mathcal{G} \cup \widehat{\mathcal{G}} \cup \mathcal{P}$. Define

$$Y_j(\infty) := (\ \frac{1}{s^r} G_j(s)^{-1}\ )|_{s\to\infty}\ ,$$
$$\hat{Y}_j(\infty) := (\ \frac{1}{s} \hat{G}_j(s)^{-1}\ )|_{s\to\infty}\ ; \tag{5}$$

$Y_j(\infty)^{-1} = (s^r G_j(s))|_{s\to\infty}$; $\hat{Y}_j(\infty)^{-1} = (s\hat{G}_j(s))|_{s\to\infty}$. Designate an arbitrary member $G_o \in \mathcal{G}$ in the set with the largest number r of blocking-zeros at infinity as the nominal plant. Obviously, this plant may be one from $\widehat{\mathcal{G}}$ if $r = 1$ or from $\mathcal{P}$ if $r = 0$. By (5), $Y_o(\infty)^{-1} = (\, s^r\, G_o(s)\,)\,|_{s\to\infty}$. Define F_j and $\hat{F}_j$ as

$$F_j := Y_j(\infty)\, Y_o(\infty)^{-1} = (G_j^{-1} G_o)(\infty)\ ,$$
$$\hat{F}_j := \hat{Y}_j(\infty)\, Y_o(\infty)^{-1} = (s^{r-1}\, \hat{G}_j^{-1} G_o)(\infty)\ . \tag{6}$$

Lemma 1: (*Necessary existence condition for simultaneous integral-action controllers*): If all plants in $\mathcal{G} \cup \widehat{\mathcal{G}}$ can be simultaneously stabilized using an integral-action controller, then for all G_j, $\hat{G}_j$,

$$\det\, F_j = \det\,(\, G_j^{-1}\, G_o\,)(\infty)\ >\ 0\ ,$$
$$\det\, \hat{F}_j = \det\,(\, s^{r-1}\, \hat{G}_j^{-1}\, G_o\,)(\infty)\ >\ 0\ . \tag{7}$$

Theorem 1: (*Sufficient existence condition for simultaneous integral-action controllers*): If the constant matrices $F_j = (\,G_j^{-1}\,G_o\,)(\infty)$ and $\hat{F}_j = \det(\,s^{r-1}\,\hat{G}_j^{-1}\,G_o\,)(\infty)$ are diagonal and positive-definite for all $G_j \in \mathcal{G}$, $\hat{G}_j \in \widehat{\mathcal{G}}$, then all plants in $\mathcal{G}\cup\widehat{\mathcal{G}}\cup\mathcal{P}$ can be simultaneously stabilized using an integral-action controller. □

Corollary 1: If $\mathcal{G} \cup \widehat{\mathcal{G}}$ is empty or contains only one plant, G_o, then all plants $\mathcal{P} \cup \{\,G_o\,\}$ can be simultaneously stabilized using an integral-action controller.

Remarks 1: **1)** The necessary condition (7) in Lemma 1 is only for those plants that have a blocking-zero at infinity, i.e., it is not required for $P_j \in \mathcal{P}$. For example, let $G_o = \frac{-1}{s-1}$, $P = \frac{s+1}{s-1}$; $(sP^{-1}G_o)(\infty) = -1 \not> 0$; the simple integral-action controller $C = \frac{-4(s+2)}{s}$ stabilizes both G_o and P.
2) Condition (7) is not necessary for existence of simultaneously stabilizing controllers for the plants in $\mathcal{G} \cup \widehat{\mathcal{G}}$ unless integral-action is required. For example, $G_o = \frac{-1}{s-1}$ and $G_1 = \frac{1}{s+4}$, which violate (7), are simultaneously stabilizable by the constant $C = -2$, but by Lemma 1, they are not simultaneously stabilizable using any integral-action controllers. **3)** For SISO plants, the sufficient condition in Theorem 1 is equivalent to F_j, $\hat{F}_j$ being positive, i.e., the strictly-proper plants having the same high frequency gain sign. **4)** By Corollary 1, any finite set of plants that have no $\mathcal{U}$-zeros (including infinity), with the exception of one plant with any number of blocking-zeros at infinity, are simultaneously stabilizable using an integral-action controller assuming no additional conditions on these plants. □

Proposition 1 provides a synthesis procedure that explicitly constructs simultaneously stabilizing integral-action controllers for the set $\mathcal{G} = \mathcal{G} \cup \widehat{\mathcal{G}} \cup \mathcal{P}$ of MIMO plants. This construction is therefore the proof of existence under the sufficient condition of Theorem 1. We assume that the constant matrices $F_j = (\,G_j^{-1}\,G_o\,)(\infty)$ are diagonal and positive-definite for all G_j, and that $\hat{F}_j = (\,s^{r-1}\,\hat{G}_j^{-1}\,G_o\,)(\infty)$ are positive-definite for all $\hat{G}_j$. Define

$$F_j = \operatorname{diag}\begin{bmatrix} f_{j1} & f_{j2} & \cdots & f_{jm}\end{bmatrix}\,,\ \mathsf{f}_j := \max_{1\le i\le m} f_{ji}\,. \quad (8)$$

Proposition 1: (*Simultaneous integral-action controller synthesis*): Consider the finite set $\mathcal{G}\cup\widehat{\mathcal{G}}\cup\mathcal{P}$ of MIMO plants that have r, 1, or 0 blocking-zeros at infinity.
a) Suppose that $r > 1$. Choose an arbitrary plant $G_o \in \mathcal{G}$, with $Y_o(\infty)^{-1} = (\,s^r\,G_o(s)\,)\,|_{s\to\infty}$. Let $\varphi(s)$ be any monic r-th order Hurwitz polynomial (roots in $\mathbb{C}\setminus\mathcal{U}$). Let

$$C_r = \beta\,\frac{\varphi(s)}{(s+\alpha)^r - \alpha^r}\,Y_o(\infty)\ . \quad (9)$$

If $F_j = (G_j^{-1}G_o)(\infty)$ defined as in (6) is diagonal, positive definite for all $G_j \in \mathcal{G}$, define Ψ_j as

$$\Psi_j := s\,[\,\frac{1}{\varphi}G_j^{-1}(s)Y_o(\infty)^{-1}F_j^{-1} - I\,]\ . \quad (10)$$

If $\hat{F}_j = (s^{r-1}\,\hat{G}_j^{-1}G_o)(\infty)$ is positive definite for all $\hat{G}_j \in \widehat{\mathcal{G}}$, define $\widehat{\Psi}_j$ as

$$\widehat{\Psi}_j := s\,[\frac{[\,(s+\alpha)^r - \alpha^r\,]}{s\,\varphi(s)}\hat{G}_j^{-1}(s)Y_o(\infty)^{-1} - \hat{F}_j]\ . \quad (11)$$

Under these assumptions,
i) C_r is an integral-action controller that simultaneously stabilizes all plants $G_j \in \mathcal{G}$ for $\alpha, \beta \in \mathbb{R}_+$ satisfying

$$\alpha\ >\ r\,\max_{G_j\in\mathcal{G}}\|\,\Psi_j\,\|\ ,\quad \beta \ge \alpha^r\,\max_{G_j\in\mathcal{G}}\mathsf{f}_j\ . \quad (12)$$

ii) C_r is an integral-action controller that simultaneously stabilizes all plants G_j, $\hat{G}_j$ in $\mathcal{G}\cup\widehat{\mathcal{G}}$ for $\alpha \in \mathbb{R}_+$ satisfying (12) and $\beta \in \mathbb{R}_+$ satisfying (12) and

$$\beta > \max_{\hat{G}_j\in\widehat{\mathcal{G}}}\|\,\widehat{\Psi}_j\,\|\ . \quad (13)$$

iii) C_r is an integral-action controller that simultaneously stabilizes all plants G_j, $\hat{G}_j$, P_j in $\mathcal{G}\cup\widehat{\mathcal{G}}\cup\mathcal{P}$ for $\alpha \in \mathbb{R}_+$ satisfying (12), $\beta \in \mathbb{R}_+$ satisfying (12)-(13) and

$$\beta > \max_{P_j\in\mathcal{P}}\|\,P_j^{-1}Y_o(\infty)^{-1}\frac{[\,(s+\alpha)^r - \alpha^r\,]}{\varphi(s)}\,\|\ . \quad (14)$$

b) Suppose $r = 1$, i.e., $\mathcal{G} = \emptyset$. Choose an arbitrary plant $G_o \in \widehat{\mathcal{G}}$, $Y_o(\infty)^{-1} = (\,s\,G_o(s)\,)\,|_{s\to\infty}$. Let $\varphi(s) = (s+g)$ be any monic 1-st order Hurwitz polynomial ($g > 0$). Let

$$C_1 = \beta\,\frac{(s+g)}{s}\,Y_o(\infty)\ . \quad (15)$$

If $\hat{F}_j = (\hat{G}_j^{-1}G_o)(\infty)$ is positive definite for all $\hat{G}_j \in \widehat{\mathcal{G}}$, define $\widehat{\Psi}_j$ as

$$\widehat{\Psi}_j := s\,[\frac{1}{(s+g)}\hat{G}_j^{-1}(s)Y_o(\infty)^{-1} - \hat{F}_j]\ . \quad (16)$$

Under these assumptions,
i) C_1 is an integral-action controller that simultaneously stabilizes all plants $\hat{G}_j$ in $\widehat{\mathcal{G}}$ for $\beta \in \mathbb{R}_+$ satisfying (13).
ii) C_1 is an integral-action controller that simultaneously stabilizes all plants $\hat{G}_j$, P_j in $\widehat{\mathcal{G}}\cup\mathcal{P}$ for $\beta \in \mathbb{R}_+$ satisfying (13) and

$$\beta > \max_{P_j\in\mathcal{P}}\|\,\frac{s}{s+g}P_j^{-1}Y_o(\infty)^{-1}\,\|\ . \quad (17)$$

c) Suppose $r = 0$, i.e., $\mathcal{G} = \emptyset$, $\widehat{\mathcal{G}} = \emptyset$. Let $K \in \mathbb{R}^{m\times m}$ be any nonsingular matrix, and $\varphi(s) = (s+g)$ be any monic 1-st order Hurwitz polynomial ($g > 0$). Let

$$C_0 = \beta\,\frac{(s+g)}{s}\,K\ . \quad (18)$$

Then C_0 is an integral-action controller that simultaneously stabilizes all plants $P_j \in \mathcal{P}$ for $\beta \in \mathbb{R}_+$ satisfying

$$\beta\ >\ \max_{P_j\in\mathcal{P}}\|\frac{s}{s+g}P_j^{-1}(s)\,K^{-1}\,\|\ . \quad (19)$$

□

Remarks 2: The integral-action controller C_r in (9) that simultaneously stabilizes the plants in the set $\mathcal{G}\cup\widehat{\mathcal{G}}\cup\mathcal{P}$ is simple. The transfer-function of C_r is bi-proper and has a stable inverse since φ is a strictly Hurwitz polynomial. The poles of C_r are the roots of $\chi(s) = (s+\alpha)^r - \alpha^r$. One root is at $s = 0$ (provides integral-action); the remaining $r-1$ roots are all in the open left-half plane $\mathbb{C}\setminus\mathcal{U}$. The zeros of C_r can be chosen completely arbitrarily in the open left-half complex plane since the choice of $\varphi(s)$ is free. □

Example 1: Consider the plants $G_j \in \mathcal{G}$, with $r = 4$: $G_o = \frac{k_o}{(s-p_1)(s-p_2)(s^2+p_3^2)}$, $G_j = \frac{k_j(s+z_j)^{v_j}}{(s-p_1)(s-p_2)(s^2+p_3^2)(s-p_j)^{v_j}}$. Pick $k_o = k_1 = 3$, $k_2 = 1$, $k_3 = 2.5$, $k_4 = 2$; $p_1 = 1$, $p_2 = 2$, $p_3 = 3$; $z_1 = 4$, $z_2 = 5$, $z_3 = 1$, $z_4 = 6$; $p_1 = 5$, $p_2 = -7$, $p_3 = 0.5$, $p_4 = -4$; $v_1 = 1$, $v_2 = 3$, $v_3 = 4$, $v_4 = 1$. With $Y_o(\infty)^{-1} = k_o = 3$, $F_j = k_o/k_j > 0$, $\max \mathsf{f}_j = 3$, design a fourth order controller for the five plants in $\mathcal{G}$ following Proposition 1-(i). Let $\varphi(s) = (s+2)^4$. By (12), we choose $\alpha = 90 > 4\max\{11, 20, 7.4093, 17, 13\}$ and $\beta = 1.9683 \times 10^8 = 3\alpha^4$ and obtain the controller $C_r = \frac{65610000(s+2)^4}{s(s+180)((s+90)^2+8100)}$ as in (9), which has integral-action due to the pole at $s = 0$ and has three other poles at $\{-180, -90 \pm j90\}$. Now include the plants $\hat{G}_j \in \widehat{\mathcal{G}}$ to be simultaneously stabilized with the plants in $\mathcal{G}$: $\hat{G}_1 = \frac{\hat{k}_1}{s-p_1}$, $\hat{G}_2 = \frac{\hat{k}_2}{s-p_2}$, $\hat{G}_3 = \frac{\hat{k}_3(s+1)}{s^2+p_3^2}$, with $\hat{k}_1 = 9$, $\hat{k}_2 = 2$, $\hat{k}_3 = 1$. By (11), $\max\{\|\widehat{\Psi}_j\|\} = \max\{6.8711\times10^4, 4.2087\times10^5, 2.1271\times10^6\}$, the choice of β in the controller designed for the set $\mathcal{G}$ satisfies (13); hence, C_r simultaneously stabilizes the eight plants in $\mathcal{G} \cup \widehat{\mathcal{G}}$. Now include the plants $P_j \in \mathcal{P}$ to be simultaneously stabilized with the plants in $\mathcal{G} \cup \widehat{\mathcal{G}}$: $P_1 = \frac{-k_j(s+z_j)^{v_j}}{(s-p_j)^{v_j}}$, $j = 1, 2$, $P_3 = \frac{10(s+10)^2}{s^2+10}$. Note that P_1, P_2 do not have the same high frequency gain sign as G_o. Condition (14) is satisfied for $\beta > \max\{1.4594 \times 10^5, 9.3326 \times 10^5, 3.1182 \times 10^3\}$. The choice of β in the controller designed for the set $\mathcal{G}$ satisfies (14), and hence, C_r simultaneously stabilizes the eleven plants in the set $\mathcal{G} \cup \widehat{\mathcal{G}} \cup \mathcal{P}$. We can include any number of plants with four, one, or no zeros at infinity and adjust α, β for the additional plants to achieve simultaneous stabilization with integral-action. □

B. Plants with transmission zeros at infinity

In this section we consider the set $\mathcal{G}^t$ of MIMO plants with transmission-zeros at infinity that may not appear in every entry of the transfer-matrix with the same multiplicity. Let $G_j \in \mathcal{G}^t \subset \mathbf{R_p}^{m\times m}$ have an LCF $G_j = Y_j^{-1}X$ such that $\text{rank}X(\infty) < m$ but $\text{rank}X(s) = m$ for $s \in \mathbb{C}_+$. Write

$$X = \frac{1}{(s+a)^r} X_t \ , \tag{20}$$

where $a \in \mathbb{R}_+$, $r \geq 0$ is the number of blocking-zeros at infinity for each $G_j \in \mathcal{G}^t$, and $\text{rank}X_t(\infty) < m$ but $X_t(\infty) \neq 0$; i.e., X_t book-keeps the transmission-zeros at infinity that $G_j \in \mathcal{G}^t$ may have in addition to the r blocking-zeros at infinity. With $n_{k\ell}$ and $d_{k\ell}$ as polynomials, write

$$X_t^{-1} = \left[\begin{array}{c} \dfrac{n_{k\ell}}{d_{k\ell}} \end{array} \right]_{k,\ell\in\{1,\ldots,m\}} ; \tag{21}$$

X_t^{-1} has no poles in the closed right-half complex plane $\mathbb{C}_+$ (i.e., the polynomials $d_{k\ell}$ are Hurwitz) but may have poles at infinity. Define the integers $\rho_{k\ell}$ as $\rho_{k\ell} := \delta n_{k\ell} - \delta d_{k\ell}$, if $\delta n_{k\ell} > \delta d_{k\ell}$, $\rho_{k\ell} := 0$, if $\delta n_{k\ell} \leq \delta d_{k\ell}$, and for $\ell = 1, \ldots, m$, define ρ_ℓ and r_ℓ as

$$\rho_\ell := \max_{1\leq k\leq m} r_{k\ell} \ , \quad r_\ell := r + \rho_\ell \ . \tag{22}$$

Let $a \in \mathbb{R}_+$; for $\ell = 1, \ldots, m$, define $\lambda_\ell := \frac{1}{(s+a)^{\rho_\ell}}$ and

$$\begin{aligned} \Lambda &:= \text{diag}\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{bmatrix} \\ &= \text{diag}\begin{bmatrix} \frac{1}{(s+a)^{\rho_1}} & \frac{1}{(s+a)^{\rho_2}} & \cdots & \frac{1}{(s+a)^{\rho_m}} \end{bmatrix} \ . \end{aligned} \tag{23}$$

Although X_t^{-1} may be improper, $X_t^{-1}\Lambda$ is stable since $\frac{n_{k\ell}}{d_{k\ell}(s+a)^\ell} \in \mathbf{S}$. Define $Y_j(\infty)$ as

$$Y_j(\infty) := (X(s)G_j(s)^{-1})|_{s\to\infty} = (\frac{1}{s^r}X_t(s)G_j(s)^{-1})|_{s\to\infty} \ , \tag{24}$$

i.e., $Y_j(\infty)^{-1} = (\, s^r G_j(s) X_t^{-1}(s)\,)|_{s\to\infty}$. Designate an arbitrary member $G_o \in \mathcal{G}^t$ as the nominal plant. This plant may be one from $\widehat{\mathcal{G}}$ if $r = 1$ or from $\mathcal{P}$ if $r = 0$. By (24), $Y_o(\infty)^{-1} = \big(\, s^r\, G_o(s) X_t^{-1}(s)\,\big)\,|_{s\to\infty}$. Define F_j as

$$F_j := Y_j(\infty)\, Y_o(\infty)^{-1} = (X_t^{-1} G_j^{-1} G_o X_t)(\infty) \ . \tag{25}$$

Proposition 2: (*Simultaneous integral-action controller synthesis for plants with transmission-zeros at infinity*):
Consider the finite set $\mathcal{G}^t$ of MIMO plants. Choose an arbitrary plant $G_o \in \mathcal{G}^t$, with $Y_o(\infty)^{-1} = \big(\, s^r\, G_o(s) X_t^{-1}(s)\,\big)\,|_{s\to\infty}$. For $j = 1, \ldots, m$, let $\varphi_\ell(s)$ be any monic r_ℓ-th order Hurwitz polynomial (with r_ℓ roots in $\mathbb{C} \setminus \mathcal{U}$) and let $g_\ell \in \mathbb{R}_+$. Let

$$C_t = \beta\, X_t^{-1} \Lambda\, \text{diag}\left[\begin{array}{cccc} c_1 & c_2 & \cdots & c_m \end{array}\right] Y_o(\infty) \ , \tag{26}$$

where, for $\ell = 1, \ldots, m$,

$$c_\ell = \frac{\varphi_\ell(s)}{(s+\alpha)^{r_\ell} - \alpha^{r_\ell}} \ , \text{ if } r_\ell > 1, \ c_\ell = \frac{(s+g_\ell)}{s} \ , \text{ if } r_\ell = 1. \tag{27}$$

If $F_j = (Y_j Y_o^{-1})(\infty)$ defined as in (6) is diagonal, positive definite for all $G_j \in \mathcal{G}^t$, define Ψ_{tj} as

$$\Psi_{tj} := s[(s+a)^r Y_j(s) Y_o(\infty)^{-1} \Lambda^{-1} F_j^{-1} \text{diag}\left[\frac{1}{\varphi_\ell(s)}\right]_{\ell=1}^{m} - I]. \tag{28}$$

Under these assumptions, C_t is an integral-action controller that simultaneously stabilizes all plants $G_j \in \mathcal{G}^t$ for $\alpha, \beta \in \mathbb{R}_+$ satisfying

$$\alpha \ > \max_{1\leq\ell\leq m} r_\ell \max_{G_j\in\mathcal{G}^t} \|\, \Psi_{tj}\, \| \ , \quad \beta \geq \max_{1\leq\ell\leq m} \alpha^{r_\ell} \max_{G_j\in\mathcal{G}^t} \mathsf{f}_j \ . \tag{29}$$

□

Remarks 3: **1)** *(Parametrization of all simultaneously stabilizing integral-action controllers)*: The integral-action controllers shown in Propositions 1 and 2, which simultaneously stabilize the plants in $\mathcal{G} \cup \widehat{\mathcal{G}} \cup \mathcal{P}$ or $\mathcal{G}^t$ are low order controllers. Although the synthesis methods offer flexibility in the choice of parameters, the fact that the order is low restricts achievable design objectives. Once the existence of simultaneous integral-action controllers is established through the proposed controllers, a parametrization of other simultaneous integral-action stabilizers with arbitrary order can be obtained as follows: Under the assumptions of Proposition 1, suppose that C_r is the integral-action controller in (9) or in (15) if $r = 1$. Let $G_o \in \mathcal{G}$ be any member of the set chosen as the nominal plant. Then all integral-action controllers simultaneously stabilizing the plants $G_j \in \mathcal{G}$ are

$$C = (\, C_r + \frac{1}{(s+a)^r} G_o^{-1} Q C_r)(I - \frac{1}{(s+a)^r} Q C_r)^{-1} \,, \quad (30)$$

where $Q \in \mathcal{M}(\mathbf{S})$ is such that

$$[\, I + \frac{1}{(s+a)^r}(I + C_r G_j)^{-1} C_r (G_j G_o^{-1} - I) Q \,] \quad \text{is unimodular for all } G_j \in \mathcal{G} \,. \quad (31)$$

For example, $Q \in \mathcal{M}(\mathbf{S})$ can be chosen such that $\|Q\| < \|(I + C_r G_j)^{-1} C_r (G_j G_o^{-1} - I)\|^{-1}$ to satisfy this unimodularity condition in (31). The controllers in (30) also stabilize all $\hat{G}_j \in \widehat{\mathcal{G}}$ if $Q \in \mathcal{M}(\mathbf{S})$ satisfies (31) and $[I + \frac{1}{(s+a)}(I + C_r \hat{G}_j)^{-1} C_r (\hat{G}_j G_o^{-1} - I) Q\,]$ is unimodular for all $\hat{G}_j \in \widehat{\mathcal{G}}$. The simultaneously stabilizing controllers in (30) have integral-action if and only if $Q(0) = 0$. Although the controller in (9) has r-th order transfer-function, the order of the controllers in (30) are unrestricted. The parametrization in (30) can be used to select controllers to achieve other design objectives that may not be achievable with the order restriction of C_r. Similarly, under the assumptions of Proposition 2, suppose that C_t is the integral-action controller in (26). Let $G_o \in \mathcal{G}^t$ be any member of the set chosen as the nominal plant. Then all integral-action controllers simultaneously stabilizing the plants $G_j \in \mathcal{G}^t$ are given by

$$\begin{aligned} C = (\, & C_t + Y_o Q \operatorname{diag}\begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} Y_o(\infty)\,) \\ & \cdot (I - Y_o G_o Q \operatorname{diag}\begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} Y_o(\infty)\,)^{-1} \,, \quad (32) \end{aligned}$$

where $Q \in \mathcal{M}(\mathbf{S})$ is such that $[I + \operatorname{diag}\begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} Y_o(\infty)(I + G_j C_t)^{-1} G_j (X Y_o - Y_j X) Q\,]$ is unimodular for all $G_j \in \mathcal{G}^t$. For example, $Q \in \mathcal{M}(\mathbf{S})$ can be chosen such that $\|Q\| < \|\operatorname{diag}\begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} Y_o(\infty)(I + G_j C_t)^{-1} G_j (X Y_o - Y_j X)\|^{-1}$ to satisfy this unimodularity condition. The simultaneously stabilizing controllers in (32) have integral-action if and only if $Q(0) = 0$.

2) *(Robustness of the simultaneously stabilizing controllers)*: By standard robustness arguments, the simultaneously stabilizing controllers C_r or C_t in Propositions 1, 2, achieve robust simultaneous stability under 'sufficiently small' plant uncertainty for the plant classes considered. For the set $\mathcal{G}$, the controller C_r in (9) robustly simultaneously stabilizes the additively perturbed plants $G_j + \Delta_j$ for all $\Delta_j \in \mathbf{S}^{m \times m}$ such that $\|\,\Delta_j\,\| < \|C_r (I + G_j C_r)^{-1}\|^{-1}$. For multiplicative perturbations, C_r robustly simultaneously stabilizes the plants $G_j\,(I + \Delta_j)$ under all pre-multiplicative perturbations $\Delta_j \in \mathbf{S}^{m \times m}$ such that $\|\,\Delta_j\,\| < \|C_r G_j (I + C_r G_j)^{-1}\|^{-1}$. Similarly, C robustly simultaneously stabilizes the plants $(I + \Delta_j) G_j$ under all post-multiplicative perturbations $\Delta_j \in \mathbf{S}^{m \times m}$ such that $\|\,\Delta_j\,\| < \|G_j C_r (I + G_j C_r)^{-1}\|^{-1}$. Some of the free controller parameter choices in the synthesis may be used to maximize the allowable perturbation magnitudes. Similar robust stability conclusions apply to the plant class $\widehat{\mathcal{G}}$ or $\mathcal{P}$ with the controllers C_1 or C_0. For the plant set $\mathcal{G}^t$, the controller C_t in (26) of Proposition 2 robustly simultaneously stabilizes the perturbed plants where the uncertainties satisfy similar bounds. $\square$

Example 2: Consider the linear model of the VZ-4 doak, a vertical take-off and landing aircraft [7]: $G_j = \begin{bmatrix} \frac{s+z_j}{(s+0.8223)(s^2-0.6401s+0.5326)} & 0 \\ \frac{-1.08}{s-p_j} & \frac{1}{s-p_j} \end{bmatrix}$, $z_j > 0$. The states of the system are forward velocity, downward velocity, pitch rate and pitch angle. The outputs are pitch angle and altitude rates; the inputs are elevator angle and thrust. The nominal parameters are $z_o = 0.137$, $p_o = -0.137$ for the nominal plant G_o. The poles for each G_j are at p_j and $0.3201 \pm j0.6559$. These plants have no $\mathcal{U}$-zeros except at infinity, and therefore they can be written as $G_j = Y_j^{-1} X$, where $Y_j^{-1} = \begin{bmatrix} \frac{(s+0.8223)(s^2-0.6401s+0.5326)}{(s+z_j)(s+a)^2} & 0 \\ 0 & \frac{-(s-p_j)}{(s+a)} \end{bmatrix}^{-1}$, $X = \begin{bmatrix} \frac{1}{(s+a)^2} & 0 \\ \frac{1.08}{(s+a)} & \frac{1}{(s+a)} \end{bmatrix}$, for any $a > 0$. Following Proposition 2, we design a simultaneously stabilizing controller C_t as in (26), with $Y_o(\infty) = Y_j(\infty) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $F_j = I$. With $r = 1$, $X = \frac{1}{(s+a)} \begin{bmatrix} \frac{1}{(s+a)} & 0 \\ 1.08 & 1 \end{bmatrix} = \frac{1}{(s+a)} X_t$ as in (20); then, $X_t^{-1} = \begin{bmatrix} (s+a) & 0 \\ -1.08(s+a) & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} \frac{1}{(s+a)} & 0 \\ 0 & 1 \end{bmatrix}$, where $\rho_1 = 1$, $\rho_2 = 0$, $r_1 = 2$, $r_2 = 0$. By (28), $\Psi_{tj} = s[(s+a) \begin{bmatrix} \frac{(s+0.8223)(s^2-0.6401s+0.5326)}{(s+z_j)(s+a)^2} & 0 \\ 0 & \frac{-(s-p_j)}{(s+a)} \end{bmatrix} \cdot \begin{bmatrix} (s+a) & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s^2+10s+24)} & 0 \\ 0 & \frac{1}{(s+5)} \end{bmatrix} - I\,]$, $\|\Psi_{tj}\| = \max\{\|\frac{s[\,(s+0.8223)(s^2-0.6401s+0.5326) - \varphi_1 (s+z_j)\,]}{\varphi_1 (s+z_j)}\|, \|\frac{s[\,s-p_j-\varphi_2]}{\varphi_2}\|\}$. By (26), $C_t = \beta \begin{bmatrix} 1 & 0 \\ -1.08 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & -c_2 \end{bmatrix} = \beta \begin{bmatrix} \frac{\varphi_1}{s(s+2\alpha)} & 0 \\ \frac{-1.08\varphi_1}{s(s+2\alpha)} & \frac{-(s+g_2)}{s} \end{bmatrix}$. Choose $\varphi_1 = (s^2 + 10s + 24)$, $g_2 = 5$, which gives $c_1 = \frac{(s^2+10s+24)}{s(s+2\alpha)}$, $c_2 = \frac{(s+5)}{s}$. For the nominal plant G_o, we have $\|\Psi_{to}\| = \max\{9.9548, 4.8630\}$. By (29), $\alpha > 19.9096$ and $\beta \geq \alpha^2$. If we choose $\alpha = 30$, $\beta = \alpha^2$, then the controller becomes $C_t = 900 \begin{bmatrix} \frac{(s^2+10s+24)}{s(s+60)} & 0 \\ \frac{-1.08(s^2+10s+24)}{s(s+60)} & \frac{-(s+5)}{s} \end{bmatrix}$. The low order integral-action controller C_t simultaneously stabilizes all $G_j \in \mathcal{G}^t$ for all z_j, p_j such that $2\|\Psi_{tj}\| < \alpha = 30$. $\square$

IV. Conclusions

This work identified some important classes of any finite number of plants that can be simultaneously stabilized using a common low order controller. The plant classes considered here have restrictions on their zeros in the region of instability, while the poles are completely unconstrained. These restrictions are due to the difficulties involving simultaneous stabilization of three or more plants with order-restricted tracking controllers. Systematic synthesis procedures are proposed for each plant class, where the controller parameters and the design choices are explicitly defined. The proposed designs allow freedom in the parameters, which should be used to satisfy additional performance criteria that the design may require. In each of the illustrative examples we selected a set of parameters out of infinitely many satisfying the conditions of Propositions 1, 2.

While asymptotic tracking of constant reference inputs is achieved due to the integral term, performance objectives beyond tracking (and equivalently disturbance rejection) were not considered within the scope of this note. The goal of this study was to establish simultaneous stabilizability using low order controllers, and it was shown that these controllers achieve robust stability under sufficiently small additive and multiplicative plant uncertainty.

APPENDIX: PROOFS

Proof of Proposition 1: **a)** Let C_r be as in (9); then $C_r^{-1} \in \mathcal{M}(\mathbf{S})$. **i)** Define $\chi(s) := [(s+\alpha)^r - \alpha^r]$. By (1), C_r stabilizes each $G_j \in \mathcal{G}$ as in (3) if and only if $M_j = X + Y_j C_r^{-1} = \frac{1}{(s+a)^r} I + Y_j(s) Y_o(\infty)^{-1} \frac{\chi(s)}{\beta\,\varphi(s)} = (\beta F_j^{-1} W_j(s)^{-1} + \frac{(s+a)^r \chi(s)}{\varphi(s)} Y_j(s) Y_o(\infty)^{-1} F_j^{-1} W_j(s)^{-1}) \frac{W_j(s)}{\beta(s+a)^r} F_j = (I + s[\frac{(s+a)^r}{\varphi(s)} Y_j(s) Y_o(\infty)^{-1} F_j^{-1} - I] \frac{\chi(s)}{s} W_j(s)^{-1}) \cdot \frac{W_j(s)}{\beta(s+a)^r} F_j = (I + \Psi \frac{\chi(s)}{s} W_j(s)^{-1}) \frac{W_j(s)}{\beta(s+a)^r} F_j$ is unimodular, where $W_j(s) := (\chi(s) I + \beta F_j^{-1})$, and $W_j(s)^{-1} \in \mathcal{M}(\mathbf{S})$ since $f_{ji} > 0$ and $\beta \geq \alpha^r \mathsf{f}_j$. Since $\beta \geq \alpha^r f_{ji}$, we have $\| \frac{\chi(s)}{s[\chi(s) + f_{ji}^{-1}\beta]} \| \leq \| \frac{\chi(s)}{s\,(s+\alpha)^r} \| = \frac{r}{\alpha}$ implies $\| \frac{\chi(s)}{s} W_j(s)^{-1} \| = \| \mathrm{diag} \left[\frac{(s+\alpha)^r - \alpha^r}{s[(s+\alpha)^r - \alpha^r + f_{ji}^{-1}\beta]} \right]_{i=1}^{m} \| \leq \frac{r}{\alpha}$. Therefore, $\| \Psi \frac{\chi(s)}{s} W_j(s)^{-1} \| \leq \frac{r}{\alpha} \| \Psi \| < 1$ for α, β satisfying (12); hence, M_j is unimodular for each $G_j \in \mathcal{G}$. **ii)** By (i), the controller C_r in (9) stabilizes each $G_j \in \mathcal{G}$ for α, β as in (12). In addition, C_r also stabilizes each $\hat{G}_j \in \widehat{\mathcal{G}}$ if and only if $\hat{M}_j = \hat{X} + \hat{Y}_j C_r^{-1} = \frac{1}{(s+a)} I + \hat{Y}_j(s) Y_o(\infty)^{-1} \frac{\chi(s)}{\beta\,\varphi(s)} = (\beta \hat{W}_j^{-1} + \frac{\chi(s)}{\varphi(s)}(s+a) \hat{Y}_j(s) Y_o(\infty)^{-1} \hat{W}_j^{-1}) \frac{\hat{W}_j}{\beta(s+a)} = (I + s[\frac{\chi(s)]}{s\varphi(s)}(s+a) \hat{Y}_j(s) Y_o(\infty)^{-1} - \hat{F}_j] \hat{W}_j^{-1}) \frac{\hat{W}_j}{\beta(s+a)} = (I + \widehat{\Psi} \hat{W}_j^{-1}) \frac{\hat{W}_j}{\beta(s+a)}$ is unimodular, where $\hat{W}_j := (s\hat{F}_j + \beta I)$ and $\hat{W}_j^{-1} \in \mathcal{M}(\mathbf{S})$ since $\beta > 0$ and $\hat{F}_j$ is positive definite. Since $\| \hat{W}_j^{-1} \| = \| (s\hat{F}_j + \beta I)^{-1} \| = \frac{1}{\beta}$, we have $\| \widehat{\Psi}_j \hat{W}_j(s)^{-1} \| \leq \frac{1}{\beta} \| \widehat{\Psi} \| < 1$ for β satisfying (13); hence, $\hat{M}_j$ is unimodular for $\hat{G}_j \in \widehat{\mathcal{G}}$. **iii)** By (i)-(ii), the controller C_r in (9) stabilizes each $G_j \in \mathcal{G}$ and $\hat{G}_j \in \widehat{\mathcal{G}}$ for α, β as in (12)-(13). In addition, C_r also stabilizes each $P_j \in \mathcal{P}$ if and only if $\tilde{M}_j = I + P_j^{-1} C_r^{-1} = I + P_j^{-1} Y_o(\infty)^{-1} \frac{[(s+\alpha)^r - \alpha^r]}{\beta\varphi(s)}$ is unimodular. For β satisfying (14), we have $\| P_j^{-1} Y_o(\infty)^{-1} \frac{\chi(s)}{\beta\,\varphi(s)} \| < 1$ and hence, $\tilde{M}_j$ is unimodular for each $P_j \in \mathcal{P}$. **b)** The controller C_1 in (15) is the same as C_r in (9) for $r = 1$. Similarly, $\widehat{\Psi}_j$ in (16) is the same as (11). The proof follows similar steps as in the proof (ii)-(iii) of part (a) above. **c)** The controller C_0 in (18) is the same as C_1 in (15) with $Y_o(\infty)$ replaced by an arbitrary constant nonsingular matrix K. □

Proof of Proposition 2: Let C_t be as in (26); then an RCF $C_t = ND^{-1}$ is given by $N = \beta X_t^{-1} \Lambda \in \mathcal{M}(\mathbf{S})$, $D = Y_o(\infty)^{-1} \mathrm{diag} \left[\, c_1^{-1} \;\; c_2^{-1} \;\; \cdots \;\; c_m^{-1} \, \right] \in \mathcal{M}(\mathbf{S})$. Define $\chi_\ell(s) := [(s+\alpha)^{r_\ell} - \alpha^{r_\ell}]$. By (1), C_t stabilizes each $G_j \in \mathcal{G}^t$ if and only if $M_j = \beta X X_t^{-1} \Lambda + Y_j Y_o(\infty)^{-1} \mathrm{diag} \left[c_1^{-1} \;\; c_2^{-1} \cdots \;\; c_m^{-1} \right] = \frac{\beta}{(s+a)^r} \Lambda + Y_j Y_o(\infty)^{-1} \mathrm{diag} \left[\frac{\chi_\ell(s)}{\varphi_\ell(s)} \right]_{\ell=1}^{m} = (\beta F_j^{-1} W_j^{-1} + (s+a)^r Y_j Y_o(\infty)^{-1} \cdot \mathrm{diag} \left[\frac{\chi_\ell(s)}{\varphi_\ell(s)} \right]_{\ell=1}^{m} \Lambda^{-1} F_j^{-1} W_j(s)^{-1}) \frac{W_j}{(s+a)^r} F_j \Lambda = (I + [(s+a)^r Y_j Y_o(\infty)^{-1} \mathrm{diag} \left[\frac{\chi_\ell(s)}{\varphi_\ell(s)} \right]_{\ell=1}^{m} \Lambda^{-1} F_j^{-1} - \mathrm{diag} \left[\chi_\ell(s) \right]_{\ell=1}^{m}] W_j(s) {-} 1) \frac{W_j(s)}{(s+a)^r} F_j \Lambda = (I + s[(s+a)^r Y_j(s) Y_o(\infty)^{-1} \mathrm{diag} \left[\frac{1}{\varphi_\ell(s)} \right]_{\ell=1}^{m} \Lambda^{-1} F_j^{-1} - I] \mathrm{diag} \left[\frac{\chi_\ell(s)}{s} \right]_{\ell=1}^{m} W_j(s)^{-1}) \frac{W_j(s)}{(s+a)^r} F_j \Lambda = (I + \Psi_{tj} \mathrm{diag} \left[\frac{\chi_\ell(s)}{s} \right]_{\ell=1}^{m} W_j(s)^{-1}) \frac{W_j(s)}{(s+a)^r} F_j \Lambda$ is unimodular, where $W_j(s) := (\mathrm{diag} \left[(s+\alpha)^{r_\ell} - \alpha^{r_\ell} \right]_{\ell=1}^{m} + \beta F_j^{-1}) = \mathrm{diag} \left[\chi_\ell(s) + \beta / f_{j\ell} \right]_{\ell=1}^{m}$, and $W_j(s)^{-1} \in \mathcal{M}(\mathbf{S})$ since $f_{j\ell} > 0$ and $\beta \geq \max_{1 \leq \ell \leq m} \alpha^{r_\ell} \mathsf{f}_j$. Since $\beta \geq \alpha^{r_\ell} f_{j\ell}$, we have $\| \frac{\chi_\ell(s)}{s[\chi_\ell + f_{j\ell}^{-1}\beta]} \| \leq \| \frac{\chi_\ell(s)}{s(s+\alpha)^{r_\ell}} \| = \frac{r_\ell}{\alpha}$ implies $\| \frac{\chi_\ell(s)}{s} W_j^{-1} \| = \| \mathrm{diag} \left[\frac{\chi_\ell(s)}{s[\chi_\ell(s) + f_{j\ell}^{-1}\beta]} \right]_{\ell=1}^{m} \| \leq \max_{1 \leq \ell \leq m} \dfrac{r_\ell}{\alpha}$. Therefore, $\| \Psi_{tj} \mathrm{diag} \left[s^{-1} [(s+\alpha)^{r_\ell} - \alpha^{r_\ell}] \right]_{\ell=1}^{m} W_j^{-1} \| \leq \max_{1 \leq \ell \leq m} \dfrac{r_\ell}{\alpha} \| \Psi_{tj} \| < 1$ for α, β satisfying (29) and hence, M_j is unimodular for each $G_j \in \mathcal{G}^t$. □

REFERENCES

[1] B.R. Barmish, K. Wei, "Simultaneous stabilizability of single-input single-output systems," in: *Modelling, Identification and Robust Control*, C.I. Byrnes, A. Lindquist (Eds.), Elsevier, Amsterdam, pp. 431-443, 1986.

[2] V. Blondel, G. Campion, and M. Gevers, "A sufficient condition for simultaneous stabilization," *IEEE Trans. Automatic Control*, 38, pp. 1264-1266, 1993.

[3] V. Blondel, M. Gevers, R. Mortini, and R. Rupp, "Simultaneous stabilization of three or more plants: Conditions on the positive real axis do not suffice," *SIAM Journal on Control and Optimization*, 32:2, pp. 572-590, 1994.

[4] V. D. Blondel, *Simultaneous Stabilization of Linear Systems*, Springer Verlag, Heidelberg, 1994.

[5] B. A. Francis and W. A. Wonham, "The internal model principle for linear multivariable regulators," *Applied Mathematics & Optimization*, 2:2, pp. 170-195, 1975.

[6] A. N. Gündeş, "Simultaneous stabilization of MIMO systems with integral action controllers," *Automatica*, 44:4, pp. 1156-1160, 2008.

[7] R.K. Heffley, W.F. Jewell, "Aircraft handling qualities data" *NASA Contractor Report, NASA CR-2144*, pp. 243-247, December 1972.

[8] Y. Jia, J. Ackermann, "Condition and algorithm for simultaneous stabilization of linear plants," *Automatica*, 37, pp. 1425-1434, 2001.

[9] P. Kabamba, C. Yang, "Simultaneous controller design for linear time invariant systems," *IEEE Trans. Auto. Contr.*, 36, pp. 106-111, 1991.

[10] J. Stoustrup, V. D. Blondel, "Fault tolerant control: a simultaneous stabilization result," *IEEE Trans. Automatic Control*, 49:2, pp. 305-310, 2004.

[11] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, Cambridge, MA: MIT Press, 1985.

[12] D. Youla, J. Bongiorno, C. Lu, "Single-loop feedback stabilization of linear multivariable plants," *Automatica*, 10, pp. 159-173, 1974.