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ISSUES IN ALGEBRAIC CONTROL THEORY

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ABSTRACT

A unified view of recent results in the algebraic theory of linear, time-invariant multiinput-multioutput control systems is presented, with emphasis on the unity-feedback system (one-degree-of-freedom design) and the more general two-input two-output plant and compensator configuration (four-degrees-of-freedom design). The issues of stability, parametrization of all stabilizing compensators, achievable input-output maps and decoupling are discussed.

I. INTRODUCTION

This is a review paper presenting the algebraic theory of two linear, time-invariant (l.t-i), multiinput-multioutput (MIMO) control systems: the classical unity-feedback system $S(P,C)$ and the more general system configuration $\Sigma(\hat{P},\hat{C})$. Due to the general algebraic setting, the results apply to lumped as well as distributed, continuous-time as well as discrete-time systems.

The unity-feedback configuration $S(P,C)$ is studied in Section III. The system $S(P,C)$ is called H-stable if and only if all closed-loop input-output (I/O) maps are H-stable. The H-stability condition for $S(P,C)$ is stated in Theorem 3.4 in terms of coprime factorizations of P and C. The class of all compensators that H-stabilize the plant P is parametrized in Theorem 3.7; compensator design using the configuration $S(P,C)$ is called one-degree-of-freedom design due to the single free parameter matrix Q of the H-stabilizing compensator [Hor.1]. Although a right-coprime or a left-coprime factorization of the plant are commonly used in obtaining this parametrization, it is also possible to start with a bicoprime factorization and reduce $N_{pr}D^{-1}N_{pl}$ to $N_pD_p^{-1}$ or to $\tilde{D}_p^{-1}\tilde{N}_p$. The class of all achievable maps for $S(P,C)$ is obtained by using the class of all stabilizing compensators; all closed-loop I/O maps in the H-stabilized $S(P,C)$ are affine maps in Q.

The system configuration $\Sigma(\hat{P},\hat{C})$ shown in Figure 3 represents the most general interconnection of two physical systems, a plant $\hat{P}$ and a compensator $\hat{C}$. This system is studied in Section IV; the plant and the compensator each have two (vector-)inputs and two (vector-)outputs. The measured output y of $\hat{P}$ is used in feedback, but the output z is the actual output of the plant (the output in the performance specifications); the point is that z and y are not the same. The input v is considered as a disturbance, noise or an external command applied directly to the plant. The compensator output y', which is utilized by the plant in feedback, can be considered as the ideal actuator inputs; the output z' of $\hat{C}$ can be used for performance monitoring or fault diagnosis. The input v' of $\hat{C}$ is considered as the independent control input like commands or initial conditions. The signals u and u', which appear at the interconnection of $\hat{P}$ and $\hat{C}$, model possible additive disturbances, noise, interference and loading.

The conditions for H-stability of $\Sigma(\hat{P},\hat{C})$ are stated in Theorem 4.4. Intuitively, only those plants which have "instabilities that the feedback-loop can remove" can be considered for H-stabilization; these plants are called Σ-admissible. The restriction on the class of H-stabilizable $\hat{P}$ is due to the feedback being applied only through the second inputs and outputs. The class of Σ-admissible $\hat{P}$ is given in Theorem 4.8; the class of all H-stabilizing compensators for Σ-admissible plants is given in Theorem 4.10. The 2-2 block of $\hat{C}$ is essentially in a feedback configuration like $S(P,C)$.

In the unity-feedback configuration $S(P,C)$, the class of all C that H-stabilize P is parametrized by one parameter matrix Q; including this parameter matrix Q that comes from C, the set of all $\hat{C}$ that H-stabilize $\hat{P}$ is parametrized by *four* H-stable matrices and hence, we call the system $\Sigma(\hat{P},\hat{C})$ a four-degrees-of-freedom design (or four-parameter design) [Net.1]. $\Sigma(\hat{P},\hat{C})$ can obviously be reduced to two parameter design by taking $C_{11}=0$ and $C_{12}=0$. The class of all achievable maps for $\Sigma(\hat{P},\hat{C})$ involves the four compensator parameters; *each* closed-loop I/O map achieved by the H-stabilized $\Sigma(\hat{P},\hat{C})$ depends on one and *only one* of these four parameter matrices Q_{11}, Q_{12}, Q_{21}, Q. Several performance specifications can be imposed on the closed-loop performance of $\Sigma(\hat{P},\hat{C})$.

In Section V, we consider the decoupling problem; namely, find $\hat{C}$ such that, for the given $\hat{P}$, the I/O map $H_{zv'}: v' \mapsto z$ of $\Sigma(\hat{P},\hat{C})$ is diagonal. Assuming that N_{12} is nonsingular, it is always possible to choose $Q_{21} \in \mathcal{M}(H)$ such that $H_{zv'} = N_{12}Q_{21}$ is diagonal. Diagonalization with this configuration does not involve the feedback-loop and the parameter Q of C; hence, decoupling the I/O map $H_{zv'}$ is independent of the I/O maps that are affine functions in Q. On the other hand, in the unity-feedback configuration $S(P,C)$, diagonalizing the map $H_{yu'}$: $u' \mapsto y$ would depend on the choice for Q such that $N_p(U_p + Q\tilde{D}_p)$ is diagonal, and hence, diagonalizing the map $H_{yu'}$ in $S(P,C)$ may not be possible for certain plants.

II. ALGEBRAIC BACKGROUND

2.1. Notation [Lan.1, Vid.1]: H is a principal ring (i.e., an entire commutative ring in which every ideal is principal). $\mathcal{M}(H)$ is the set of matrices with elements in H. $J \subset H$ is the group of units of H. $I \subset H$ is a multiplicative subset, $0 \notin I$, $1 \in I$. $G = H/I := \{ n/d : n \in H, d \in I \}$ is the ring of fractions of H associated with I. G_S is the Jacobson radical of G;
$G_S := \{ x \in G : (1+xy)^{-1} \in G, \text{ for all } y \in G \}$.

2.2. Example (Rational functions in s) : Let $\mathcal{U} \supset \mathbb{C}_+$ be a closed subset of $\mathbb{C}$, symmetric about the real axis, and let $\mathbb{C}\setminus\mathcal{U}$ be nonempty; let $\mathcal{U}_e := \mathcal{U} \cup \{\infty\}$. The ring of proper scalar rational functions (with real coefficients) which are analytic in $\mathcal{U}$, denoted by $R_{\mathcal{U}}(s)$, is a principal ring. Let H be $R_{\mathcal{U}}(s)$; by definition of J, $f \in J$ implies that f is a proper rational function, which has neither poles nor zeros in $\mathcal{U}_e$. We choose I to be the multiplicative subset of $R_{\mathcal{U}}(s)$ such that $f \in I$ implies that $f(\infty)$ is a nonzero constant in $\mathbb{R}$; equivalently, $I \subset R_{\mathcal{U}}(s)$ is the set of proper, but not *strictly proper*, real rational functions which are analytic in $\mathcal{U}$. Then $R_{\mathcal{U}}(s)/I$ is the ring of *proper* rational functions $\mathbb{R}_p(s)$. The Jacobson radical of $\mathbb{R}_p(s)$ is the set of strictly proper rational functions $\mathbb{R}_{sp}(s)$.

2.3. Definitions (Coprime factorizations in H):
(i) The pair (N_p, D_p), where $N_p, D_p \in \mathcal{M}(H)$, is called *right-coprime* (r.c.) iff there exist $U_p, V_p \in \mathcal{M}(H)$ such that $V_pD_p + U_pN_p = I$; (ii) the pair (N_p, D_p) is called a *right-fraction representation* (r.f.r.) of $P \in \mathcal{M}(G)$ iff D_p is square, $\det D_p \in I$ and $P = N_pD_p^{-1}$; (iii) the pair (N_p, D_p) is called a *right-coprime-fraction representation* (r.c.f.r.) of $P \in \mathcal{M}(G)$ iff (N_p, D_p) is an r.f.r. of P and (N_p, D_p) is r.c. (iv) The pair $(\tilde{D}_p, \tilde{N}_p)$, where $\tilde{D}_p, \tilde{N}_p \in \mathcal{M}(H)$, is called *left-coprime* (l.c.) iff there exist $\tilde{U}_p, \tilde{V}_p \in \mathcal{M}(H)$ such that

$\tilde N_p \tilde U_p + \tilde D_p \tilde V_p = I$; (v) the pair $(\tilde D_p, \tilde N_p)$ is called a *left-fraction representation* (l.f.r.) of $P \in \mathcal{M}(G)$ iff $\tilde D_p$ is square, $\det \tilde D_p \in \mathcal{I}$ and $P = \tilde D_p^{-1} \tilde N_p$; (vi) the pair $(\tilde D_p, \tilde N_p)$ is called a *left-coprime-fraction representation* (l.c.f.r.) of $P \in \mathcal{M}(G)$ iff $(\tilde D_p, \tilde N_p)$ is an l.f.r. of P and $(\tilde D_p, \tilde N_p)$ is l.c. (vii) The triple (N_{pr}, D, N_{pl}), where N_{pr}, D, $N_{pl} \in \mathcal{M}(H)$, is called a *bicoprime-fraction representation* (b.c.f.r.) of $P \in \mathcal{M}(G)$ iff the pair (N_{pr}, D) is *right-coprime*, the pair (D, N_{pl}) is *left-coprime*, $\det D \in \mathcal{I}$ and $P = N_{pr} D^{-1} N_{pl}$. Note that $P \in \mathcal{M}(G)$ is sometimes given as $P = N_{pr} D^{-1} N_{pl} + S_p$, where $S_p \in \mathcal{M}(H)$ and (N_{pr}, D, N_{pl}) is a bicoprime (b.c.) triple. In this case, the b.c.f.r. is given by (N_{pr}, D, N_{pl}, S_p) [Vid.1]. □

Every $P \in \mathcal{M}(G)$ has an r.c.f.r. (N_p, D_p), an l.c.f.r. $(\tilde D_p, \tilde N_p)$, and a b.c.f.r. (N_{pr}, D, N_{pl}) in H.

2.4. Generalized Bezout Identity for (N_p, D_p) and $(\tilde D_p, \tilde N_p)$: Let (N_p, D_p) be an r.c. pair and let $(\tilde D_p, \tilde N_p)$ be an l.c. pair, and let $\tilde N_p D_p = \tilde D_p N_p$, where $N_p \in H^{n_o \times n_i}$, $D_p \in H^{n_i \times n_i}$, $\tilde D_p \in H^{n_o \times n_o}$, $\tilde N_p \in H^{n_o \times n_i}$; then there are matrices $V_p, U_p, \tilde U_p, \tilde V_p \in \mathcal{M}(H)$ such that

$$\begin{bmatrix} V_p & U_p \\ -\tilde N_p & \tilde D_p \end{bmatrix} \begin{bmatrix} D_p & -\tilde U_p \\ N_p & \tilde V_p \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{2.1}$$

2.5. Definition (Doubly-coprime fraction representation): (i) If the generalized Bezout identity (2.1) holds, then $((N_p, D_p), (\tilde D_p, \tilde N_p))$ is called a *doubly-coprime* pair. (ii) If $P = N_p D_p^{-1} = \tilde D_p^{-1} \tilde N_p$, then $((N_p, D_p), (\tilde D_p, \tilde N_p))$ is called a *doubly-coprime-fraction* representation of P.

2.6. Generalized Bezout identities for (N_{pr}, D, N_{pl}): Let (N_{pr}, D, N_{pl}) be a b.c. triple, where $N_{pr} \in H^{n_o \times n}$, $D \in H^{n \times n}$, $N_{pl} \in H^{n \times n_i}$; then we have two generalized Bezout identities: (i) For the r.c. pair (N_{pr}, D), there are matrices $V_{pr}, U_{pr}, \tilde X, \tilde Y, \tilde U, \tilde V \in \mathcal{M}(H)$ such that

$$\begin{bmatrix} V_{pr} & U_{pr} \\ -\tilde X & \tilde Y \end{bmatrix} \begin{bmatrix} D & -\tilde U \\ N_{pr} & \tilde V \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_o} \end{bmatrix}; \tag{2.2}$$

equation (2.2) is of the form

$$M_r M_r^{-1} = I_{n+n_o}. \tag{2.3}$$

(ii) For the l.c. pair (D, N_{pl}) there are matrices $V_{pl}, U_{pl}, X, Y, U, V \in \mathcal{M}(H)$ such that

$$\begin{bmatrix} D & -N_{pl} \\ U & V \end{bmatrix} \begin{bmatrix} V_{pl} & X \\ -U_{pl} & Y \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_i} \end{bmatrix}; \tag{2.4}$$

equation (2.4) is of the form

$$M_l M_l^{-1} = I_{n+n_i}. \tag{2.5}$$

2.7. Proposition: Let $P \in \mathcal{M}(G)$. Let (N_{pr}, D, N_{pl}) be a b.c.f.r. of P; hence, equations (2.2)-(2.4) hold; then

$$(N_p, D_p) := (N_{pr} X, Y) \text{ is an r.c.f.r. of } P, \tag{2.6}$$

$$(\tilde D_p, \tilde N_p) := (\tilde Y, \tilde X N_{pl}) \text{ is an l.c.f.r. of } P, \tag{2.7}$$

where $X, Y, \tilde X, \tilde Y \in \mathcal{M}(H)$ are defined in (2.2)-(2.4).

2.8. Comments: (i) Using equations (2.2)-(2.4) we obtain a generalized Bezout identity for the doubly-coprime pair $((N_{pr} X, Y), (\tilde Y, \tilde X N_{pl}))$:

$$\begin{bmatrix} V + U V_{pr} N_{pl} & U U_{pr} \\ -\tilde X N_{pl} & \tilde Y \end{bmatrix} \begin{bmatrix} Y & -U_{pl} \tilde U \\ N_{pr} X & \tilde V + N_{pr} V_{pl} \tilde U \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix} \tag{2.8}$$

Note the similarity between equations (2.8) and (2.1). (ii) If, instead of $N_{pr} D^{-1} N_{pl}$, the plant is given by $P = N_{pr} D^{-1} N_{pl} + S_p$, where $S_p \in \mathcal{M}(H)$, then an r.c.f.r. and an l.c.f.r. are given by:

$$(N_p, D_p) := (N_{pr} X + S_p Y, Y), (\tilde D_p, \tilde N_p) := (\tilde Y, \tilde X N_{pl} + \tilde Y S_p),$$

and the Bezout identities in (2.8) are replaced by:

$$(V + U V_{pr} N_{pl} - U U_{pr} S_p) Y + U U_{pr} (N_{pr} X + S_p Y) = I_{n_i}$$

$$(-\tilde X N_{pl} - \tilde Y S_p)(-U_{pl} \tilde U) + \tilde Y (\tilde V + N_{pr} V_{pl} \tilde U - S_p U_{pl} \tilde U) = I_{n_o}.$$

2.9. Example: Let H be $R_{\mathcal{U}}(s)$ as in Example 2.2. Let $P \in \mathbf{R}_p(s)^{n_o \times n_i}$ be represented by its state-space representation $\dot x = Ax + Bu$, $y = Cx$, where (C, A, B) is $\mathcal{U}_e$-stabilizable and $\mathcal{U}_e$-detectable. Then $P = (s+a)^{-1} C [(s+a)^{-1}(sI - A)]^{-1} B$, where $-a \in \mathbf{C} \setminus \mathcal{U}_e$, $-a \in \mathbf{R}$. The pair $((s+a)^{-1} C, (s+a)^{-1}(sI - A))$ is r.c., the pair $((s+a)^{-1}(sI - A), B)$ is l.c. and $\det[(s+a)^{-1}(sI - A)] \in \mathcal{I}$. Therefore, $(N_{pr}, D, N_{pl}) = ((s+a)^{-1} C, (s+a)^{-1}(sI - A), B)$ is a b.c.f.r. of P. Choose $K \in \mathbf{R}^{n_i \times n}$ and $F \in \mathbf{R}^{n \times n_o}$ such that $(A - BK)$ and $(A - FC)$ have eigenvalues in $\mathbf{C} \setminus \mathcal{U}_e$. Let $G_K := (sI_n - A + BK)^{-1}$; let $G_F := (sI_n - A + FC)^{-1}$; then G_K, $G_F \in \mathcal{M}(R_{\mathcal{U}}(s)) \cap \mathcal{M}(\mathbf{R}_{sp}(s))$ and hence, $(s+a)(sI_n - A + BK)^{-1} = (s+a) G_K \in \mathcal{M}(R_{\mathcal{U}}(s))$ and $(s+a)(sI_n - A + FC)^{-1} = (s+a) G_F \in \mathcal{M}(R_{\mathcal{U}}(s))$. For this b.c.f.r., (2.2) and (2.4) become:

$$\begin{bmatrix} (s+a) G_F & (s+a) G_F F \\ -C G_F & I_{n_o} - C G_F F \end{bmatrix} \begin{bmatrix} (s+a)^{-1}(sI_n - A) & -F \\ (s+a)^{-1} C & I_{n_o} \end{bmatrix} = I_{n+n_o};$$

$$\begin{bmatrix} (s+a)^{-1}(sI_n - A) & -B \\ (s+a)^{-1} K & I_{n_i} \end{bmatrix} \begin{bmatrix} (s+a) G_K & (s+a) G_K B \\ -K G_K & I_{n_i} - K G_K B \end{bmatrix} = I_{n+n_i}.$$

We obtain a Bezout identity for this case from (2.8):

$$\begin{bmatrix} I_{n_i} + K G_F B & K G_F F \\ -C G_F B & I_{n_o} - C G_F F \end{bmatrix} \begin{bmatrix} I_{n_i} - K G_K B & -K G_K F \\ C G_K B & I_{n_o} + C G_K F \end{bmatrix} = I_{n_i + n_o}.$$

Clearly, $(C G_K B, (I_{n_i} - K G_K B))$ is an r.c. pair and $((I_{n_o} - C G_F F), C G_F B)$ is an l.c. pair.

III. THE UNITY-FEEDBACK SYSTEM $S(P, C)$

We consider the system $S(P, C)$ shown in Figure 1.

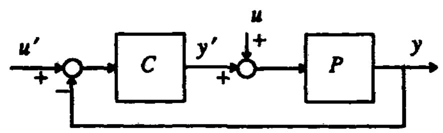

Figure 1. The unity-feedback system $S(P, C)$.

3.1. Assumptions:

(A) The plant $P \in G^{n_o \times n_i}$. Let (N_p, D_p) be an r.c.f.r., $(\tilde D_p, \tilde N_p)$ be an l.c.f.r., (N_{pr}, D, N_{pl}) be a b.c.f.r. of P, where $N_p \in H^{n_o \times n_i}$, $D_p \in H^{n_i \times n_i}$, $\tilde D_p \in H^{n_o \times n_o}$, $\tilde N_p \in H^{n_o \times n_i}$, $N_{pr} \in H^{n_o \times n}$, $D \in H^{n \times n}$, $N_{pl} \in H^{n \times n_i}$.

(B) The compensator $C \in G^{n_i \times n_o}$. Let $(\tilde D_c, \tilde N_c)$ be an l.c.f.r. and (N_c, D_c) be an r.c.f.r. of C, where $\tilde D_c \in H^{n_i \times n_i}$, $\tilde N_c \in H^{n_i \times n_o}$, $N_c \in H^{n_i \times n_o}$, $D_c \in H^{n_o \times n_o}$.

Let $\bar y := \begin{bmatrix} y \\ y' \end{bmatrix}$, $\bar u := \begin{bmatrix} u \\ u' \end{bmatrix}$. The map $H_{\bar y \bar u} : \bar u \mapsto \bar y$ is the I/O map of $S(P, C)$. In terms of P and C, $H_{\bar y \bar u}$ is given by

$$H_{\bar y \bar u} = \begin{bmatrix} P(I_{n_i} + CP)^{-1} & P(I_{n_i} + CP)^{-1} C \\ -CP(I_{n_i} + CP)^{-1} & (I_{n_i} + CP)^{-1} C \end{bmatrix}. \tag{3.1}$$

3.2. Analysis of $S(P, C)$: The system $S(P, C)$ can be analyzed by using an r.c.f.r., an l.c.f.r., or a b.c.f.r. of P and C. We show the analysis for a b.c.f.r. of P and an l.c.f.r. of C: Let $P = N_{pr} D^{-1} N_{pl}$, $C = \tilde{D}_c^{-1} \tilde{N}_c$, where (N_{pr}, D, N_{pl}) is b.c., $(\tilde{D}_c, \tilde{N}_c)$ is l.c. (see Figure 2); ξ_x denotes the pseudo-state of P.

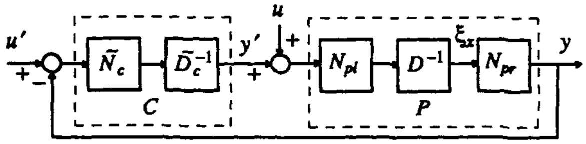

Figure 2. $S(P, C)$ with $P = N_{pr} D^{-1} N_{pl}$ and $C = \tilde{D}_c^{-1} \tilde{N}_c$.

$S(P, C)$ is then described by equations (3.2)-(3.3):

$$\begin{bmatrix} D & \vdots & -N_{pl} \\ \tilde{N}_c N_{pr} & \vdots & \tilde{D}_c \end{bmatrix} \begin{bmatrix} \xi_x \\ y' \end{bmatrix} = \begin{bmatrix} N_{pl} & \vdots & 0 \\ 0 & \vdots & \tilde{N}_c \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} N_{pr} & \vdots & 0 \\ 0 & \vdots & I_{n_i} \end{bmatrix} \begin{bmatrix} \xi_x \\ y' \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}. \tag{3.3}$$

Equations (3.2)-(3.3) are of the form $D_{H3}\xi_3 = N_{L3}\bar{u}$, $N_{R3}\xi_3 = \bar{y}$. If $\det D_{H3} \in \mathcal{I}$, then the system is well-posed; by elementary row and column operations on the matrices in equations (3.2)-(3.3), it is easy to see that (N_{R3}, D_{H3}, N_{L3}) is a b.c.f.r. of $H_{\bar{y}\bar{u}}$.

3.3. Definition (H-stability): The system $S(P, C)$ is said to be *H-stable* iff $H_{\bar{y}\bar{u}} \in \mathfrak{m}(H)$.

3.4. Theorem (H-stability of $S(P, C)$): Consider $S(P, C)$. Let Assumptions 3.1 (A) and (B) hold; then (i)-(v) below are equivalent:

(i) $S(P, C)$ is H-stable;

(ii) $D_{H1} := \tilde{D}_c D_p + \tilde{N}_c N_p$ is H-unimodular; (3.4)

(iii) $D_{H2} := \tilde{D}_p D_c + \tilde{N}_p N_c$ is H-unimodular; (3.5)

(iv) $D_{H3} := \begin{bmatrix} D & -N_{pl} \\ \tilde{N}_c N_{pr} & \tilde{D}_c \end{bmatrix}$ is H-unimodular; (3.6)

(v) $D_{H4} := \begin{bmatrix} D & -N_{pl} N_c \\ N_{pr} & D_c \end{bmatrix}$ is H-unimodular. (3.7)

3.5. Comments: (i) Post-multiplying D_{H3} in (3.6) by the H-unimodular matrix M_l^{-1} defined in (2.4)-(2.5), we obtain

$$D_{H3} M_l^{-1} = \begin{bmatrix} I_n & 0 \\ \tilde{N}_c N_{pr} V_{pl} - \tilde{D}_c U_{pl} & \tilde{N}_c N_{pr} X + \tilde{D}_c Y \end{bmatrix}.$$

But D_{H3} is H-unimodular if and only if $D_{H3} M_l^{-1}$ is H-unimodular; hence, condition (3.6) holds if and only if

$$\tilde{D}_c Y + \tilde{N}_c N_{pr} X \quad \text{is } H\text{-unimodular}. \tag{3.8}$$

The H-unimodularity condition (3.8) is the same as (3.4) since $(N_{pr} X, Y)$ is an r.c.f.r. of P by Proposition 2.7. Similarly, pre-multiplying D_{H4} in (3.7) by the H-unimodular matrix M_r defined (2.2)-(2.3), we conclude that (3.7) holds if and only if

$$\tilde{X} N_{pl} N_c + \tilde{Y} D_c \quad \text{is } H\text{-unimodular}. \tag{3.9}$$

Note that condition (3.9) is the same as (3.5) since $(\tilde{Y}, \tilde{X} N_{pl})$ is an l.c.f.r. of P by Proposition 2.7. (ii) If condition (3.4) (equivalently, (3.5)) holds, then by normalization we obtain

$$\tilde{D}_c D_p + \tilde{N}_c N_p = I_{n_i}, \text{ and } \tilde{N}_p N_c + \tilde{D}_p D_c = I_{n_o}. \tag{3.10}$$

With $P = N_p D_p^{-1} = \tilde{D}_p^{-1} \tilde{N}_p$, $C = \tilde{D}_c^{-1} \tilde{N}_c = N_c D_c^{-1}$, equation (3.10) is equivalent to

$$\begin{bmatrix} \tilde{D}_c & \tilde{N}_c \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \begin{bmatrix} D_p & -N_c \\ N_p & D_c \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ 0 & I_{n_o} \end{bmatrix}. \tag{3.11}$$

3.6. Definition (H-stabilizing compensator C): (i) C is called an *H-stabilizing compensator for P* (later abbreviated as: C *H-stabilizes* P) iff $C \in G^{n_i \times n_o}$ satisfies Assumption 3.1 (B) *and* the system $S(P, C)$ is H-stable. (ii) The set $\mathbf{S}(P) := \{ C : C \ H\text{-stabilizes } P \}$ is called the *set of all H-stabilizing compensators for P*.

3.7. Theorem (Set of all H-stabilizing compensators for P): Let $P \in \mathfrak{m}(G_S)$ and let P satisfy Assumption 3.1 (A); then the set $\mathbf{S}(P)$ of all H-stabilizing compensators C for P is given by equation (3.12) and equivalently, by equation (3.13) below:

$$\mathbf{S}(P) = \{ C = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p) : Q \in \mathfrak{m}(H) \}; \tag{3.12}$$

$$\mathbf{S}(P) = \{ C = (\tilde{U}_p + D_p Q)(\tilde{V}_p - N_p Q)^{-1} : Q \in \mathfrak{m}(H) \}; \tag{3.13}$$

where the matrices V_p, U_p, $\tilde{V}_p$, $\tilde{U}_p$ in equations (3.12)-(3.13) satisfy the generalized Bezout identity (2.1). Equations (3.12) and (3.13) give a *parametrization* of all H-stabilizing compensators for P; in each case, the map $Q \mapsto C$ is bijective and, for the same $Q \in \mathfrak{m}(H)$, (3.12) and (3.13) give the same C.

3.8. Comments: (i) (All H-stabilizing compensators based on a b.c.f.r. of P): By Proposition 2.7, $(N_{pr} X, Y)$ is an r.c.f.r. and $(\tilde{Y}, \tilde{X} N_{pl})$ is an l.c.f.r. of P; then set $\mathbf{S}(P)$ of all H-stabilizing compensators is given by:

$$\mathbf{S}(P) = \{ (V + UV_{pr} N_{pl} - Q\tilde{X} N_{pl})^{-1}(UU_{pr} + Q\tilde{Y}) \}, \tag{3.14}$$

$$\mathbf{S}(P) = \{ (U_{pl}\tilde{U} + YQ)(\tilde{V} + N_{pr} V_{pl} \tilde{U} - N_{pr} XQ)^{-1} \}, \tag{3.15}$$

where $Q \in \mathfrak{m}(H)$ and the matrices in equations (3.14)-(3.15) satisfy the generalized Bezout identities (2.2) and (2.4).

A generalized Bezout identity for the doubly-coprime pair $((N_{pr} X, Y), (\tilde{Y}, \tilde{X} N_{pl}))$ is given by (2.8); comparing (2.8) and (2.2), it is easy to see that (3.14) is equivalent to (3.12) and (3.15) is equivalent to (3.13). **(ii) (All H-stabilizing compensators for H-stable P):** If $P \in \mathfrak{m}(H)$, then the set $\mathbf{S}(P)$ of all H-stabilizing compensators is given by:

$$\mathbf{S}(P) = \{ C = (I_{n_i} - QP)^{-1} Q : Q \in \mathfrak{m}(H) \},$$

$$\mathbf{S}(P) = \{ C = Q(I_{n_o} - PQ)^{-1} : Q \in \mathfrak{m}(H) \}.$$

(iii) (All H-stabilizing compensators when $P \in \mathfrak{m}(G)$): In Theorem 3.7, if we assume that $P \in \mathfrak{m}(G)$ but not $\mathfrak{m}(G_S)$, then in equations (3.12)-(3.13) (and equivalently, (3.14)-(3.15)) we choose $Q \in \mathfrak{m}(H)$ such that $\det(V_p - Q\tilde{N}_p) \in \mathcal{I}$ (equivalently, $\det(\tilde{V}_p - N_p Q) \in \mathcal{I}$).

(iv) (All P such that $S(P, C)$ is H-stable): Let $C \in \mathfrak{m}(G_S)$, $C = \tilde{D}_c^{-1} \tilde{N}_c = N_c D_c^{-1}$, be given; let $(\tilde{D}_c, \tilde{N}_c)$ be l.c. and (N_c, D_c) be r.c. Under these conditions, the set of all $P \in \mathfrak{m}(G)$ for which $S(P, C)$ is H-stable is given by:

$\{ P = (\tilde{U}_c + D_c Q_p)(\tilde{V}_c - N_c Q_p)^{-1} : Q_p \in \mathfrak{m}(H) \} =$
$\{ P = (V_c - Q_p \tilde{N}_c)^{-1}(U_c + Q_p \tilde{D}_c) : Q_p \in \mathfrak{m}(H) \}$,

where V_c, U_c, $\tilde{V}_c$, $\tilde{U}_c \in \mathfrak{m}(H)$ satisfy a generalized Bezout identity for the doubly-coprime pair $((N_c, D_c), (\tilde{D}_c, \tilde{N}_c))$. If $C \in \mathfrak{m}(G)$, then $Q_p \in \mathfrak{m}(H)$ should be chosen so that $\det(\tilde{V}_c - N_c Q_p) \in \mathcal{I}$ (equivalently, $\det(V_c - Q_p \tilde{N}_c) \in \mathcal{I}$).

3.9. Achievable I/O maps of $S(P, C)$: The set $\mathbf{A}(P) := \{ H_{\bar{y}\bar{u}} : C \ H\text{-stabilizes } P \}$ is called the set of *all achievable I/O maps* of the unity-feedback system $S(P, C)$.

By Theorem 3.7, the compensator C H-stabilizes P if and only if $C \in \mathbf{S}(P)$. Substituting $\tilde{D}_c^{-1}\tilde{N}_c = (V_p - Q\tilde{N}_p)^{-1}(U_p + Q\tilde{D}_p)$ or $N_c D_c^{-1} = (\tilde{U}_p + D_p Q)(\tilde{V}_p - N_p Q)^{-1}$ for C into (3.1), we obtain the set of all achievable I/O maps:

$$\mathrm{A}(P) = \{\ H_{\bar{y}\bar{u}} = \begin{bmatrix} N_p(V_p - Q\tilde{N}_p) & N_p(U_p + Q\tilde{D}_p) \\ -(\tilde{U}_p + D_p Q)\tilde{N}_p & D_p(U_p + Q\tilde{D}_p) \end{bmatrix}\ \},$$

where $Q \in \mathcal{M}(H)$. Note that each closed-loop map of $S(P, C)$ is an affine map in the parameter matrix $Q \in \mathcal{M}(H)$.

Compensator design using $S(P, C)$ is called one-degree-of-freedom design or one-parameter design since all achievable maps are parametrized by the single parameter matrix Q.

IV. THE FEEDBACK SYSTEM $\Sigma(\hat{P}, \hat{C})$

Consider the feedback system $\Sigma(\hat{P}, \hat{C})$ shown in Figure 3.

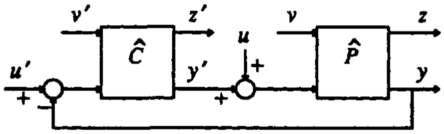

Figure 3. The feedback system $\Sigma(\hat{P}, \hat{C})$.

4.1. Assumptions:

(A) The $(\eta_o + n_o)\mathrm{x}(\eta_i + n_i)$ plant $\hat{P} \in \mathcal{M}(G)$ is partitioned as

$$\hat{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P \end{bmatrix} \in G^{(\eta_o+n_o)\mathrm{x}(\eta_i+n_i)}, \text{ where } P \in G^{n_o \mathrm{x} n_i}.$$

(B) The compensator $\hat{C} \in G^{(\eta_o'+n_i)\mathrm{x}(\eta_i'+n_o)}$ is partitioned as

$$\hat{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C \end{bmatrix} \in G^{(\eta_o'+n_i)\mathrm{x}(\eta_i'+n_o)}, \text{ where } C \in G^{n_i \mathrm{x} n_o}.$$

4.2. Fact: (i) Let the plant $\hat{P}$ satisfy Assumption 4.1 (A); then $\hat{P}$ has an r.c.f.r. $(N_{\hat{P}}, D_{\hat{P}})$ and an l.c.f.r. $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ which satisfy equations (4.1)-(4.2) below:

$$\text{(i)}\ (N_{\hat{P}}, D_{\hat{P}}) =: \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_p \end{bmatrix}, \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_p \end{bmatrix} \right), \tag{4.1}$$

$$\text{(ii)}\ (\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}}) =: \left(\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_p \end{bmatrix}, \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix} \right), \tag{4.2}$$

where (N_p, D_p) is an r.f.r. of P, and $(\tilde{D}_p, \tilde{N}_p)$ is an l.f.r. of P.
(ii) Let the compensator $\hat{C}$ satisfy Assumption 4.1 (B); then $\hat{C}$ has an l.c.f.r. $(\tilde{D}_{\hat{C}}, \tilde{N}_{\hat{C}})$ and an r.c.f.r. $(N_{\hat{C}}, D_{\hat{C}})$ which satisfy equations (4.3)-(4.4) below:

$$(\tilde{D}_{\hat{C}}, \tilde{N}_{\hat{C}}) =: \left(\begin{bmatrix} \tilde{D}'_{11} & \tilde{D}'_{12} \\ 0 & \tilde{D}_c \end{bmatrix}, \begin{bmatrix} \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{21} & \tilde{N}_c \end{bmatrix} \right), \tag{4.3}$$

$$(N_{\hat{C}}, D_{\hat{C}}) =: \left(\begin{bmatrix} N'_{11} & N'_{12} \\ N'_{21} & N_c \end{bmatrix}, \begin{bmatrix} D'_{11} & 0 \\ D'_{21} & D_c \end{bmatrix} \right), \tag{4.4}$$

where $(\tilde{D}_c, \tilde{N}_c)$ is an l.f.r. of C, and (N_c, D_c) is an r.f.r. of C. □

Any *other* r.c.f.r. of $\hat{P}$ is given by $(N_{\hat{P}}R, D_{\hat{P}}R)$, where $(N_{\hat{P}}, D_{\hat{P}})$ is the r.c.f.r in (4.1) and $R \in \mathcal{M}(H)$ is H-unimodular. Similarly, any other l.c.f.r. of $\hat{P}$ is given by $(L\tilde{D}_{\hat{P}}, L\tilde{N}_{\hat{P}})$, where $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ is the l.c.f.r. in (4.2) and $L \in \mathcal{M}(H)$ is H-unimodular. The pair (N_p, D_p) in (4.1) is *not* necessarily r.c.; the pair $(\tilde{D}_p, \tilde{N}_p)$ in (4.2) is *not* necessarily l.c.

Let $\hat{y} := \begin{bmatrix} z \\ y \\ z' \\ y' \end{bmatrix}$, $\hat{u} := \begin{bmatrix} v \\ u \\ v' \\ u' \end{bmatrix}$. The map $H_{\hat{y}\hat{u}} : \hat{u} \mapsto \hat{y}$ is the I/O map of $\Sigma(\hat{P}, \hat{C})$. In terms of $\hat{P}$ and $\hat{C}$, $H_{\hat{y}\hat{u}}$ is given by

$$\begin{bmatrix} P_{11} - P_{12}T^{-1}CP_{21} & P_{12}T^{-1} & P_{12}T^{-1}C_{21} & P_{12}T^{-1}C \\ \hat{T}P_{21} & PT^{-1} & PT^{-1}C_{21} & PT^{-1}C \\ -C_{12}\hat{T}P_{21} & -C_{12}PT^{-1} & C_{11} - C_{12}PT^{-1}C_{21} & C_{12}\hat{T} \\ -T^{-1}CP_{21} & T^{-1} - I_{n_i} & T^{-1}C_{21} & T^{-1}C \end{bmatrix}$$

where $T := (I_{n_i} + CP)$ and $\hat{T} := (I_{n_o} - PT^{-1}C)$.

4.2. Analysis of $\Sigma(\hat{P}, \hat{C})$: We analyze the system $\Sigma(\hat{P}, \hat{C})$ by factorizing $\hat{P}$ as $N_{\hat{P}}D_{\hat{P}}^{-1}$ and $\hat{C}$ as $\tilde{D}_{\hat{C}}^{-1}\tilde{N}_{\hat{C}}$; $\hat{\xi}_p$ denotes the pseudo-state of $\hat{P}$. $\Sigma(\hat{P}, \hat{C})$ is then described by (4.5)-(4.6):

$$\begin{bmatrix} D_{11} & 0 & 0 & 0 \\ D_{21} & D_p & 0 & -I_{n_i} \\ \tilde{N}'_{12}N_{21} & \tilde{N}'_{12}N_p & \tilde{D}'_{11} & \tilde{D}'_{12} \\ \tilde{N}_c N_{21} & \tilde{N}_c N_p & 0 & \tilde{D}_c \end{bmatrix} \begin{bmatrix} \hat{\xi}_p \\ \cdots \\ z' \\ y' \end{bmatrix} = \begin{bmatrix} I_{\eta_i+n_i} & 0 \\ 0 & \tilde{N}_{\hat{C}} \end{bmatrix} \begin{bmatrix} v \\ u \\ \cdots \\ v' \\ u' \end{bmatrix} \tag{4.5}$$

$$\begin{bmatrix} N_{\hat{P}} & 0 \\ 0 & I_{\eta_o'+n_i} \end{bmatrix} \begin{bmatrix} \hat{\xi}_p \\ z' \\ y' \end{bmatrix} = \begin{bmatrix} z \\ y \\ z' \\ y' \end{bmatrix}. \tag{4.6}$$

(4.5)-(4.6) are of the form $\hat{D}_H\hat{\xi} = \hat{N}_L\hat{u}$, $\hat{N}_R\hat{\xi} = \hat{y}$; it is easy to see that $(\hat{N}_R, \hat{D}_H, \hat{N}_L)$ is a b.c. triple. If $\det\hat{D}_H \in I$, then the I/O map $H_{\hat{y}\hat{u}}$ is given by $H_{\hat{y}\hat{u}} = \hat{N}_R\hat{D}_H^{-1}\hat{N}_L \in \mathcal{M}(G)$.

4.3. Definition (H-stability): The system $\Sigma(\hat{P}, \hat{C})$ is said to be H-*stable* iff $H_{\hat{y}\hat{u}} \in \mathcal{M}(H)$.

4.4. Theorem (H-stability of $\Sigma(\hat{P}, \hat{C})$): Consider $\Sigma(\hat{P}, \hat{C})$. Let Assumptions 4.1 (A)-(B) hold; then (i)-(iii) below are equivalent:

(i) $\Sigma(\hat{P}, \hat{C})$ is H-*stable*;

(ii) $\hat{D}_H$ is H-unimodular; (4.7)

(iii) D_{11} is H-unimodular , and (4.8)

$\tilde{D}'_{11}$ is H-unimodular , and (4.9)

$\tilde{D}_c D_p + \tilde{N}_c N_p$ is H-unimodular. (4.10)

4.5. Comments: (i) Condition (4.7) of Theorem 4.4 is equivalent to $\det\hat{D}_H \in J$; by equation (4.3),

$$\det\hat{D}_H = \det D_{11}\det\tilde{D}'_{11}\det(\tilde{D}_c D_p + \tilde{N}_c N_p). \tag{4.11}$$

Now $\det\hat{D}_H \in J$ if and only if each of the three factors in (4.11) is in J; hence, by (4.1) and (4.3), $\det\hat{D}_H \in J$ if and only if $\det D_{11} = \det D_{\hat{P}}(\det D_p)^{-1} \in J$ (equivalently, $\det D_{\hat{P}} \approx \det D_p$), and $\det\tilde{D}'_{11} = \det\tilde{D}_{\hat{C}}(\det\tilde{D}_c)^{-1} \in J$ (equivalently, $\det\tilde{D}_{\hat{C}} \approx \det\tilde{D}_c$), and $\det(\tilde{D}_c D_p + \tilde{N}_c N_p) \in J$ (equivalently, $\det(\tilde{D}_c D_p + \tilde{N}_c N_p) \approx 1$). Due to (4.11), condition (4.7) of is equivalent to conditions (4.8)-(4.9)-(4.10). (ii) By normalization, conditions (4.8)-(4.9)-(4.10) of can be written as:

$$D_{11} = I_{\eta_i} \text{ and } \tilde{D}'_{11} = I_{\eta_o'} \text{ and } \tilde{D}_c D_p + \tilde{N}_c N_p = I_{n_i}. \tag{4.12}$$

The last condition in equation (4.12) is in fact a right-Bezout identity for the r.c.f.r. (N_p, D_p) of P and a left-Bezout identity for the l.c.f.r. $(\tilde{D}_c, \tilde{N}_c)$ of C. (iii) From equation (4.11), using $\det(I_{n_i} + CP) = \det(I_{n_o} + PC)$, we can express $\det\hat{D}_H$ also as:

$$\det\hat{D}_H = \det D_{11}\det D_p\det\tilde{D}'_{11}\det\tilde{D}_c\det(I_{n_o} + PC). \tag{4.13}$$

Now using equations (4.1)-(4.4), we obtain $\det D_{\hat{P}} \approx \det\tilde{D}_{\hat{P}}$ (equivalently, $\det D_{11}\det D_p \approx \det\tilde{D}_{11}\det\tilde{D}_p$) and $\det\tilde{D}_{\hat{C}} \approx \det D_{\hat{C}}$ (equivalently, $\det\tilde{D}'_{11}\det\tilde{D}_c \approx \det D'_{11}\det D_c$); hence we obtain

$$\det\hat{D}_H = \det\tilde{D}_{11}\det D'_{11}\det(\tilde{D}_p D_c + \tilde{N}_p N_c). \tag{4.14}$$

Therefore, if we analyze the system $\Sigma(\hat{P}, \hat{C})$ with $\hat{P}$ factorized as $\tilde{D}_{\hat{P}}^{-1}\tilde{N}_{\hat{P}}$ and $\hat{C}$ factorized as $N_{\hat{C}}D_{\hat{C}}^{-1}$, by normalization, condition (iii) of Theorem 4.4 is equivalent to

$$\tilde{D}_{11} = I_{\eta_o} \text{ and } D'_{11} = I_{\eta_i'} \text{ and } \tilde{D}_p D_c + \tilde{N}_p N_c = I_{n_o}.$$

(iv) Conditions (4.8)-(4.9)-(4.10) can be interpreted as follows: $\Sigma(\hat{P}, \hat{C})$ is H-stabilized if and only if 1) the only source of "instability" in the plant $\hat{P}$ is D_p (equivalently, $\tilde{D}_p$) 2) *and* the only source of "instability" in the compensator $\hat{C}$ is $\tilde{D}_c$ (equivalently,

$\tilde{D}_p$) 2) *and* the only source of "instability" in the compensator $\hat{C}$ is $\tilde{D}_c$ (equivalently, D_c) 3) *and* the *feedback-loop* (with P and C) is H-stable. Note that the H-stability of the "feedback-loop" is equivalent to the H-stability of the unity-feedback system $S(P, C)$. □

4.6. Definition (H-stabilizing compensator $\hat{C}$): (i) $\hat{C}$ is called an *H-stabilizing compensator for* $\hat{P}$ (later abbreviated as: $\hat{C}$ *H-stabilizes* $\hat{P}$) iff $\hat{C} \in \mathcal{M}(G)$ satisfies Assumption 4.1 (B) *and* the system $\Sigma(\hat{P}, \hat{C})$ is H-stable. (ii) The set

$$\hat{\mathrm{S}}(\hat{P}) := \{\ \hat{C} : \hat{C}\ H\text{-stabilizes}\ \hat{P}\ \}$$

is called the *set of all H-stabilizing compensators for* $\hat{P}$.

4.7. Definition (Σ-admissibility): $\hat{P} \in \mathcal{M}(G)$ is called *Σ-admissible* iff $\hat{P}$ can be H-stabilized by some $\hat{C} \in \mathcal{M}(G)$. □

Let $(N_{\hat{P}}, D_{\hat{P}})$ be an r.c.f.r. of $\hat{P}$; by Theorem 4.4, $\hat{P}$ is Σ-admissible if and only if two conditions are satisfied: 1) $\det D_{\hat{P}} \simeq \det D_p$ and 2) (N_p, D_p) is a right-*coprime*-fraction representation of P . In terms of the l.c.f.r. $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ of $\hat{P}$, again by Theorem 4.4, $\hat{P}$ is Σ-admissible if and only if 1) $\tilde{D}_{\hat{P}} \simeq \det\tilde{D}_p$ and 2) $(\tilde{D}_p, \tilde{N}_p)$ is a left-*coprime*-fraction representation of P .

4.8. Theorem (Class of Σ-admissible $\hat{P}$): Let $\hat{P} \in \mathcal{M}(G)$ satisfy Assumption 4.1 (A); then $\hat{P}$ is Σ-admissible if and only if $\hat{P}$ has an r.c.f.r. in the form given by equation (4.15) and an l.c.f.r. given by equation (4.16) below:

$$(N_{\hat{P}}, D_{\hat{P}}) = (\begin{bmatrix} \hat{N}_{11} & N_{12} \\ \tilde{V}_p\tilde{N}_{21} & N_p \end{bmatrix}, \begin{bmatrix} I_{\eta_i} & 0 \\ -\tilde{U}_p\tilde{N}_{21} & D_p \end{bmatrix}), \qquad (4.15)$$

$$(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}}) = (\begin{bmatrix} I_{\eta_o} & -N_{12}U_p \\ 0 & \tilde{D}_p \end{bmatrix}, \begin{bmatrix} \hat{N}_{11} & N_{12}V_p \\ \tilde{N}_{21} & \tilde{N}_p \end{bmatrix}), \qquad (4.16)$$

where (N_p, D_p) is an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ is an l.c.f.r. of P; the pairs (N_p, D_p) and $(\tilde{D}_p, \tilde{N}_p)$, with $U_p, V_p, \tilde{U}_p, \tilde{V}_p$, satisfy (2.1); $\hat{N}_{11}, N_{12}, \tilde{N}_{21} \in \mathcal{M}(H)$ are free parameter matrices.

4.9. Comments: (i) Suppose that (N_p, D_p) is an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ is an l.c.f.r. of P , and that the generalized Bezout identity (2.1) holds. We generate the class of all Σ-admissible plants by choosing three completely free matrices $\hat{N}_{11}, N_{12}, \tilde{N}_{21} \in \mathcal{M}(H)$ and forming the r.c. pair $(N_{\hat{P}}, D_{\hat{P}})$ in equation (4.15) or the l.c. pair $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ in equation (4.16); with this assignment of $(\tilde{D}_{\hat{P}}, \tilde{N}_{\hat{P}})$ and $(N_{\hat{P}}, D_{\hat{P}})$, $\hat{P} := N_{\hat{P}}D_{\hat{P}}^{-1} = \tilde{D}_{\hat{P}}^{-1}\tilde{N}_{\hat{P}}$ is a Σ-admissible plant. Note that $\det D_{\hat{P}} \in I$ (equivalently, $\det\tilde{D}_{\hat{P}} \in I$) follows from $\det D_p \in I$ (equivalently, $\det\tilde{D}_p \in I$). (ii) Theorem 4.8 states that the class of all Σ-admissible plants is parametrized by only *three* free matrices $\hat{N}_{11}, N_{12}, \tilde{N}_{21} \in \mathcal{M}(H)$. (iii) Suppose that we are given a $\hat{P} \in \mathcal{M}(G)$ satisfying Assumption 4.1 (A), and that the coprime-fraction representations $N_pD_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p$ of P satisfy (2.1); then $\hat{P}$ is Σ-admissible if and only if $P_{11} - P_{12}D_pU_pP_{21} \in \mathcal{M}(H)$ *and* $P_{12}D_p \in \mathcal{M}(H)$ *and* $\tilde{D}_pP_{21} \in \mathcal{M}(H)$.

4.10. Theorem (Set of all H-stabilizing compensators for $\hat{P}$): Let $\hat{P} \in \mathcal{M}(G)$ be Σ-admissible with $P \in \mathcal{M}(G_S)$; let (N_p, D_p) be an r.c.f.r. and $(\tilde{D}_p, \tilde{N}_p)$ be an l.c.f.r. of P, and let the generalized Bezout identity (2.1) hold. Under these conditions, the set $\hat{\mathrm{S}}(\hat{P})$ of all H-stabilizing compensators $\hat{C}$ for $\hat{P}$ is given by equation (4.17) and equivalently, by equation (4.18) below:

$$\hat{\mathrm{S}}(\hat{P}) = \{\hat{C} = \begin{bmatrix} I_{\eta_o'} & -Q_{12}\tilde{N}_p \\ 0 & V_p - Q\tilde{N}_p \end{bmatrix}^{-1} \begin{bmatrix} Q_{11} & Q_{12}\tilde{D}_p \\ Q_{21} & U_p + Q\tilde{D}_p \end{bmatrix}\}, \quad (4.17)$$

$$\hat{\mathrm{S}}(\hat{P}) = \{\hat{C} = \begin{bmatrix} Q_{11} & Q_{12} \\ D_pQ_{21} & \tilde{U}_p + D_pQ \end{bmatrix} \begin{bmatrix} I_{\eta_i'} & 0 \\ -N_pQ_{21} & \tilde{V}_p - N_pQ \end{bmatrix}^{-1}\} \quad (4.18)$$

where $Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)$. Equations (4.17) and (4.18) give a parametrization of all H-stabilizing compensators for $\hat{P}$; each of these equations defines a bijection from $Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)$ to $\hat{C} \in \hat{\mathrm{S}}(\hat{P})$. For the same $(Q_{11}, Q_{12}, Q_{21}, Q)$, equations (4.17)-(4.18) give the same $\hat{C} \in \hat{\mathrm{S}}(\hat{P})$.

4.11. Comments: (i) If H is the ring of proper stable rational functions $R_{\mathcal{U}}(s)$ as in Example 2.2, then the Σ-admissibility of $\hat{P}$ implies that every $\mathcal{U}$-pole of P_{11}, P_{12}, P_{21} is a $\mathcal{U}$-pole of $P = N_pD_p^{-1}$, with *at most* the same McMillan degree [Vid.1, Net.1]. Similarly, for $\hat{C}$ to be an H-stabilizing compensator for $\hat{P}$, the $\mathcal{U}$-poles of C_{11}, C_{12}, C_{21} must be "contained" in the $\mathcal{U}$-poles of $C = \tilde{D}_c^{-1}\tilde{N}_c$, and C must be chosen so that the feedback-loop is H-stable. (ii) The class of all H-stabilizing compensators is *parametrized* by four matrices, $Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)$; the matrix Q parametrizes the class of all C that H-stabilizes the loop $S(P, C)$. Design with the unity-feedback system $S(P, C)$ is *one-degree-of-freedom* design because only one parameter matrix is available for design. In contrast, for the more general system $\Sigma(\hat{P}, \hat{C})$, there are *four-degrees-of-freedom* because $\hat{C}$ has four completely free matrices in H, which can be chosen to meet performance specifications. In Section V, we use the parameter Q_{21} to diagonalize the input-output map $H_{zv'} : v' \mapsto z$.

4.12. Achievable I/O maps of $\Sigma(\hat{P}, \hat{C})$: The set $\hat{\mathrm{A}}(\hat{P}) := \{\ H_{\hat{y}\hat{u}} : \hat{C}\ \ H\text{-stabilizes}\ \hat{P}\ \}$ is called the set of *all achievable I/O maps* of the system $\Sigma(\hat{P}, \hat{C})$.

Substituting for $\hat{C}$ from the expression in equations (4.17) and (4.18) into the closed-loop I/O map $H_{\hat{y}\hat{u}}$, we obtain the set of all achievable I/O maps for $\Sigma(\hat{P}, \hat{C})$: $\hat{\mathrm{A}}(\hat{P}) = \{\ H_{\hat{y}\hat{u}} =$

$$\begin{bmatrix} \hat{N}_{11} - N_{12}Q\tilde{N}_{21} & N_{12}\tilde{D}_c & N_{12}Q_{21} & N_{12}\tilde{N}_c \\ D_c\tilde{N}_{21} & N_p\tilde{D}_c & N_pQ_{21} & N_p\tilde{N}_c \\ -Q_{12}\tilde{N}_{21} & -Q_{12}\tilde{N}_p & Q_{11} & Q_{12}\tilde{D}_p \\ -N_c\tilde{N}_{21} & -N_c\tilde{N}_p & D_pQ_{21} & D_p\tilde{N}_c \end{bmatrix}$$

$: Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)\ \}$, where $D_c := (\tilde{V}_p - N_pQ)$, $\tilde{D}_c := (V_p - Q\tilde{N}_p)$, $N_c := (\tilde{U}_p + D_pQ)$, $\tilde{N}_c := (U_p + Q\tilde{D}_p)$. Each closed-loop map achieved by $\Sigma(\hat{P}, \hat{C})$ depends on only one of *four* free *parameters* $Q_{11}, Q_{12}, Q_{21}, Q \in \mathcal{M}(H)$; in fact, each of these maps is an affine function of *one* parameter only.

If $P_{11} = 0$ and $P_{21} = I_{n_o}$, then v can be viewed as an additive disturbance at the output y; the disturbance-to-output map $H_{yv} : v \mapsto y$ is given by $(\tilde{V}_p - N_pQ)\tilde{N}_{21} = (\tilde{V}_p - N_pQ)\tilde{D}_p$, which depends on the parameter $Q \in \mathcal{M}(H)$. On the other hand, the external-input to output maps $H_{zv'} = N_{12}Q_{21}$ and $H_{yv'} = N_pQ_{21}$ depend on a *different* parameter Q_{21}. Consequently, output shaping and disturbance rejection can be achieved simultaneously, since $H_{zv'}$ and $H_{yv'}$ are *decoupled* from H_{yv}.

V. ACHIEVABLE DIAGONAL MAPS

We now consider the problem of achieving a diagonal I/O map for a Σ-admissible plant $\hat{P}$; more precisely, we require the closed-loop map $H_{zv'} : v' \mapsto z$ from the external-input v' to the output z of the H-stabilized $\Sigma(\hat{P}, \hat{C})$ to be diagonal. We obtain the class of all achievable diagonal maps $H_{zv'}$.

Suppose that $\hat{P} \in \mathcal{M}(G)$, satisfying Assumption 4.1 (A), is a Σ-admissible plant. We assume that and $\eta_i' = n_i = \eta_o$; consequently, $P_{12} \in G^{n_i \times n_i}$ is square since there are n_i inputs v' and n_i outputs z. Furthermore, we assume that $N_{12} \in H^{n_i \times n_i}$ is nonsingular (i.e., $\det N_{12} \neq 0$).

We define two diagonal (nonsingular) matrices Δ_L and Δ_R as follows: (i) Let $\Delta_{Lk} \in H$ be a greatest-common-divisor (g.c.d.) of the elements of the k-th row of N_{12} . Let

$\Delta_{Lk} \in H$ be a greatest-common-divisor (g.c.d.) of the elements of the k-th row of N_{12}. Let

$$\Delta_L := diag\ [\,\Delta_{L1}, \cdots, \Delta_{Ln_i}\,], \tag{5.1}$$

$$N_{12} =: \Delta_L \hat{N}_{12}. \tag{5.2}$$

By construction, $\det\Delta_L \not\equiv 0$. The diagonal elements Δ_{Lk} of Δ_L are unique except for factors in J. (ii) By assumption, $\det N_{12} = \det\Delta_L \det\hat{N}_{12} \not\equiv 0$; hence, $\det\hat{N}_{12} \not\equiv 0$. Write the ij-th entry of $\hat{N}_{12}^{-1}$ as $\dfrac{m_{ij}}{d_{ij}}$, where (m_{ij}, d_{ij}) is a coprime pair in H; note that $d_{ij} \not\equiv 0$ since the denominator of each entry is a factor of $\det\hat{N}_{12}$ (i.e., $\det\hat{N}_{12} = d_{ij} a_{ij}$ for some $a_{ij} \in H$). Let $\Delta_{Rj} \in H$ be a least-common-multiple (l.c.m.) of $\{d_{1j}, \cdots, d_{n_i j}\}$ (i.e., a l.c.m. of the denominators of the elements in the j-th column of $\hat{N}_{12}^{-1}$). Let

$$\Delta_R := diag\ [\,\Delta_{R1}, \cdots, \Delta_{Rn_i}\,]; \tag{5.3}$$

$\det\Delta_R \not\equiv 0$ since $d_{ij} \not\equiv 0$. The entries Δ_{Rj} of Δ_R are unique except for factors in J. Note that if $\hat{N}_{12}^{-1} \in \mathcal{M}(H)$, then $\Delta_R = I_{n_i}$. Now for some $b_{ij} \in H$, $\Delta_{Rj} = d_{ij} b_{ij}$; therefore the ij-th element of $\hat{N}_{12}^{-1}\Delta_R$ is $\dfrac{m_{ij}}{d_{ij}}\Delta_{Rj} = m_{ij} b_{ij} \in H$, and hence,

$$\hat{N}_{12}^{-1}\Delta_R \in \mathcal{M}(H). \tag{5.4}$$

Intuitively, if H is $R_u(s)$ as in Example 2.2, then we interpret the diagonal matrices Δ_L and Δ_R as follows: Δ_{Lk} extracts the $\mathcal{U}_e$-zeros that are common to all elements in the k-th row of N_{12}; Δ_L "book-keeps" the $\mathcal{U}_e$-zeros of $P_{12} = N_{12} D_p^{-1}$ that appear in each entry of some row of N_{12}. Clearly, P_{12} may have other $\mathcal{U}_e$-zeros that cannot be extracted by Δ_L; these $\mathcal{U}_e$-zeros are the $\mathcal{U}_e$-zeros of $\det\hat{N}_{12}$ (equivalently, the $\mathcal{U}_e$-poles of $\hat{N}_{12}^{-1}$). Now the diagonal matrix Δ_R makes $\hat{N}_{12}^{-1}\Delta_R$ H-stable, i.e., cancels these $\mathcal{U}_e$-poles. Let $s \in \mathcal{U}_e$ be a zero of Δ_R (hence a $\mathcal{U}_e$-zero of $\det\hat{N}_{12}$); the multiplicity of $s \in \mathcal{U}_e$ in $\det\Delta_R$ may exceed its multiplicity in $\det\hat{N}_{12}$. If $\det\hat{N}_{12} \in H^{n_i \times n_i}$ has n zeros at $s \in \mathcal{U}_e$, then $\det\Delta_R$ has *at most* n^{n_i} zeros at $s \in \mathcal{U}_e$; so Δ_R has at most as many $\mathcal{U}_e$-zeros as $(\det\hat{N}_{12})I_{n_i}$.

5.1. Definition (Achievable diagonal $H_{zv'}$): The set $\hat{\mathrm{A}}_{zv'}(\hat{P}) := \{\, H_{zv'} : \hat{C}$ *H-stabilizes* $\hat{P}$ and the map $H_{zv'}$ is diagonal and nonsingular $\}$ is called the set of *all achievable diagonal nonsingular maps* $H_{zv'} : v' \mapsto z$.

5.2. Theorem (Class of all achievable diagonal $H_{zv'}$): Let $\hat{P} \in \mathcal{M}(G)$ be Σ-admissible, and let $P \in \mathcal{M}(G_s)$; let $N_{12} \in H^{n_i \times n_i}$ be nonsingular. Under these conditions, $\hat{\mathrm{A}}_{zv'}(\hat{P}) = \{\, \Delta_L \Delta_R \hat{Q}_{21} : \hat{Q}_{21} \in \mathcal{M}(H)$ is diagonal and nonsingular $\}$, where Δ_L and Δ_R are the diagonal, nonsingular matrices defined by equations (5.1) and (5.3).

5.3. Comments: (i) The map $H_{zv'} = \Delta_L \Delta_R \hat{Q}_{21}$ (where $\hat{Q}_{21} \in \mathcal{M}(H)$) is an achievable map of $\Sigma(\hat{P}, \hat{C})$ if and only if the compensator parameter Q_{21} is chosen as

$$Q_{21} = \hat{N}_{21}^{-1}\Delta_R \hat{Q}_{21}; \tag{5.5}$$

where $\hat{Q}_{21} \in H^{n_i \times n_i}$ is diagonal and nonsingular. By equation (5.4), $Q_{21} \in \mathcal{M}(H)$. Therefore, to achieve diagonalization, from the set $\hat{\mathrm{S}}(\hat{P})$ of all H-stabilizing compensators $\hat{C}$, we must choose $C_{21} = \tilde{D}_c^{-1} Q_{21} = (V_p - Q\tilde{N}_p)^{-1} Q_{21}$ as

$$C_{21} = (V_p - Q\tilde{N}_p)^{-1}\hat{N}_{12}^{-1}\Delta_R \hat{Q}_{21}, \tag{5.6}$$

where the matrix $\hat{Q}_{21} \in H^{n_i \times n_i}$ is diagonal and nonsingular. In (5.6), $Q \in H^{n_i \times n_o}$ is a free parameter and is *not* used in diagonalizing the I/O map H_{zv}. (ii) If H is $R_u(s)$ as in Example 2.2, then the "cost" of diagonalizing the map $H_{zv'}$ is that the number of $\mathcal{U}_e$-zeros are increased. Since Δ_L is a factor of N_{12}, $H_{zv'}$ must have zeros at the $\mathcal{U}_e$-zeros of Δ_L; the multiplicity of a $\mathcal{U}_e$-zero of $H_{zv'}$ may be larger than its multiplicity in $\det N_{12}$ due to Δ_R. If Δ_L represents all $\mathcal{U}_e$-zeros of P_{12} (equivalently, if $\hat{N}_{12}^{-1} \in \mathcal{M}(H)$) and if $\hat{Q}_{21}$ is chosen so that it has no $\mathcal{U}_e$-zeros, then the $\mathcal{U}_e$-zeros of the diagonal $H_{zv'}$ have the same multiplicity as in $\det N_{12}$ since $\Delta_R = I_{n_i}$. The parameter Q_{21} is now restricted to be $\hat{N}_{12}^{-1}\Delta_R \hat{Q}_{21}$ and hence, can no longer be assigned arbitrarily; the only freedom left is the diagonal nonsingular matrix $\hat{Q}_{21} \in \mathcal{M}(H)$. (iii) Although we chose to diagonalize the map $H_{zv'}$, we could also diagonalize $H_{yv'} : v' \mapsto y$, the map from the same external-input v' to the output y of $\hat{P}$ (y is the output used in the feedback-loop). In that case, assuming that $n_o = n_i$ and that $N_p \in H^{n_i \times n_i}$ is nonsingular, we define $\Delta_{Rp}, \Delta_{Lp}, \hat{N}_p$ from N_p as we did above to obtain Δ_L, Δ_R and $\hat{N}_{12}$ from N_{12}; the set of all achievable nonsingular maps $H_{yv'}$ is then $\hat{\mathrm{A}}_{yv'}(\hat{P})$, where $\hat{\mathrm{A}}_{yv'}(\hat{P}) = \{\, \Delta_{Lp}\Delta_{Rp}\hat{Q}_{21} : \hat{Q}_{21} \in \mathcal{M}(H)$ is diagonal and nonsingular $\}$. The compensator parameter Q_{21} should be chosen as $\hat{N}_p^{-1}\Delta_{Rp}\hat{Q}_{21}$. (iv) In the unity-feedback system $S(P, C)$, diagonalizing the map $H_{yu'} : u' \mapsto y$ would depend on the choice for Q such that $N_p(U_p + Q\tilde{D}_p)$ is diagonal, and hence, diagonalizing the map $H_{yu'}$ in $S(P, C)$ may not be possible for certain plants. If $P \in \mathcal{M}(H)$, P is square and nonsingular, then the compensator $(I_{n_i} - QP)^{-1}Q$ achieves the diagonalization requirement if $Q \in \mathcal{M}(H)$ is chosen as $Q = \hat{P}^{-1}\Delta_R \hat{Q}$, where $P = \Delta_L \hat{P}$. (The matrices Δ_L and Δ_R are similarly defined for P instead of N_{12}).

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