

Circuits, k -Ports, Hidden Modes, and Stability of Interconnected k -Ports

CHARLES A. DESOER, FELLOW, IEEE, AND A. NAZLI GÜNDEŞ

Abstract—This paper exclusively considers lumped k -ports and circuits which contain linear time-invariant elements, independent sources, and controlled sources. The k -ports are represented by their hybrid matrices. Tableau equations of circuits are used as a special case of polynomial matrix description. The hidden modes of a circuit are determined by inspection from its tableau equations in the Hermite row form. The same form is used also to determine the exponential stability of the circuit and that of the k -port. Finally, necessary and sufficient conditions for exponential stability of interconnected k -ports are given; the hidden modes of the interconnection are studied. The paper is self-contained.

I. INTRODUCTION

THIS paper investigates the dynamics of k -ports obtained from lumped, linear, time-invariant circuits, of circuits resulting from k -ports driven by independent sources and of interconnections of two k -ports. In all these cases, the circuits may include R , L , C 's, ideal transformers, independent and controlled sources. As in [Bel. 1] we describe our circuits by polynomial equations; more specifically we use tableau equations.

Belevitch was the first to systematically use polynomial equations to derive properties of circuits. Later Rosenbrock [Ros. 1] applied similar methods to control problems. More recently, Callier and Civalieri [Cal. 2] used the Polynomial Matrix Description (PMD) to state conditions for complete controllability and observability of n -ports and circuits based on the Hermite Normal Form and the Smith Canonical Form.

In this paper we use tableau equations to study the relationship between a k -port and a circuit obtained by driving the k -port by independent sources. The k -ports are represented by appropriate hybrid matrices (see, e.g., [Chu. 1], [Chu. 2], [Chu. 3]). In Section VI we determine the hidden modes of the circuit by inspection from its tableau equations and relate the hidden modes of each individual circuit to those of the interconnected circuit. The tableau equations bookkeep the behavior of all branch voltages and all branch currents; therefore, the stability results account for the exponential stability of the circuit as a whole instead of that of a chosen set of output variables.

Roughly speaking, the stability results are as follows: the exponential stability of the k -port is necessary for, but does not guarantee, the stability of the circuit driven by inde-

pendent sources at the ports of the k -port. The interconnected circuit is not exponentially stable if the individual circuits in the interconnection have unstable hidden modes.

The first three sections give the construction of the k -port and the circuit, the formulation of tableau equations and the stability results based on them. Section IV relates the tableau equations to the PMD and is followed by an example. The concepts of hidden modes and exponential stability of the interconnection of two k -ports are treated in Sections VII and VIII.

Notation

\mathbb{R} (\mathbb{C}) field of real (complex) numbers; $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$, equivalently, the closed right-half of the complex plane; $\mathbb{C}_- := \{s \in \mathbb{C} : \text{Re}(s) < 0\}$, equivalently, the open left-half of the complex plane; $\mathbb{R}[p]$ euclidean ring of polynomials in p with real coefficients; $\mathbb{R}(p)$ field of rational functions in p with real coefficients; $A^{m \times n}$ set of $m \times n$ arrays of elements belonging to the set A (e.g., $\mathbb{R}^{m \times n}$, $\mathbb{R}[p]^{m \times n}$, $\mathbb{R}(p)^{m \times n}$, \dots); $\text{rk}(A)$ the rank of matrix A ; $\mathcal{Z}[f]$ the list of zeros of the function f ; $\mathcal{P}[H]$ the list of poles of the matrix function H ; \cup as in $(A \cup B)$ the concatenation of the lists A and B ; $\mathbf{1}_k$ the $k \times k$ identity matrix.

1. Generating the k -Port K from the Given Circuit \mathcal{N}

The given circuit \mathcal{N} is an arbitrary interconnection of lumped, linear, time-invariant circuit elements including independent sources. It has a connected graph of b branches and n nodes.

Assumption 1.1. The circuit \mathcal{N} is uniquely solvable.

The uncommitted circuit \mathcal{K} is obtained from \mathcal{N} as follows: k 1-ports are generated by soldering-iron entries to some nodes of \mathcal{N} and by pliers entries to some branches of \mathcal{N} [Chu. 4, p. 232]. A branch, whose nature is not yet specified, is connected to each port. These branches are called the uncommitted port-branches [Chu. 1, p. 260] and they are assigned the voltage and current reference directions of the port where they are connected (see Fig. 1). The resulting circuit is called \mathcal{K} : it has k port-branches, b internal branches (the same as the branches of \mathcal{N}), and $n := n_k - 1$ nodes excluding datum, where $n_k \geq n_t$.

In the circuit \mathcal{K} , let i and v denote the b -vectors of internal branch currents and of voltages. The superscript p distinguishes the port-branch variables from the internal branch variables: i and v have associated reference directions whereas i^p and v^p have nonassociated reference

Manuscript received June 1, 1984. This work was supported by the National Science Foundation under grant ECS-8119763.

The authors are with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

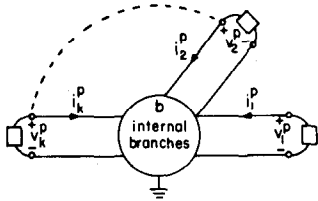


Fig. 1. The uncommitted circuit \mathcal{N} with b internal branches and k port-branches.

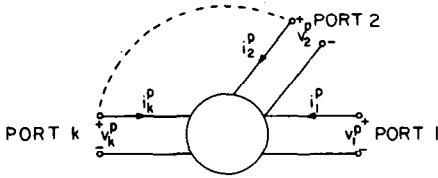


Fig. 2. The k -port K .

directions as far as the port-branches are concerned (see Fig. 1).

Let us now 1) remove all the *uncommitted port-branches* from the circuit \mathcal{N} and 2) put all its internal branches and internal nodes inside a black-box; the result is a k -port called K (see Fig. 2). Note that the port-variables i^p and v^p are the only measurable variables of the k -port K .

Definition 1.2 [Bel. 1, p. 66]: An n -port is said to be *well-defined* iff there is at least one way of choosing n independent sources to terminate its n ports such that the port-variables of the circuit thus formed are uniquely solvable for all values of the independent sources. Equivalently, an n -port is well defined iff it has at least one hybrid representation.

Fact 1.3: Assumption 1.1 implies that the k -port K is well defined.

Proof: One way of choosing the k independent sources is to connect independent current (voltage) sources to the ports of K that were created by soldering-iron entries (pliers-entries, resp.) on the circuit \mathcal{N} . The circuit thus obtained is zero-input equivalent to \mathcal{N} , which is uniquely solvable by Assumption 1.1. \square

II. THE HYBRID REPRESENTATION FOR THE k -PORT K

The Circuit \mathcal{X}_h

Let k_1 ($k_2 := k - k_1$) ports of the k -port K be driven by independent voltage (current) sources and call these ports the “voltage-ports” (“current-ports”, resp.).

Assumption 2.1: The hybrid representation corresponding to this partitioning of the k -ports of K exists. At least one such hybrid representation is guaranteed to exist by Fact 1.3.

The circuit obtained by driving the k -port K by k_1 independent voltage sources and k_2 independent current-sources is called \mathcal{X}_h (see Fig. 3). \mathcal{X}_h has the same digraph as the uncommitted circuit \mathcal{N} : the uncommitted port-branches of the circuit \mathcal{N} are now specified as k_1 voltage-port branches and k_2 current-port branches in the circuit \mathcal{X}_h .

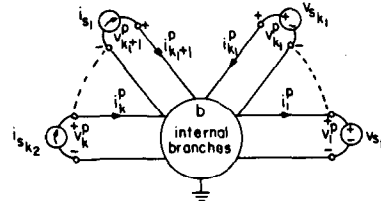


Fig. 3. The circuit \mathcal{X}_h with b internal branches, k_1 voltage-port branches and k_2 current-port branches.

Partition the port-branch variables i^p and v^p of the circuit \mathcal{X}_h as follows:

$$v^p := \begin{bmatrix} v_1^p \\ v_2^p \end{bmatrix} \quad \text{where } v_1^p := [v_{s_1} \ v_{s_2} \ \dots \ v_{s_{k_1}}]^T =: v_s$$

$$i^p := \begin{bmatrix} i_1^p \\ i_2^p \end{bmatrix} \quad \text{where } i_2^p := [i_{s_1} \ i_{s_2} \ \dots \ i_{s_{k_2}}]^T =: i_s.$$

Therefore, v_1^p and i_2^p (i_1^p and v_2^p) are the port-branch variables of the voltage-ports (current-ports, resp.). The *tableau equations* for the circuit \mathcal{X}_h can be written using the node voltages (e), the internal branch voltages and currents (v and i), and the port-branch variables (v^p and i^p).

Tableau Equations for the Circuit \mathcal{X}_h

For tableau equations we use the form

$$T(p)w(t) = u(t)$$

where d/dt is denoted by p . For the hybrid circuit \mathcal{X}_h

$$n \begin{bmatrix} 0 & 0 & A_b & 0 & A_p \\ -A_b^T & 1_b & 0 & 0 & 0 \\ -A_p^T & 0 & 0 & 1_k & 0 \\ 0 & M(p) & N(p) & 0 & 0 \\ 0 & 0 & 0 & 1_{k_1} & 0 & 0 & 0 & 0 & 1_{k_2} \end{bmatrix}$$

$$\begin{bmatrix} e(t) \\ v(t) \\ i(t) \\ -v_1^p(t) \\ -v_2^p(t) \\ i_1^p(t) \\ i_2^p(t) \end{bmatrix} \begin{matrix} n \\ b \\ b \\ k \\ k \\ k \\ k \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -u_s \\ -v_s \\ i_s \end{bmatrix} \begin{matrix} n \\ b \\ k \\ b \\ k_1 \\ k_2 \end{matrix} \quad (2.1)$$

$T(p) \in \mathbb{R}[p]^{(2b+n+2k) \times (2b+n+2k)}$ is the tableau matrix, $[A_b \ ; \ A_p] \in \mathbb{R}^{n \times (b+k)}$ is the reduced incidence matrix of \mathcal{X}_h where A_b (A_p) corresponds to the internal branches (port-branches, resp.). (For an example, see (5.1) below.)

In the tableau equations (2.1), the first n are KCL equations, the next $b+k$ are KVL equations, and the equations $M(p)v(t) + N(p)i(t) = u_s$, $M, N \in \mathbb{R}[p]^{b \times b, 1}$

¹Standard procedures guarantee that the elements of $M(p)$ and $N(p)$ are at most of degree 1. [Vla. 1, p. 101].

correspond to the b internal branch equations with u_s representing the *internal independent sources*. The last k are the branch equations of the independent sources connected at the ports. Note the minus sign in front of the port voltages due to nonassociated reference directions.

The Tableau Matrix $T(p)$

The circuit \mathcal{X}_h , and in fact any uniquely solvable circuit obtained by connecting independent sources to the ports of the k -port K , has the same digraph and internal branches as the uncommitted circuit \mathcal{X} . Therefore, the first $2b + n + k$ tableau equations of the circuit \mathcal{X}_h are identical to those of any other well-defined circuit obtained from the k -port K . Their respective tableau matrices differ only in the last k rows.

Fact 2.2. Let Assumptions 1.1 and 2.1 hold. Then

- the tableau matrix $T(p)$ is nonsingular;
- the first $2b + n + k$ rows of $T(p)$ are linearly independent in the module $(\mathbb{R}[p]^{2b+n+2k}, \mathbb{R}[p])$. [Sig. 1, ch. 6].

Proof: a) The circuit \mathcal{X}_h is uniquely solvable (Assumption 2.1) and thus its tableau equations have a unique solution; equivalently, the polynomial $\det T(p) \neq 0$.

b) Let \mathcal{X}_{h_p} be the particular circuit obtained by connecting independent current (voltage) sources to the ports of the k -port K that were created by soldering-iron entries (pliers-entries, resp.). Then the circuit \mathcal{X}_{h_p} is uniquely solvable by Assumption 1.1 since it is zero-input equivalent to the given circuit \mathcal{X} . Let $T_p(p)$ be the tableau matrix for the circuit \mathcal{X}_{h_p} which differs only in the last k rows from the tableau matrix $T(p)$ of the circuit \mathcal{X}_h . Since \mathcal{X}_{h_p} is uniquely solvable, $T_p(p)$ is nonsingular with its $2b + n + 2k$ rows (or any subset of its rows) linearly independent in the module $(\mathbb{R}[p]^{2b+n+2k}, \mathbb{R}[p])$. Then the first $2b + n + k$ rows of $T(p)$, which are identical to those of $T_p(p)$, are linearly independent in the same module. \square

Let $\tilde{R}(p) \in \mathbb{R}[p]^{(2b+n+k) \times (2b+n+2k)}$ denote the rectangular matrix that corresponds to the first $2b + n + k$ rows of $T(p)$. Then $\tilde{R}(p)$ is full row-rank by Fact 2.2.

The Hermite Row Form of $\tilde{R}(p)$

By elementary column operations (in the ring $\mathbb{R}[p]$) performed on the rectangular matrix $\tilde{R}(p)$, let us now

- make a change of variables from $-v^p$ to v^p , and
- reorder

$$\left[\begin{array}{c} v^p \\ i^p \end{array} \right] = \left[\begin{array}{c} v_1^p \\ v_2^p \\ i_1^p \\ i_2^p \end{array} \right]$$

as

$$\left[\begin{array}{c} i_1^p \\ v_2^p \\ v_1^p \\ i_2^p \end{array} \right]$$

in the first $2b + n + k$ tableau equations in (2.1).

With all *internal* independent sources of the circuit \mathcal{X}_h turned-off, these equations read

$$\tilde{R}(p) \left[\begin{array}{c} x \\ i_1^p \\ v_2^p \\ v_1^p \\ i_2^p \end{array} \right] = \mathbf{0} \quad \text{where } x := \left[\begin{array}{c} e \\ v \\ i \end{array} \right]. \quad (2.2)$$

Fact 2.3: Let Assumptions 1.1 and 2.1 hold. Then the polynomial matrix $\tilde{R}(p)$ defined in (2.2) can be put in the following *Hermite row-form* by elementary row operations in the ring $\mathbb{R}[p]$:

$$2b + n \left[\begin{array}{c|c|c} U & W & -\tilde{W} \\ \hline \mathbf{0} & A & -B \end{array} \right] \left[\begin{array}{c} x \\ i_1^p \\ v_2^p \\ v_1^p \\ i_2^p \end{array} \right] = \mathbf{0} \quad (2.3)$$

where $U(p) \in \mathbb{R}[p]^{(2b+n) \times (2b+n)}$ is an upper triangular nonsingular matrix with nonzero monic polynomials in p of degree at most 1 on the main diagonal, $A(p), B(p) \in \mathbb{R}[p]^{k \times k}$. (The matrix on the left of (2.3) is called $\hat{R}(p) \in \mathbb{R}[p]^{\nu \times (\nu+k)}$ where $\nu := 2b + n + k$.)

Comment: From the Hermite row form, all entries of $U(p)$ above the main diagonal are (possibly zero) constants.

Proof: The square tableau matrix $T(p)$ for the circuit \mathcal{X}_h in (2.1) has polynomial entries of degree at most 1 since $d/dt := p$. The last k equations in (2.1) specify the nature of the port-branches and hence, the last k rows of $T(p)$ are all zeros except in the last $2k$ columns. $T(p)$ can be reduced to an upper triangular Hermite row form by elementary row operations in the ring $\mathbb{R}[p]$ without using the last k rows to bring zeros below the main diagonal in the first $2b + n$ rows. Since $T(p)$ is nonsingular, its Hermite row form has nonzero diagonal entries. By elementary row operations performed on the first $2b + n + k$ rows of $T(p)$ we obtain $\hat{R}(p)$, the Hermite row form of the submatrix $\tilde{R}(p)$ which corresponds to the first $2b + n + k$ rows of $T(p)$. \square

The Hybrid Matrix H for the k -Port K

The last k equations in (2.3), namely,

$$A(p) \left[\begin{array}{c} i_1^p \\ v_2^p \end{array} \right] - B(p) \left[\begin{array}{c} v_1^p \\ i_2^p \end{array} \right] = \mathbf{0} \quad (2.4)$$

are the constraints imposed on the port-variables by the k -port K .

In the circuit \mathcal{X}_h ,

$$\left[\begin{array}{c} v_1^p \\ i_2^p \end{array} \right] = \left[\begin{array}{c} v_s \\ i_s \end{array} \right]$$

are the “independent” port-variables. Since the choice of independent sources corresponds to a uniquely solvable circuit \mathcal{X}_h , the hybrid representation is well-defined and the polynomial matrix $A \in \mathbb{R}[p]^{k \times k}$ is nonsingular. Then

from equation (2.3), the dependent port-variables

$$\begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix}$$

are represented in terms of the independent port-variables:

$$\begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix} = A^{-1}(p)B(p) \begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix} = H(p) \begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix} \quad (2.5)$$

where $H := A^{-1}B \in \mathbb{R}(p)^{k \times k}$ is the hybrid-matrix for the k -port K .

III. STABILITY OF THE CIRCUIT \mathcal{X}_h AND OF THE k -PORT K

If the hybrid matrix H of some k -port is not proper (i.e., has a pole at ∞), then some bounded inputs produce unbounded outputs even if H is analytic in \mathbb{C}_+ . To wit, in the one-port case, with a first-order pole at ∞ , the output due to the bounded input $\sin(\omega_0 t^2)$ includes the term $2\omega_0 t \cos(\omega_0 t^2)$ which is not bounded on \mathbb{R}_+ . Consequently, our definition for the exponential stability of a k -port will be a variant on the "bounded-input bounded-output" stability definition.

The stability of the k -port K , which is a black-box with access only to its ports, does not guarantee that the circuit \mathcal{X}_h is also exponentially stable. The internal behavior of the circuit \mathcal{X}_h is not available for measurement at the k ports of K unless all modes of the circuit \mathcal{X}_h are both controllable (i.e., excitable) by the given inputs at the ports and appear at the ports under zero input but for some appropriate initial condition.

Let

$$\begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix} = \begin{bmatrix} v_s \\ i_s \end{bmatrix}$$

be the input to the circuit \mathcal{X}_h and let

$$\begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix}$$

be the output. Then the hybrid matrix $H \in \mathbb{R}(p)^{k \times k}$ of the k -port K is the network function of the circuit \mathcal{X}_h from the given inputs to the outputs.

Definition 3.1: The k -port K with the hybrid matrix H is said to be exponentially stable iff bounded inputs with bounded support (say on $[0, T]$) create zero-state responses which go to zero exponentially as t approaches ∞ .

Definition 3.2: The network function $H \in \mathbb{R}(p)^{k \times k}$ is said to be exponentially stable iff $\mathcal{P}[H] \subset \hat{\mathbb{C}}_-$.

Comment: In most control applications the following definition is adopted: the transfer function (network function) H is exponentially stable iff H is proper and $\mathcal{P}[H] \subset \hat{\mathbb{C}}_-$. (For input/output properties of exponentially stable transfer functions so defined, see [Cal. 1, p. 127].)

Definition 3.3: $\lambda \in \mathbb{C}$ is called a natural frequency of the circuit \mathcal{X}_h iff, for some initial condition, the zero-input response of the circuit \mathcal{X}_h is of the form

$$\begin{bmatrix} x \\ v^p \\ i^p \end{bmatrix} = \alpha e^{\lambda t} \quad \text{where } \alpha \in \mathbb{C}^{(2b+n+2k)}.$$

Any zero-input response $t \mapsto \alpha e^{\lambda t}$ is called a mode of the circuit \mathcal{X}_h associated with the natural frequency λ .

Standard ordinary differential equation calculation show that λ is a natural frequency of the circuit \mathcal{X}_h iff $\det T(\lambda) = 0$ where $T(p)$ is the tableau matrix of \mathcal{X}_h .

Remark: Associated with a given natural frequency, say λ , there may be several modes:

$$\alpha_1 e^{\lambda t}, \alpha_2 e^{\lambda t}, \dots, \alpha_k e^{\lambda t}$$

where the vectors $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}^{(2b+n+2k)}$ are linearly independent members of the nullspace of $T(\lambda)$.

Definition 3.4: The circuit \mathcal{X}_h is said to be exponentially stable iff, for all initial conditions, the zero-input response (i.e., all branch voltages, all node voltages and all branch currents) goes to zero exponentially as $t \rightarrow \infty$. Equivalently, all natural frequencies of the circuit \mathcal{X}_h have negative real parts, $(\sigma(\mathcal{X}_h) \subset \hat{\mathbb{C}}_-)$, where $\sigma(\mathcal{X}_h)$ is the list of natural frequencies of the circuit \mathcal{X}_h .

Analysis and Stability Theorems

With

$$\begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix}$$

as the input to the circuit \mathcal{X}_h , rewrite (2.3) in the form

$${}_{2b+n} \begin{bmatrix} U & W \\ \mathbf{0} & A \end{bmatrix} {}_{k} \begin{bmatrix} x \\ i_1^p \\ v_2^p \end{bmatrix} = \begin{bmatrix} \bar{W} \\ B \end{bmatrix} \begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix}. \quad (3.1)$$

The matrix on the left of (3.1) is called $\hat{T}(p) \in \mathbb{R}[p]^{(2b+n) \times (2b+n+k)}$.

Comment: We refer to $\hat{T}(p)$ as the tableau matrix of the circuit \mathcal{X}_h although it is obtained by elementary operations from the original tableau matrix of equations (2.1). Since the zeros of $\det T(p)$ are the natural frequencies of the circuit \mathcal{X}_h , it is important to note that $\det \hat{T}(p)$ is equal to $\det T(p)$ times a nonzero constant and $\mathcal{Z}[\det \hat{T}] = \mathcal{Z}[\det T]$.

Let $\hat{L}(\cdot) \in \mathbb{R}[p]^{k \times k}$ be any g.c.l.d. (greatest common left divisor) of the polynomial matrices A and B of (3.1) [Cal. 1, p. 24], [Kal. 1, p. 376], [Ros. 1, p. 70]. Equivalently, there exists polynomial matrices A, B such that

$$A = \hat{L}\bar{A}, \quad B = \hat{L}\bar{B} \quad (3.2)$$

and the pair (\bar{A}, \bar{B}) is left-coprime. Then the hybrid matrix $H = A^{-1}B$ of the k -port K can also be expressed as

$$H = \bar{A}^{-1}\bar{B} \quad (3.3)$$

where $\mathcal{P}[H] = \mathcal{Z}[\det \bar{A}]$.

With these notations in mind, we have the

Theorem 3.5. The k -port K specified by the hybrid matrix H is exponentially stable iff the hybrid matrix (network function) H is exponentially stable.

Proof: The k -port K is a black-box which allows access only to the port-variables v^p, i^p of the circuit \mathcal{X}_h ; thus the modes of the k -port K are those modes of the circuit \mathcal{X}_h which are available for measurement at the k ports. Therefore, the k -port K is characterized by the network function H from the given inputs at the ports to the output port-variables. Using Laplace transforms it is

easily shown that for bounded inputs with bounded support (say on $[0, T]$), the resulting outputs go to zero exponentially as $t \rightarrow \infty$ iff this network function \mathbf{H} is exponentially stable; equivalently, iff $\mathcal{P}[\mathbf{H}] = \mathcal{Z}[\det \bar{\mathbf{A}}] \subset \dot{\mathbf{C}}_-$. (Note that the inputs are identically zero for $t > T$). \square

Theorem 3.6. Let Assumptions 1.1 and 2.1 hold. Then the circuit \mathcal{X}_h with tableau equations (3.1) is exponentially stable iff the characteristic polynomial $\chi(p) := \det \hat{\mathbf{T}}(p)$ has no zeros in \mathbf{C}_+ ; equivalently a) $\det \mathbf{U}(p)$ has no zeros in \mathbf{C}_+ (i.e., all diagonal entries of $\mathbf{U}(p)$ are strictly Hurwitz), and b) given any g.c.l.d. $\hat{\mathbf{L}}$ of (\mathbf{A}, \mathbf{B}) , $\det \hat{\mathbf{L}}(p)$ has no zeros in \mathbf{C}_+ and c) the k -port \mathbf{K} is exponentially stable (equivalently, $\mathcal{P}[\mathbf{H}] \subset \dot{\mathbf{C}}_-$).

Proof: $\sigma(\mathcal{X}_h)$, the list of all natural frequencies of the circuit \mathcal{X}_h , is given by $\sigma(\mathcal{X}_h) = \mathcal{Z}[\chi] = \mathcal{Z}[\det \hat{\mathbf{T}}]$. With $\mathbf{A} = \hat{\mathbf{L}}\bar{\mathbf{A}}$, and from (3.1) we obtain

$$\begin{aligned} \chi(p) &= \det \hat{\mathbf{T}}(p) = \det \mathbf{U}(p) \det \mathbf{A}(p) \\ &= \det \mathbf{U}(p) \det \hat{\mathbf{L}}(p) \det \bar{\mathbf{A}}(p). \end{aligned} \quad (3.4)$$

Hence,

$$\mathcal{Z}[\chi] = \mathcal{Z}[\det \mathbf{U}] \cup \mathcal{Z}[\det \hat{\mathbf{L}}] \cup \mathcal{Z}[\det \bar{\mathbf{A}}]. \quad (3.5)$$

Furthermore,

$$\mathcal{P}[\mathbf{H}] = \mathcal{P}[\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}] = \mathcal{Z}[\det \bar{\mathbf{A}}] \quad (3.6)$$

since $\bar{\mathbf{A}}, \bar{\mathbf{B}}$ are left-coprime. The conclusion follows from (3.5) and (3.6). \square

Definition 3.7: A mode of the circuit \mathcal{X}_h is said to be a *hidden mode* iff it is not a mode of the k -port \mathbf{K} ; equivalently, it is not controllable by the inputs

$$\begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix}$$

and/or not observable at the output port-variables

$$\begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix}.$$

Comment: We shall see later that 1) the list of "hidden modes" of the circuit \mathcal{X}_h is $\mathcal{Z}[\det \mathbf{U}] \cup \mathcal{Z}[\det \hat{\mathbf{L}}]$. If $z \in \mathbf{C}$ is in this list and $\det \bar{\mathbf{A}}(z) \neq 0$, then no input with bounded support, (say on $[0, T]$), can create a response at the output port-variables which contains the term e^{zt} for all $t > T$.

2) If there are no hidden modes, or if all the hidden modes are *exponentially stable*, then the exponential stability of the k -port \mathbf{K} is equivalent to the exponential stability of the circuit \mathcal{X}_h .

IV. THE POLYNOMIAL MATRIX DESCRIPTION (PMD) OF THE CIRCUIT \mathcal{X}_h

Let us rewrite equation (3.1) for the circuit \mathcal{X}_h in the following form:²

$$\underbrace{\begin{bmatrix} \mathbf{U} & \mathbf{W} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}}_{\mathbf{D}(p)} \underbrace{\begin{bmatrix} \mathbf{x} \\ i_1^p \\ v_2^p \end{bmatrix}}_{\xi(t)} = \underbrace{\begin{bmatrix} -\bar{\mathbf{W}} \\ -\bar{\mathbf{B}} \end{bmatrix}}_{\mathbf{N}_l(p)} \underbrace{\begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix}}_{\mathbf{u}(t)} \quad (4.1)$$

$$\underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{1}_k \end{bmatrix}}_{\mathbf{N}_r(p)} \underbrace{\begin{bmatrix} \mathbf{x} \\ i_1^p \\ v_2^p \end{bmatrix}}_{\xi(t)} = \underbrace{\begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix}}_{\mathbf{y}(t)}. \quad (4.2)$$

Equations (4.1)–(4.2) define the Polynomial Matrix Description (PMD) $\mathcal{D} = [\mathbf{D}, \mathbf{N}_l, \mathbf{N}_r, \mathbf{0}]$ of the circuit \mathcal{X}_h [Cal. 1, sec. 3.2], [Kai. 1, sec. 6.2.3], [Ros. 1, sec. 2.2]. In equations (4.1)–(4.2),

- a) $\mathbf{D}(p) \in \mathbb{R}[p]^{\nu \times \nu}$, $\mathbf{N}_l(p) \in \mathbb{R}[p]^{\nu \times k}$, $\mathbf{N}_r(p) \in \mathbb{R}[p]^{k \times \nu}$;
- b) $\mathbf{D}(\cdot)$ is nonsingular;
- c) $\mathbf{u}(\cdot): \mathbb{R}_+ \rightarrow \mathbb{R}^k$, $\xi(\cdot): \mathbb{R}_+ \rightarrow \mathbb{R}^\nu$, $\mathbf{y}(\cdot): \mathbb{R}_+ \rightarrow \mathbb{R}^k$ are called the *input*, *pseudostate*, and *output* of the PMD;
- d) if we want to avoid δ -functions, $\mathbf{u}(\cdot)$ must be piecewise sufficiently differentiable [Cal. 1, p. 93].

Call $\mathbf{H}(s) \in \mathbb{R}(s)^{k \times k}$ the transfer function of the PMD. Then $\mathbf{H}(s) = \mathbf{N}_r(s)\mathbf{D}^{-1}(s)\mathbf{N}_l(s)$ and the Laplace transform of the zero-state response is $\hat{\mathbf{y}}(s) = \mathbf{H}(s)\hat{\mathbf{u}}(s)$.

Decoupling zeros of the PMD: [Cal. 1, sec. 3.2], [Ros. 1, p. 64]:

Let $\tilde{\mathbf{L}}(\cdot) \in \mathbb{R}[p]^{\nu \times \nu}$ be any g.c.l.d. of $(\mathbf{D}, \mathbf{N}_l)$. Equivalently, there are polynomial matrices $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{N}}_l$ such that

$$\mathbf{D} = \tilde{\mathbf{L}}\tilde{\mathbf{D}} \quad \text{and} \quad \mathbf{N}_l = \tilde{\mathbf{L}}\tilde{\mathbf{N}}_l \quad (4.3)$$

and $(\tilde{\mathbf{D}}, \tilde{\mathbf{N}}_l)$ is left-coprime.

Definition 4.1: A point $z_i \in \mathbf{C}$ is called an *input-decoupling zero* (i-d zero) of the PMD \mathcal{D} described by (4.1)–(4.2) iff given any g.c.l.d. $\tilde{\mathbf{L}}(\cdot)$ of $(\mathbf{D}, \mathbf{N}_l)$, $\det \tilde{\mathbf{L}}(z_i) = 0$ [Cal. 1, p. 101].

Rank Test 4.2 [Cal. 1, p. 101, ex. 37], [Ros. 1, ch. 2]: $z_i \in \mathbf{C}$ is an i-d zero of the PMD \mathcal{D} described by (4.1)–(4.2) iff $\text{rk}[\mathbf{D}(z_i) \ ; \ \mathbf{N}_l(z_i)] < \nu$.

It can be shown that every *i-d zero* of the PMD \mathcal{D} is associated with an *uncontrollable* mode of the circuit \mathcal{X}_h [Cal. 1, p. 101].

Let $\mathbf{R}(\cdot) \in \mathbb{R}[p]^{\nu \times \nu}$ be any greatest common right divisor (g.c.r.d) of $(\mathbf{N}_r, \mathbf{D})$. Equivalently, there are polynomial matrices $\bar{\mathbf{N}}_r, \hat{\mathbf{D}}$ such that

$$\mathbf{D} = \hat{\mathbf{D}}\mathbf{R} \quad \mathbf{N}_r = \bar{\mathbf{N}}_r\mathbf{R} \quad (4.4)$$

and the pair $(\bar{\mathbf{N}}_r, \hat{\mathbf{D}})$ is right-coprime.

Definition 4.3: A point $z_0 \in \mathbf{C}$ is called an *output-decoupling zero* (o-d zero) of the PMD \mathcal{D} described by (4.1)–(4.2) iff given any g.c.r.d. $\mathbf{R}(\cdot)$ of $(\mathbf{N}_r, \mathbf{D})$, $\det \mathbf{R}(z_0) = 0$ [Cal. 1, p. 104], [Ros. 1, p. 65].

Rank Test 4.4 [Cal. 1, p. 104, ex 26], [Ros. 1, ch. 2]: $z \in \mathbf{C}$ is an o-d zero of the PMD \mathcal{D} described by (4.1)–(4.2) iff

$$\text{rk} \begin{bmatrix} \mathbf{D}(z_0) \\ \mathbf{N}_r(z_0) \end{bmatrix} < \nu.$$

Fact 4.5: $z_0 \in \mathbf{C}$ is an o-d zero of the PMD \mathcal{D} described by (4.1)–(4.2) iff $\det \mathbf{U}(z_0) = 0$.

Proof: Using the rank test 4.4, z_0 is an o-d zero iff

$$\text{rk} \begin{bmatrix} \mathbf{U} & \mathbf{W} \\ \mathbf{0} & \mathbf{A} \\ \mathbf{0} & \mathbf{1}_k \end{bmatrix} (z_0) < \nu \Leftrightarrow \text{rk} \mathbf{U}(z_0) < \nu - k$$

and the conclusion follows. \square

²Note that what is called $\mathbf{D}(p)$ in this context was previously called $\hat{\mathbf{T}}(p)$.

VI. PHYSICAL INTERPRETATION OF THE MODES OF THE CIRCUIT \mathcal{X}_h AND OF THE k -PORT K

Consider the PMD equations (4.1)–(4.2) for the circuit \mathcal{X}_h . We now determine the hidden modes of the circuit from (4.1) by inspection and give a physical interpretation for each natural frequency.

The uncontrollable and unobservable hidden modes of the circuit \mathcal{X}_h

Proposition 6.1: Consider (4.1). If the i th diagonal entry of the upper triangular matrix U is the monic first degree polynomial $(p - \lambda_1)$ and if the i th rows of the matrices $W(\lambda_1)$ and $\bar{W}(\lambda_1)$ are both zero, then associated with the natural frequency λ_1 , there is an *uncontrollable and unobservable* mode.

Proof: By assumption, $\det U(\lambda_1) = 0 \Rightarrow \det D(\lambda_1) = \det T(\lambda_1) = 0$ and λ_1 is a natural frequency of the circuit \mathcal{X}_h . By the rank test 4.4, λ_1 is an o-d zero since U drops rank at λ_1 . By the rank test 4.2, λ_1 is an i-d zero since the i th rows of $U(\lambda_1)$, $W(\lambda_1)$, and $\bar{W}(\lambda_1)$ are zero. Therefore, λ_1 is both an o-d zero and an i-d zero and the conclusion follows. \square

In the example of Section V, $\lambda_1 = 1/2$ corresponds to an uncontrollable and unobservable mode since row 19 of equation (5.2) is

$$(p - 1/2)i_2 = 0.$$

The controllable but unobservable hidden modes of the circuit \mathcal{X}_h

Proposition 6.2: Consider (4.1). If the j th diagonal entry of the upper triangular matrix U is $(p - \lambda_2)$ and if $\text{rk}[D(\lambda_2) \vdots N_l(\lambda_2)] = \nu$, then associated with the natural frequency λ_2 there is an *unobservable* mode that is controllable.

Proof: By assumption, $\det U(\lambda_2) = 0$; hence λ_2 is a natural frequency of the circuit \mathcal{X}_h . By the rank test 4.4, λ_2 is an o-d zero, but since $\text{rk}[D(\lambda_2) \vdots N_l(\lambda_2)] = \nu$, λ_2 is not an i-d zero and the conclusion follows. \square

In the example of Section V, $\lambda_2 = 1/5$ and $\lambda_3 = 1$ correspond to unobservable but controllable modes. From rows 21 and 30 of equation (5.2)

$$\begin{aligned} \left(p - \frac{1}{5}\right)i_4 - g_m p v_2^p &= 0 \text{ and } (p + 1)v_2^p = i_2^p = i_s \\ \Rightarrow i_4 &= \frac{1}{p - 1/5} \frac{g_m p}{(p + 1)} i_s. \end{aligned}$$

From row 28, $(p - 1)i_{11} + (1 - p)i_1^p = -v_1^p = -v_s$. Note that $\lambda_4 = 1 = \lambda_3$ is a multiple natural frequency: λ_4 is an i-d zero but λ_3 is an o-d zero.

The Observable but Uncontrollable Hidden Modes of the Circuit \mathcal{X}_h

Proposition 6.3: Consider (4.1). Let $\det A(\lambda_4) = 0$. Then associated with the natural frequency λ_4 there is an *uncontrollable* mode that is observable iff $\text{rk}[A(\lambda_4) \vdots B(\lambda_4)] < k$.

Proof: The assumption implies that $\det D(\lambda_4) = \det \hat{T}(\lambda_4) = 0$ and that λ_4 is a natural frequency of the

circuit \mathcal{X}_h . Since $\mathcal{P}[\det \hat{L}]$ is the complete list of the i-d zeros that are not o-d zeros, λ_4 is associated with an uncontrollable but observable mode iff $\det[\hat{L}(\lambda_4)] = 0$, where \hat{L} is as in (3.2). But $\det[\hat{L}(\lambda_4)] = 0 \Leftrightarrow \text{rk}[A(\lambda_4) \vdots B(\lambda_4)] < k$ and the conclusion follows. \square

Comment: If, in addition, $\det U(\lambda_4) = 0$, λ_4 is a repeated natural frequency and associated with λ_4 there is another mode that is *unobservable*.

In the example of section V, $\lambda_4 = 1$ corresponds to an uncontrollable but observable mode. From row 29 of equation (5.2)

$$(p - 1)i_1^p = -(p - 1)v_1^p.$$

(The first rows of both A and B only have $(p - 1)$.)

The Observable and controllable modes of the Circuit \mathcal{X}_h

Proposition 6.4: Consider (4.1) and (3.4)–(3.5). If $\det \bar{A}(\lambda_5) = 0$ then associated with the natural frequency λ_5 there is a mode that is both observable and controllable. Hence, this mode is a mode of the k -port K .

Proof: By assumption $\lambda_5 \in \sigma(\mathcal{X}_h) = \mathcal{P}[\chi]$. By the rank tests,

$$\text{rk}[D(\lambda_5) \vdots N_l(\lambda_5)] = \nu \text{ and } \text{rk}\left[\frac{D(\lambda_5)}{N_r(\lambda_5)}\right] = \nu.$$

Therefore, λ_5 is not a decoupling zero. \square

Comment: If λ_5 is a repeated natural frequency, then associated with λ_5 , there may also be another mode that is hidden if $\det U(\lambda_5) = 0$ and/or if $\det \hat{L}(\lambda_5) = 0$.

In the example of Section V, $\lambda_5 = -1$ corresponds to a controllable and observable mode: from row 30 of equation (5.2)

$$(p + 1)v_2^p = i_2^p = i_s.$$

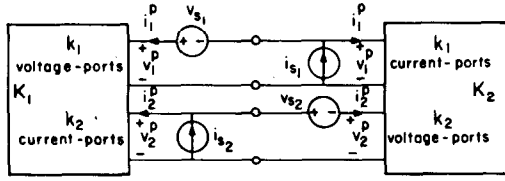
Comment: The impulse response of the k -port K contains a term of the form $p(t)e^{\lambda t}$ for some λ (with $p(t) \in \mathbb{R}[t]$) $\Leftrightarrow \lambda \in \mathcal{P}[H]$. The *zero-input response* of the circuit \mathcal{X}_h may contain terms of the form $p'(t)e^{\lambda t}$ such terms are created by initial conditions inside the k -port K that cannot be set up by appropriate port excitations.

VII. INTERCONNECTION OF TWO k -PORTS

Construction

For $\alpha = 1, 2$, let the linear, time-invariant circuits \mathcal{N}_α be uniquely solvable and have no independent sources. Let Assumption 1.1 hold for the circuits \mathcal{N}_α , and let \mathcal{N}_α have b_α branches and n_α nodes.

The uncommitted circuits \mathcal{X}_α with b_α internal branches, n_α nodes, and k port-branches are obtained from the given circuits \mathcal{N}_α as \mathcal{X} was obtained from \mathcal{N} in Section I. Continue the procedure of Section I to obtain the k -ports K_α from the circuits \mathcal{X}_α . From Fact 1.3, the k -ports K_1 and K_2 are well-defined.


 Fig. 7. The interconnected circuit \mathcal{X}_i .

Next partition the k ports such that k_1 ports of the k -port K_1 (K_2) are voltage-ports (current-ports) and the rest are current-ports (voltage-ports, resp.) and assume that the corresponding hybrid representations exist. Therefore the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} obtained similarly as the circuit \mathcal{X}_h are uniquely solvable. Observe that if we use a hybrid representation for the k -port K_1 with the k_1 voltage-ports and the $k_2 := k - k_1$ current-ports chosen above, then for the k -port K_2 we use the (dual) hybrid representation with the k_1 current-ports and the k_2 voltage-ports chosen above.

Connect the k_1 voltage-ports (k_2 current-ports) of the k -port K_1 to the k_1 current-ports (k_2 voltage-ports, resp.) of the k -port K_2 . Call the resulting circuit the interconnected circuit \mathcal{X}_i (see Fig. 7).³

Let x_1 (x_2) denote the $2b_1 + n_1$ ($2b_2 + n_2$) internal-branch variables and

$$\begin{pmatrix} v^p \\ i^p \end{pmatrix}_1 \begin{pmatrix} v^p \\ i^p \end{pmatrix}_2$$

denote the $2k$ port-branch variables of the circuit \mathcal{X}_{h_1} (\mathcal{X}_{h_2} , resp.). Then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ represents the internal-branch variables and

$$\left(\begin{pmatrix} v^p \\ i^p \end{pmatrix}_1^T, \begin{pmatrix} v^p \\ i^p \end{pmatrix}_2^T \right)^T$$

are the driving-port variables of the circuit \mathcal{X}_i . The independent-source drive of the interconnection is

$$u_1 := \begin{pmatrix} v_{s1} \\ i_{s2} \end{pmatrix} \quad \text{and} \quad u_2 := \begin{pmatrix} -i_{s1} \\ -v_{s2} \end{pmatrix}.$$

We call $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ the driving-point input of the interconnected circuit \mathcal{X}_i .

VIII. TABLEAU EQUATIONS, PMD, AND STABILITY OF THE CIRCUIT \mathcal{X}_i

For each of the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} , we write tableau equations as in (2.1) and put each of the tableau equations into the Hermite row-form (2.3). These tableau equations lead us to the PMD's for the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} similar to the PMD \mathcal{D} of (4.1)–(4.2).

The PMD for the circuit \mathcal{X}_{h_1}

From the tableau equations for the circuit \mathcal{X}_{h_1} , we obtain its PMD $\mathcal{D}_1 = [D_1, N_1, N_r, \mathbf{0}]$

$$2b_1 + n_1 \underbrace{\begin{bmatrix} 2b_1 + n_1 & k \\ U_1 & W_1 \\ \mathbf{0} & A_1 \end{bmatrix}}_{D_1(p)} \underbrace{\begin{bmatrix} x_1 \\ i_1^p \\ v_2^p \end{bmatrix}}_{\xi_1(t)} = \underbrace{\begin{bmatrix} k \\ \tilde{W}_1 \\ B_1 \end{bmatrix}}_{N_1(p)} \underbrace{\begin{bmatrix} v_1^p \\ i_2^p \end{bmatrix}}_{e_1(t)} \quad (8.1)$$

$$k \underbrace{\begin{bmatrix} \mathbf{0} & \vdots & \mathbf{1}_k \end{bmatrix}}_{N_r(p)} \underbrace{\begin{bmatrix} x_1 \\ i_1^p \\ v_2^p \end{bmatrix}}_{\xi_1(t)} = \underbrace{\begin{bmatrix} i_1^p \\ v_2^p \end{bmatrix}}_{y_1(t)} \quad (8.2)$$

where $D_1(p) \in \mathbb{R}[p]^{v_1 \times v_1}$, $N_1(p) \in \mathbb{R}[p]^{v_1 \times k}$, $N_r(p) \in \mathbb{R}[p]^{k \times v_1}$, $v_1 := 2b_1 + n_1 + k$. Since the circuit \mathcal{X}_{h_1} is uniquely solvable, $\det D_1(p) = \det U_1(p) \det A_1(p) \neq 0$.

From

$$A_1 \begin{pmatrix} i_1^p \\ v_2^p \end{pmatrix} = B_1 \begin{pmatrix} v_1^p \\ i_2^p \end{pmatrix}_1$$

we obtain the hybrid matrix $H_1 := A_1^{-1} B_1 \in \mathbb{R}(p)^{k \times k}$ where A_1 is nonsingular since the hybrid representation exists by assumption. The transfer function (network function) from

$$\begin{pmatrix} v_1^p \\ i_2^p \end{pmatrix}_1$$

to

$$\begin{pmatrix} i_1^p \\ v_2^p \end{pmatrix}_1$$

is

$$H_1 = N_r D_1^{-1} N_1 = A_1^{-1} B_1. \quad (8.3)$$

The PMD for the Circuit \mathcal{X}_{h_2}

Similarly for \mathcal{X}_{h_2} , we have the PMD $\mathcal{D}_2 = [D_2, N_2, N_r, \mathbf{0}]$

$$2b_2 + n_2 \underbrace{\begin{bmatrix} 2b_2 + n_2 & k \\ U_2 & W_2 \\ \mathbf{0} & B_2 \end{bmatrix}}_{D_2(p)} \underbrace{\begin{bmatrix} x_2 \\ v_1^p \\ -i_2^p \end{bmatrix}}_{\xi_2(p)} = \underbrace{\begin{bmatrix} k \\ \tilde{W}_2 \\ A_2 \end{bmatrix}}_{N_2(p)} \underbrace{\begin{bmatrix} -i_1^p \\ v_2^p \end{bmatrix}}_{e_2(t)} \quad (8.4)$$

$$k \underbrace{\begin{bmatrix} \mathbf{0} & \vdots & \mathbf{1}_k \end{bmatrix}}_{N_r(p)} \underbrace{\begin{bmatrix} x_2 \\ v_1^p \\ -i_2^p \end{bmatrix}}_{\xi_2(t)} = \underbrace{\begin{bmatrix} v_1^p \\ -i_2^p \end{bmatrix}}_{y_2(t)} \quad (8.5)$$

³These connection rules guarantee that the interconnection is well-posed.

where $D_2(p) \in \mathbb{R}[p]^{\nu_2 \times \nu_2}$, $N_{l_2}(p) \in \mathbb{R}[p]^{\nu_2 \times k}$, $N_{r_2}(p) \in \mathbb{R}[p]^{k \times \nu_2}$, $\nu_2 := 2b_2 + n_2 + k$. Since the circuit \mathcal{X}_{h_2} is uniquely solvable, $\det D_2(p) = \det U_2(p) \det B_2(p) \neq 0$. As for the k -port K_1 we have $H_2 := B_2^{-1}A_2$, and

$$H_2 = N_{r_2} D_2^{-1} N_{l_2} = B_2^{-1} A_2. \quad (8.6)$$

The PMD of the Circuit \mathcal{X}_i

First we concatenate the PMD equations (8.1)–(8.2) and (8.4)–(8.5), and reorder variables to obtain

$$\underbrace{\begin{bmatrix} U_1 & \mathbf{0} & W_1 & \mathbf{0} \\ \mathbf{0} & U_2 & \mathbf{0} & W_2 \\ \mathbf{0} & \mathbf{0} & A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B_2 \end{bmatrix}}_{D_i(p)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \left(i_1^p \right) \\ \left(v_2^p \right)_1 \\ \left(v_1^p \right) \\ \left(-i_2^p \right)_2 \end{bmatrix}}_{\xi(t)} = \underbrace{\begin{bmatrix} \tilde{W}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{W}_2 \\ B_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix}}_{N_{l_i}(p)} \underbrace{\begin{bmatrix} \left(v_1^p \right) \\ \left(i_2^p \right)_1 \\ \left(-i_1^p \right) \\ \left(v_2^p \right)_2 \end{bmatrix}}_{e(t)} \quad (8.7)$$

$$\underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_k \end{bmatrix}}_{N_{r_i}(p)} \xi(t) = \underbrace{\begin{bmatrix} \left(i_1^p \right) \\ \left(v_2^p \right)_1 \\ \left(v_1^p \right) \\ \left(-i_2^p \right)_2 \end{bmatrix}}_{y(t)} \quad (8.8)$$

where

$$\begin{aligned} \det D_i(p) &= \det U_1(p) \det A_1(p) \det U_2(p) \det B_2(p) \\ &= \det D_1(p) \cdot \det D_2(p) \neq 0. \end{aligned}$$

Using KVL and KCL, we obtain the connection equations from Fig. 7.

$$\begin{aligned} \begin{pmatrix} v_1^p \\ i_2^p \end{pmatrix}_1 &= \begin{pmatrix} v_1^p \\ -i_2^p \end{pmatrix}_2 + \begin{pmatrix} v_{s_1} \\ i_{s_2} \end{pmatrix} \\ \begin{pmatrix} -i_1^p \\ v_2^p \end{pmatrix}_2 &= \begin{pmatrix} i_1^p \\ v_2^p \end{pmatrix}_1 + \begin{pmatrix} -i_{s_1} \\ -v_{s_2} \end{pmatrix}. \end{aligned} \quad (8.9)$$

Eliminating

$$\begin{pmatrix} v_1^p \\ i_2^p \end{pmatrix}_1 \quad \text{and} \quad \begin{pmatrix} -i_1^p \\ v_2^p \end{pmatrix}_2$$

from (8.7) to (8.8) by using (8.9), we get the PMD $\mathcal{D}_i =$

$[D_g, N_{l_i}, N_{r_i}, \mathbf{0}]$ of the circuit \mathcal{X}_i :

$$\underbrace{\begin{bmatrix} 2b_1 + n_1 & U_1 & \mathbf{0} & W_1 & -W_1 \\ 2b_2 + n_2 & \mathbf{0} & U_2 & -W_2 & W_2 \\ k & \mathbf{0} & \mathbf{0} & A_1 & -B_1 \\ k & \mathbf{0} & \mathbf{0} & -A_2 & B_2 \end{bmatrix}}_{D_g(p)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \left(i_1^p \right) \\ \left(v_2^p \right)_1 \\ \left(v_1^p \right) \\ \left(-i_2^p \right)_2 \end{bmatrix}}_{\xi(t)} = \underbrace{\begin{bmatrix} \tilde{W}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{W}_2 \\ B_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix}}_{N_{l_i}(p)} \underbrace{\begin{bmatrix} v_{s_1} \\ i_{s_2} \\ -i_{s_1} \\ -v_{s_2} \end{bmatrix}}_{u(t)} \quad (8.10)$$

$$\underbrace{k}_{k} \underbrace{[\mathbf{0} \quad \mathbf{1}_{2k}]}_{N_{r_i}(p)} \xi(t) = y(t) \quad (8.11)$$

where $N_{l_i}(p) \in \mathbb{R}[p]^{(\nu_1 + \nu_2) \times k}$, $N_{r_i}(p) \in \mathbb{R}[p]^{k \times (\nu_1 + \nu_2)}$, $\nu_1 + \nu_2 = (2b_1 + n_1 + k) + (2b_2 + n_2 + k)$.

Equations (8.10) are the tableau equations of the circuit \mathcal{X}_i modulo elementary operations. The characteristic polynomial of the circuit \mathcal{X}_i is $\chi_i(p) = \det D_g(p)$.

Fact 8.1 (Well-posedness condition): Let the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} be uniquely solvable (equivalently, $\det D_1(\cdot) \neq 0$, $\det D_2(\cdot) \neq 0$). Under these conditions, the circuit \mathcal{X}_i is uniquely solvable (equivalently, $\det D_g(\cdot) \neq 0$) $\Leftrightarrow \det(I - H_1 H_2) = \det(I - H_2 H_1) \neq 0$.

Proof: From (8.10) we obtain

$$\det D_g(p) = \det U_1(p) \det U_2(p)$$

$$\cdot \det \begin{bmatrix} A_1(p) & -B_1(p) \\ -A_2(p) & B_2(p) \end{bmatrix}.$$

By elementary column operations,

$$\begin{aligned} \det D_g(p) &= \det U_1(p) \det U_2(p) \det A_1(p) \\ &\quad \cdot \det B_2(p) \det(I - B_2^{-1} A_2 A_1^{-1} B_1) \quad (8.12) \\ &= \det D_1(p) \det D_2(p) \det(I - H_2 H_1) \quad (8.13) \end{aligned}$$

and the conclusion follows. \square

Let $\hat{L}_1(\cdot) \in \mathbb{R}[p]^{k \times k}$ be any g.c.l.d. of (A_1, B_1) and $\hat{L}_2(\cdot) \in \mathbb{R}[p]^{k \times k}$ be any g.c.l.d. of (B_2, A_2) . Equivalently \exists polynomial matrices \bar{A}_1, \bar{B}_1 and \bar{B}_2, \bar{A}_2 such that

$$\begin{aligned} A_1 &= \hat{L}_1 \bar{A}_1 & B_1 &= \hat{L}_1 \bar{B}_1, & (\bar{A}_1, \bar{B}_1) &\text{ is left-coprime} \\ B_2 &= \hat{L}_2 \bar{B}_2 & A_2 &= \hat{L}_2 \bar{A}_2, & (\bar{B}_2, \bar{A}_2) &\text{ is left-coprime.} \end{aligned} \quad (8.14)$$

Then the hybrid matrices H_1 and H_2 for the k -ports K_1

and K_2 defined by (8.2) and (8.6) resp. become

$$H_1 = \bar{A}_1^{-1} \bar{B}_1 \quad H_2 = \bar{B}_2^{-1} \bar{A}_2. \quad (8.15)$$

The network function H_{yu} from the driving-point input u to the output y is given by⁴

$$\begin{aligned} H_{yu} &:= N_r D_g^{-1} N_l \\ &= \begin{bmatrix} H_1(I - H_2 H_1)^{-1} & H_1 H_2 (I - H_1 H_2)^{-1} \\ H_2 H_1 (I - H_2 H_1)^{-1} & H_2 (I - H_1 H_2)^{-1} \end{bmatrix}. \end{aligned} \quad (8.16)$$

From (8.14) to (8.15) we have

$$\mathcal{P}[H_1] = \mathcal{Z}[\det \bar{A}_1], \quad \mathcal{P}[H_2] = \mathcal{Z}[\det \bar{B}_2]$$

and an easy calculation shows that

$$\mathcal{P}[H_{yu}] = \mathcal{Z}[\det \bar{A}_1 \det \bar{B}_2 \det (I - H_2 H_1)] \quad (8.17)$$

From (8.12) to (8.14) we obtain the characteristic polynomial $\chi_i(p)$ of the circuit \mathcal{X}_i :

$$\begin{aligned} \chi_i(p) &= \det D_g(p) \\ &= \det U_1(p) \det U_2(p) \det \hat{L}_1(p) \det \hat{L}_2(p) \\ &\quad \cdot [\det \bar{A}_1(p) \det \bar{B}_2(p) \det (I - H_2 H_1)]. \end{aligned} \quad (8.18)$$

Let $\sigma(\mathcal{X}_i)$ denote the list of all natural frequencies of the circuit \mathcal{X}_i . Then from (8.17) to (8.18),

$$\begin{aligned} \sigma(\mathcal{X}_i) &= \mathcal{Z}[\chi_i] \\ &= \mathcal{Z}[\det U_1] \cup \mathcal{Z}[\det U_2] \cup \mathcal{Z}[\det \hat{L}_1] \\ &\quad \cup \mathcal{Z}[\det \hat{L}_2] \cup \mathcal{P}[H_{yu}]. \end{aligned} \quad (8.19)$$

Stability of the circuit \mathcal{X}_i and Physical Interpretation of the Modes

Equation (8.19) will be used to give a physical interpretation of the natural frequencies of the circuit \mathcal{X}_i . First we show that the circuit \mathcal{X}_i inherits of all hidden modes of the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} and *no other* hidden modes result from the interconnection. From (8.1)–(8.2), ((8.4)–(8.5), resp.):

a) z_1 (z_2) is an o-d zero of the PMD for the circuit \mathcal{X}_{h_1} (\mathcal{X}_{h_2}) iff $\det U_1(z_1) = 0$ ($\det U_2(z_2) = 0$, resp.),

b) s_1 (s_2) is an i-d zero of the PMD for the circuit \mathcal{X}_{h_1} (\mathcal{X}_{h_2}) iff $\text{rk}[D_1(s_1) \vdots N_{l_1}(s_1)] < \nu_1$ ($\text{rk}[D_2(s_2) \vdots N_{l_2}(s_2)] < \nu_2$, resp.). Then we have the

Theorem 8.2 Consider the PMD's for the circuits \mathcal{X}_{h_1} , \mathcal{X}_{h_2} , and \mathcal{X}_i defined by (8.1)–(8.2), (8.4)–(8.5), and (8.10)–(8.11), resp.

i) The list of all o-d zeros of the PMD for the circuit \mathcal{X}_i is the concatenated list of the o-d zeros of the PMD's for the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} .

ii) The list of all i-d zeros of the PMD for the circuit \mathcal{X}_i is the concatenated list of the i-d zeros of the PMD's for the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} .

iii) The complete list of the decoupling zeros of the PMD for the circuit \mathcal{X}_i is the list

$$\mathcal{Z}[\det U_1] \cup \mathcal{Z}[\det U_2] \cup \mathcal{Z}[\det \hat{L}_1] \cup \mathcal{Z}[\det \hat{L}_2]. \quad (8.20)$$

iv) The circuit \mathcal{X}_i is exponentially stable iff the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} have *no unstable hidden modes* and $\mathcal{P}[H_{yu}] \subset \mathbb{C}_-$.

Proof: i) $z_0 \in \mathbb{C}$ is an o-d zero of the PMD for \mathcal{X}_i , (see (8.10)–(8.11)),

$$\Leftrightarrow \text{rk} \begin{bmatrix} D_g(z_0) \\ N_r(z_0) \end{bmatrix} < \nu_1 + \nu_2$$

$$\Leftrightarrow \det U_1(z_0) \det U_2(z_0) = 0$$

$$\Leftrightarrow z_0 \text{ is an o-d zero of } \mathcal{X}_{h_1} \text{ and/or of } \mathcal{X}_{h_2}$$

and the conclusion follows.

ii) $z_i \in \mathbb{C}$ is an i-d zero of the PMD for \mathcal{X}_i .

$$\Leftrightarrow \text{rk}[D_g(z_i) \vdots N_l(z_i)] < \nu_1 + \nu_2$$

$$\Leftrightarrow \text{rk} \begin{bmatrix} U_1 & \mathbf{0} & W_1 & -\bar{W}_1 & \bar{W}_1 & \mathbf{0} \\ \mathbf{0} & U_2 & -\bar{W}_2 & W_2 & \mathbf{0} & \bar{W}_2 \\ \mathbf{0} & \mathbf{0} & A_1 & -B_1 & B_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -A_2 & B_2 & \mathbf{0} & A_2 \end{bmatrix} (z_i) < \nu_1 + \nu_2$$

and by elementary column operations,

$$\Leftrightarrow \text{rk} \begin{bmatrix} U_1 & \mathbf{0} & W_1 & \mathbf{0} & \bar{W}_1 & \mathbf{0} \\ \mathbf{0} & U_2 & \mathbf{0} & W_2 & \mathbf{0} & \bar{W}_2 \\ \mathbf{0} & \mathbf{0} & A_1 & \mathbf{0} & B_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B_2 & \mathbf{0} & A_2 \end{bmatrix} (z_i) < \nu_1 + \nu_2$$

by column and row exchanges,

$$\Leftrightarrow \text{rk} \begin{bmatrix} U_1 & W_1 & \bar{W}_1 & \mathbf{0} \\ \mathbf{0} & A_1 & B_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & U_2 & W_2 & \bar{W}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & B_2 & A_2 \end{bmatrix} (z_i) < \nu_1 + \nu_2$$

$$\Leftrightarrow \text{rk} \begin{bmatrix} D_1 & N_{l_1} & \mathbf{0} \\ \mathbf{0} & D_2 & N_{l_2} \end{bmatrix} (z_i) < \nu_1 + \nu_2$$

$$\Leftrightarrow z_i \text{ is an i-d zero of } \mathcal{X}_{h_1} \text{ and/or of } \mathcal{X}_{h_2}$$

and the conclusion follows.

iii) From i), the list of all o-d zeros of the PMD for \mathcal{X}_i is given by

$$\mathcal{Z}[\det U_1] \cup \mathcal{Z}[\det U_2]. \quad (8.21)$$

⁴Note the similarity between this equation and the feedback connection equations encountered in control applications [e.g., Cal. 1, p. 221].

Let $\hat{L}_1(\cdot)$ and $\hat{L}_2(\cdot)$ be as in eqn. (8.14). Then z_1 (z_2) is an *i-d zero that is not an o-d zero* of the PMD for the circuit \mathcal{X}_{h_1} (\mathcal{X}_{h_2}) iff $\det \hat{L}_1(z_1) = 0$ ($\det \hat{L}_2(z_2) = 0$, resp.). From ii), the list of all *i-d zeros that are not o-d zeros* of the PMD for \mathcal{X}_i is given by

$$\mathcal{Z}[\det \hat{L}_1] \cup \mathcal{Z}[\det \hat{L}_2]. \quad (8.22)$$

The concatenation of the lists from (8.21) to (8.22) gives the complete list of the decoupling zeros of the PMD for \mathcal{X}_i and the conclusion follows.

iv) The circuit \mathcal{X}_i has a hidden mode associated with each decoupling zero of the circuit \mathcal{X}_{h_1} and of the circuit \mathcal{X}_{h_2} . The list of all natural frequencies $\sigma(\mathcal{X}_i)$, given by (8.19), is the concatenation of $\mathcal{P}[H_{yu}]$ and the decoupling zeros listed in (8.20). Associated with each decoupling zero $z_d \in \mathbb{C}_+$ there is an *unstable* hidden mode of the circuit \mathcal{X}_i . Since the circuit \mathcal{X}_i is exponentially stable iff $\sigma(\mathcal{X}_i) \subset \mathbb{C}_-$, the conclusion (iv) follows. \square

The physical interpretation of the natural frequencies of the circuit \mathcal{X}_i listed in (8.20) is as follows: Associated with each natural frequency from the list in (8.21), the circuit \mathcal{X}_i has an *unobservable hidden mode*. Some of these modes may also be uncontrollable as well as being unobservable. (To determine the *o-d zeros that are also i-d zeros* for each of the circuits \mathcal{X}_{h_1} and \mathcal{X}_{h_2} , see Section VI.) Associated with each natural frequency from the list in (8.22), the circuit \mathcal{X}_i has an *uncontrollable (but observable) hidden mode*. Associated with each natural frequency, say λ_{co} , from the list in (8.17), the circuit \mathcal{X}_i has a controllable and observable mode and the impulse response at the driving point of \mathcal{X}_i includes an exponential term of the form $p(t)e^{\lambda_{co}t}$ (where $p(t)$ is a polynomial).

Remark: In this paper we choose to write tableau equations as a general method of circuit analysis although all of the discussion above is also valid for Modified Node Analysis (MNA), using as circuit variables the node voltages, the additional branch currents, and the port variables v^p and i^p .

IX. CONCLUSION

This paper investigates the dynamics of lumped, linear time-invariant k -ports, and of circuits obtained from them, by using tableau equations. In a polynomial matrix description framework, the concepts of modes, hidden modes, uncontrollable and unobservable modes are explained and conditions are obtained for the exponential stability of k -ports (theorem 3.5), of circuits (theorem 3.6), and of interconnected k -ports (theorem 8.2). It is shown that the interconnection of two k -ports inherits all hidden modes of each of the individual circuits in the interconnection and that the presence of any unstable hidden modes causes the *circuits* to be exponentially unstable even though the net-

work functions from their inputs to their outputs are exponentially stable.

ACKNOWLEDGMENT

The authors are very grateful for the reviewers' careful comments.

REFERENCES

- [Bel. 1] V. Belevitch, *Classical Network Theory*. San Francisco, CA: Holden-Day 1968.
- [Cal. 1] F. M. Callier and C. A. Desoer, *Multivariable Feedback Systems*. New York-Heidelberg-Berlin: Springer-Verlag, 1982.
- [Cal. 2] F. M. Callier and P. P. Civalleri, "On the controllability and observability of linear lumped time-invariant N-ports: Hermite normal form versus Smith canonical form," *Alta Freq.*, vol. XLVII-N.1.
- [Car. 1] H. J. Carlin and A. B. Giordano, *Network Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1964.
- [Chu. 1] L. O. Chua and P.-M. Lin, *Computer-Aided Analysis and Design of Electronic Circuits*. Englewood Cliffs, NJ: Prentice-Hall, 1975.
- [Chu. 2] L. O. Chua and Y.-F. Lam, "Dimension of N-ports," *IEEE Trans. Circuits Syst.*, vol. CAS-21, May 1974.
- [Chu. 3] L. O. Chua and B. J. Leon, "Unified N-Port system theory," US Government Res. and Development Rep., vol. 41, Jan. 1966.
- [Chu. 4] L. O. Chua, *Introduction to Nonlinear Network Theory*. New York: McGraw-Hill, 1969.
- [Kai. 1] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.
- [Ros. 1] H. H. Rosenbrock, *State Space and Multivariable Theory*. New York: Wiley, 1970.
- [Sig. 1] L. E. Sigler, *Algebra*. UTM, New York-Heidelberg-Berlin: Springer-Verlag, 1976.
- [Vla. 1] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*. New York: Van Nostrand-Reinhold, 1983.

*

Charles A. Desoer (S'50-A'53-SM'57-F'64), for a photograph and biography please see page 634 of this issue.

*



A. Nazli Gündes started studying electrical engineering at Boğaziçi University, Istanbul, Turkey, and transferred to the University of California, Berkeley, where she received the B.S. (with highest honors) and the M.S. degrees. She is currently working towards the Ph.D degree at Berkeley. In graduate school she has held university fellowships, teaching and research assistantships. Her research interests are in system theory and multi-variable control.